SUMS OF TWO REGULAR ELEMENTS

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A well-known fact concerning a prime right Goldie ring R, proved in [**4**, Section 5], is that an essential right ideal is generated by the regular elements which it contains. There is a modification of that proof which shows that each element of R is the sum of at most two regular elements. This suggested that the recent results of Chatters and Ginn [**1**] concerning rings generated by their regular elements might possibly be refined a little, since their arguments actually show that elements of R are sums of at most three regular elements.

Here, as well as providing the modification mentioned above, we apply the technique to show that various classes of rings have the property that each element is the sum of two regular elements, namely

(i) any semiprime right Goldie ring which does not have $\mathbb{Z}/(2)$ as a direct summand,

(ii) any Noetherian ring with no Artinian right ideals,

(iii) any matrix ring $M_n(K)$ over an integral domain K, unless $n=1$ and $K=\mathbb{Z}/(2)$.

Throughout, if A is an ideal of a ring R then

$$\mathscr{C}(A)=\{c\in R \mid [c+A] \text{ is a regular element of } R/A\}.$$

THEOREM 1. *Let R be a prime right Goldie ring with $R\neq\mathbb{Z}/(2)$, and let $a, x\in R$ with $x\in\mathscr{C}(0)$. Then there exist $b, c\in\mathscr{C}(0)$, with $b\in xR$, such that $a=b+c$.*

Proof. Note first that xR is also a prime right Goldie ring with the same right quotient division ring, Q say, as R, and $Q\simeq M_n(D)$ for some division ring D, by Goldie's theorