HOMOMORPHISMS HAVING A GIVEN $\mathscr{H}$-CLASS AS A SINGLE CLASS

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In [1] it was shown that if S is a stable semigroup and H an $\mathscr{H}$-class of S then there is a congruence $\mathscr{E}(H)$ on S in which H is a single class. After considering some consequences of this result for abstract semigroups, we consider some analogous questions for compact semigroups.

We note first that if S is (weakly) stable then any homomorphism on S can be factored into five homomorphisms each of which has some reasonably special property. This factorization depends upon (and is defined through), a given $\mathscr{D}$-class. As a corollary, one concludes that on a stable semigroup with a finite number of $\mathscr{D}$-classes any homomorphism can be factored into homomorphisms which alternate between being one-to-one on $\mathscr{H}$-classes and having each class contained in $\mathscr{H}$. This is an extension of a result of Rhodes for finite semigroups [6].

In the case of a compact semigroup, we note that $\mathscr{E}(H_1)$, the Teissier congruence defined by H_1, need not be upper semicontinuous. However, we show that if S is a compact totally disconnected semigroup and H an arbitrary $\mathscr{H}$-class of S, there is a closed congruence having H as a single class. Thus, the result of Rhodes, in an appropriate $\mathscr{D}$-class formulation, holds for profinite semigroups.

Using some results of Malcev on the congruences on the full transformation semigroup on a set we construct a compact connected locally connected one dimensional semigroup with identity which cannot be brought to a point with a finite sequence of homomorphisms alternating in the sense above. This answers a question raised in [7, p. 159].

For convenience let us record some items which are germain in what follows. S^1 will denote S if the latter has an identity and the extended semigroup if not. The Green equivalences:

$$a \equiv b(\mathscr{L}) \Leftrightarrow S^1a = S^1b, \quad a \equiv b(\mathscr{R}) \Leftrightarrow aS^1 = bS^1$$
$$a \equiv b(\mathscr{J}) \Leftrightarrow S^1aS^1 = S^1bS^1, \; \mathscr{H} = \mathscr{L} \cap \mathscr{R}$$
$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$$

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The semigroup S is stable if for all $a, b \in S$

(1) $Sa \subseteq Sab \to Sa = Sab$

(2) $aS \subseteq baS \to aS = baS$

If S is stable then it is rather clear that S^1 is stable. However, S may be unstable while S^1 is stable. [5].

Following [5] we shall call S weakly stable if S^1 is stable. If S is weakly stable $\mathscr{D} = \mathscr{J}$ since adjunction of an identity leaves the Green equivalences unchanged. The set $S^1DS^1 \setminus D = I(D)$, where D is a $\mathscr{D}$-class, is an ideal of S.

If the subsets A and B have a nonvacuous intersection we shall write $A \mathbin{\overline{\underline{\circ}}} B$.

Malcev [4], Teissier, [8], have shown that for a subset M of an (abstract) semigroup S to be the class of some congruence on S it is necessary and sufficient that for any pairs of points x, y one has

$$xMy \mathbin{\overline{\underline{\circ}}} M \to xMy \subseteq M$$

The congruence generated by M — which we shall call the Teissier congruence associated with M — is as follows:

$a \equiv b(\mathscr{E}(M))$ if and only if there exist points
$x_1, x_2, x_3, \cdots, x_n, y_1, y_2, y_3, \cdots, y_n$ such that
$a \in x_1My_1 \mathbin{\overline{\underline{\circ}}} x_2My_2 \mathbin{\overline{\underline{\circ}}} x_3My_3 \mathbin{\overline{\underline{\circ}}} \cdots \mathbin{\overline{\underline{\circ}}} x_nMy_n \ni b$

PROPOSITION 1. *Let S be a weakly stable semigroup, D a $\mathscr{D}$-class of S, and f a homomorphism onto T. Then there is a commutative diagram of semigroups and homomorphisms.*

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ & \searrow S' \to S'' \to S''' \nearrow & \end{array}$$

such that

(1) *$S \to S'$ is one-to-one on the complement of $I(D)$ $(= S^1DS^1 \setminus D)$,*

(2) *$S' \to S''$ is one-to-one on the complement of D' — the image of D — and each nondegenerate clsss is contained in some $\mathscr{H}$-class in D'.*

(3) *$S'' \to S'''$ is one-to-one off of D''-image of D — and one-to-one on any individual $\mathscr{H}$-class.*

(4) *$S''' \to T$ is one-to-one on the ideal generated by D''' — the image of D.*

PROOF. First form S' by letting classes outside of $I(D)$ be degenerate and for $y \in I(D)$ take classes as sets $f^{-1}(t) \cap I(D)$ where $f(y) = t$. That is to say S' is S modulo the above congruence. For simplicity of notation, let us identify D with its image in S'. Now take any nonempty set of the form $C \cap H$ where C is a class of f and H is an $\mathscr{H}$-class in D. We form the Teissier congruence associated with $C \cap H$. If C_0 is any other class of f and H_0 another $\mathscr{H}$-class of D then $\mathscr{E}(C_0 \cap H_0)$ Indeed, if r and s are any two points such that

$$r(C \cap H)s \mathbin{\overline{\underline{\circ}}} C_0 \cap H_0$$

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then

$$r(C \cap H)s = C_0 \cap H_0$$

since the map $h \to rhs$ is one-to-one from H onto H_0. (Two such points necessarily exist since H and H_0 lie in the same D-class. Now by definition, the class of any point outside of $S'DS'$ is degenerate while two points a' and b' in $S'DS'$ are congruent if there exist points $x_1, \cdots, x_n, y_1, \cdots, y_n$ such that

$$a' \in x_1(C \cap H)y_1 \,\underline{\overline{\circ}}\, \cdots \,\underline{\overline{\circ}}\, x_n(C \cap H)y_n \ni b'$$

Now a set such as

$$x_i(C \cap H)y_i$$

lies entirely in D, being then a class of $\mathscr{E}(C \cap H)$, or lies entirely in $I(D)$. In the latter case it is degenerate since the sets $f^{-1}(t) \cap I(D)$ have already been collapsed by the very definition of S'. Thus a' and b', if in $I(D)$, would be one and the same point.

Now to form S''' continue the definition of f on the ideal $S''D''S''$. Clearly S''' may also be viewed as starting with S and collapsing each set $f^{-1}(t) \cap S^1DS^1$.

Finally to map S''' onto T, simply complete the definition of f.

One can obtain a (possibly) finer factorization

$$S \to S_0 \to S' \to S'' \to S''' \to T$$

by first defining S_0 by restricting $\mathscr{E}(C \cap H)$ to $I(D)$. One then continues the definition of f on S_0 to obtain S' and then proceeds as before.

To emphasize the dependence on D one may wtite the factorization as

$$\begin{array}{ccc} S & \xrightarrow{\qquad\quad f \qquad\quad} & T \\ \searrow & S'_D \to S''_D \to S'''_D & \nearrow \end{array}$$

Suppose now that D_1 and D_2 are two $\mathscr{D}$-classes of the weakly stable semigroup S. Suppose that say, D_2 is not in the ideal generated by D_1. Now D_2 may be identified with its image in S'''_{D_1}. Accordingly, we have a factorization

$$\begin{array}{ccc} S & \xrightarrow{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & T \\ \searrow & S'_{D_1} \to S''_{D_1} \to S'''_{D_1} \to (S'''_{D_1})'_{D_2} \to (S'''_{D_1})''_{D_2} \to (S'''_{D_1})'''_{D_2} & \nearrow \end{array}$$

It follows then that if $S/\mathscr{D}$ can be well ordered α, (qua set), in such a way that D_α a D_β implies that it is false that $D_\beta < D_\alpha$ in the usual partial order T is a direct limit using the construction above.

COROLLARY. *If $S/\mathscr{H}$ is finite or if S is weakly stable and $S/\mathscr{D}$ is finite then any homomorphism $f: S \to T$ can be factored*

$$S \to S_1 \to S_2 \to S_3 \to \ldots \to S_n \to T$$

where $S \to S_1$, $S_2 \to S_3$ etc. *have any nondegenerate class entirely contained in an* $\mathscr{H}$*-class of some fixed* $\mathscr{D}$*-class (for the homomorphism) and* $S_1 \to S_2$, $S_3 \to S_4$ *etc. are such that a nondgenerate class is contained in some* $\mathscr{D}$*-class and are one-to-one on any* $\mathscr{H}$*-class.*

In effect, one uses Proposition 1 on each $\mathscr{D}$-class in turn observing the partial order on $S/\mathscr{D}$.

The only remaining point is that the finiteness of $S/\mathscr{H}$ implies stability of S. Clearly it suffices to show that some power of every element lies in some subgroup. If J_0 is the smallest $\mathscr{J}$-class (in the usual partial ordering) containing some power of the element b, say b^q then $B = \langle b^q, b^{q+1}, b^{q+2}, \cdots \rangle$ lies in J_0. The $\mathscr{H}$ equivalence of S is a congruence on B. Since a finite semigroup contains an idempotent some $\mathscr{H}$-class H of J_0 must be such that $H^2 \overline{\underline{\circ}} H$ so that H is a subgroup containing some power of b.

Malcev [4, ch. 10] has shown that the lattice of congruences on a full transformation semigroup $\mathscr{T}_X$ is generated as a lattice by three kinds of congruences. If X is finite there are only two types of congruences to consider and the said lattice is a chain. From the results of Malcev it follows that a congruence on $\mathscr{T}_X$ is either a congruence $\mathscr{E}_N$ for some normal subgroup N of some H_e or is a Rees congruence. In the first case $\mathscr{E}_N$ collapses to a point the entire ideal $I(e)$. [See Theorem 10.68 of [4]).

It will be convenient to have the following lemma which is an easy consequence of Malcev' results:

LEMMA. *Let*

$$\mathscr{T}_X \xrightarrow{f} A \xrightarrow{g} B$$

be a diagram where f and g are onto homomorphisms which are both not isomorphisms. If g is one-to-one on subgroups then the above is precisely

$$\mathscr{T}_X \to \mathscr{T}_X / \mathscr{E}_N \to \mathscr{T}_X / J(H)$$

for some $\mathscr{H}$*-class* H*, where the homomorphisms are the canonical ones.*

It now follows the minimum sequence of alternating homomorphisms needed to collapse $\mathscr{T}_X$ can be given explicitly: (We write $\mathscr{T}$ for $\mathscr{T}_X$ and otherwise follow the notation of [4]. In particular I_k is the ideal of elements having rank $< k$. We use $\mathscr{E}_i$ for the Teissier congruence determined by any $\mathscr{H}$-class of rank i.) Of course the sequence cannot start with a congruence contained in M.

$$\begin{gathered} \mathscr{T} \to \mathscr{T}/I_2 \to \mathscr{T}/\mathscr{E}_2 \to \mathscr{T}/I_3 \to \\ \to \cdots \to \mathscr{T}/\mathscr{E}_i \to \mathscr{T}/I_{i+1} \to \cdots \\ \cdots \to \mathscr{T}/\mathscr{E}_n \to \mathscr{T}/I_{n+1} = \{1\} \end{gathered}$$

Following [7] we write $\#_\alpha(S) = n$ if the shortest alternating sequence needed to collapse S has length n and $\#_\alpha(S) = \infty$ if no such sequence exists.

EXAMPLE 1. *There exists a zero dimensional compact semigroup S with identity,* (on the cantor set) *such that $\#_\alpha(S) = \infty$. Thus, there is a locally connected, one dimensional, compact, connected semigroup with identity having $\#_\alpha(S) = \infty$.*

We have already noted that if card $X = n$ then $\#_\alpha(\mathscr{T}_X) = 2n-1$. Let S be the cartesian product of the semigroups $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \cdots$ etc.

To achieve the second example, one starts with the cone over S and uses the standard constructions (See [3]).

Although a homomorphism cannot in general be given by the action on the subgroups or even by its induced homomorphisms on the Schützenberger groups one can say a few things in certain cases. Suppose x is a point of a (weakly) stable semigroup such that there is an idempotent $e = e(x)$ such that $x = ex$. Then $xG_e = H_x$ for a subgroup G_e of H_e. Thus, instead of using $C \cap H_x$ for the factorization corresponding to D_x, one could equally well use the Teissier decomposition for a certain subset of G_e and restrict this to the ideal generated by D_x.

EXAMPLE 2. *Let $\mathscr{T}$ denote the full transformation semigroup on the integers. If H is an $\mathscr{H}$-class then H is the class of some congruence if and only if H is of finite rank. In particular if H has infinite rank then there exist elements $\alpha, \beta \in \mathscr{T}$ such that $\alpha H \beta$ meets both H and the minimal ideal of $\mathscr{T}$.*

PROOF. Let h be an element of $\mathscr{T} = \mathscr{T}_X$ such that $h(X)$ is infinite. Let β be any one-to-one (into) map of X such that $h(\beta(X))$ is infinite and omits an infinite number of points of $h(X)$. Let σ be an automorphism (qua set) of $h(X)$ such that $h\beta(X)$ and $\sigma h\beta(X)$ are mutually exclusive. Now σn is $\mathscr{H}$ equivalent to h since σ and h yield the same decomposition of X and σ and h have the same range, namely $h(X)$. Let $h' = \sigma h$. Now define α so that (1) α takes $h\beta(X)$ in a one-to-one manner onto $h(X)$, (2) α is constant on $h'\beta(X)$ and (3) $\alpha(X) = h(X)$. Now $\alpha h \beta$ has the same decomposition as h since β was one-to-one and α has the same decomposition as h since β was one-to-one and α was one-to-one on $h\beta(X)$. Moreover $\alpha h\beta(X) = h(X)$ so $\alpha h \beta$ and h are $\mathscr{H}$-equivalent. However, $\alpha h' \beta$ is a constant map of X and so is in its minimal ideal.

Now let $h(X)$ be finite and suppose that $\alpha h \beta$ and h are $\mathscr{H}$ equivalent. Since $\alpha h\beta(X) = h(X)$, we see that $h\beta(X) = h(X)$, and that α is one-to-one on $h(X)$. Let h' be $\mathscr{H}$ equivalent to h. Since $\beta(X)$ meets each class of the map h, it meets each class of h'. Thus $h'\beta(X) = h'(X) = h(X)$. Since α was one-to-one on $h\beta(X)$, it is the same on $h'\beta(X)$. Thus the classes of $\alpha h' \beta$ and h are the same. Since $\alpha h'\beta(X) = h(X)$, we conclude that $\alpha h' \beta$ and h are $\mathscr{H}$ equivalent.

EXAMPLE 3. *There exists a finitely generated semigroup S having an $\mathscr{H}$-class which is not the class of any congruence.*

Define S as $\langle a, b, u, v, r, s, x, y; au = b, va = b, rb = a, bs = a, sby = b\rangle$. Now a and b lie in the same $\mathscr{H}$-class H and clearly $xHy \,\overline{\underline{\circ}}\, H$. However, the element xay is not even $\mathscr{D}$-equivalent with b.

EXAMPLE 4. *There exists a two generator one relator monoid in which H_1 is not the class of any congruence. Namely,*

$$\langle x, y; (xy)^2 = 1\rangle.$$

Here H_1 consists of xy and 1. In particular, x has no left inverse.

Let A be a subset of the semigroup S and let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be elements of S. The sets $A_i = x_iAy_i$ are said to form an A-chain from a to be if (1) $a \equiv b\mathscr{E}(A)$, $a \notin x_jAy_j$ for $j > 1$, $b \notin x_jAy_j$ $j < n$, x_qAy_q does not meet x_rAy_r unless $r = q-1$, q, $q+1$. The number n will be called the length of the chain. The integer $\tau(a, b, A)$ is defined as the minimum such n for which there is an A-chain from a to b. The minimum over all pairs a, b will be denoted by $\tau(S, A)$. The maximum n over all pairs is $\lambda(S, A)$.

EXAMPLE 5. *For any n there is a finite semigroup S such that $\tau(S, H_1) = n$. Here H_1 may be chosen as the group of order 2.*

Define T as $\langle g, x_1, x_2, \cdots, x_n, \cdots, y_n;\ g^2 = \text{identity},\ x_{i+1}y_{i+1} = x_igy_i,\ i = 1, 2, \cdots, n-1.\rangle$

Now let $\sim$ be the congruence which collapses the ideal consisting of all words of length greater than four, and let S be $T/\sim$.

EXAMPLE 6. *There exists a compact zero dimensional semigroup on which the Teissier congruence is not upper semi-continuous.*

Define F as the cartesian product of $F_1, F_2, F_3, \cdots$ where for each i, $\tau(F_i) = i$. Thus if $\tau(a_i, b_i) = i$ where $a_i, b_i \in F_i$ one need only consider the sequences $(a_1, a_2, \cdots, a_i, 1, 1, 1, \cdots)$ and $(b_1, b_2, b_3, \cdots, b_i, 1, 1, 1, \cdots)$.

It is convenient to have available the following.

DEFINITION. A semigroup S is said to be *$\mathscr{H}$-invariantly embedded in T* if two points of S are $\mathscr{H}$ related in T if and only if they are so related in S.

Thus, a standard thread or more generally any compact S with $S/\mathscr{H}$ a thread is $\mathscr{H}$ invariantly embedded in any compact semigroup [3]. The bicyclic semigroup $C(p.q.)$ is $\mathscr{H}$ invariantly embedded in any semigroup [1]. If T is the union of groups then any embedding of S into T (both taken compact) is perforce $\mathscr{H}$ invariant. This fails if T is only regular or even completely 0-simple.

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LEMMA. *The canonical embedding of the inverse limit of a system of compact semigroup is an $\mathscr{H}$ invariant embedding into the cartesian product.*

PROOF. Let $S = \lim_{\leftarrow}(S_\alpha, f_{\alpha,\beta})$ in terms of the lemma. Let $\{x_\alpha\}$ and $\{y_\alpha\}$ points of S which are $\mathscr{H}$ related qua points of the product. Thus there is a $\{p_\alpha\}$ such that for each α we have $x_\alpha = y_\alpha p_\alpha$. We claim that p_α may be taken so that $\{p_\alpha\}$ is in S. For each α let P_α be the set of all points q_α such that

$$x_\alpha = y_\alpha q_\alpha$$

Then for each α, P_α is compact and if $\alpha < \beta$ then $f_{\alpha\beta}(P) \subset P_\alpha$. Thus the compact sets P_α along with $f_{\alpha\beta}$ cut down form an inverse system. Any point $\{p_\alpha\}$ in the limit will be in S and will be such that $\{x_\alpha\} = \{y_\alpha\}\{p_\alpha\}$. In the same way $\{y_\alpha\}$ is in the right ideal generated by $\{x_\alpha\}$ and so forth.

PROPOSITION 2. *Let S be a profinite semigroup (i,e. a compact zero dimensional semigroup). If H is an $\mathscr{H}$-class of S there is a closed congruence $\mathscr{C}(H)$ in which H is a single class.*

PROOF. S, viewed as the inverse limit of finite semigroups, is $\mathscr{H}$ invariantly embedded in the cartesian product of these finite semigroups. On each of these finite semigroups the appropriate Teissier congruence defines a congruence in which a given $\mathscr{H}$-class is a congruence class. The product of these congruences defines a closed congruence on the cartesian product having the product of the $\mathscr{H}$-classes, (which is an $\mathscr{H}$-class in the cartesian product) as a single class. Since S is $\mathscr{H}$-invariantly embedded in the cartesian product the restriction to S defines the desired congruence.

COROLLARY. *Let S be a compact zero dimensional compact semigroup and $f: S \to T$ a continuous homomorphism onto T. In terms of Proposition 1 there is a commutative diagram of compact semigroups and homomorphisms:*

$$\begin{array}{ccccc} S & & \xrightarrow{\quad f \quad} & & T \\ & \searrow & & \nearrow & \\ & S'' & \to & S''' & \end{array}$$

(The factorization $S \to S' \to S''$ is avaliable qua abstract homomorphism, but we now need not have S' compact).

Reasonably clean necessary and sufficient conditions for an $\mathscr{H}$-class of a compact semigroup to be a class of some congruence are not known. In this connection however, we mention that, as a corollary to [3], it follows that if H_e is connected, e lies in the centre of S and eSe under the action of H_e is one dimensional then H_e is a class of a closed congruence.

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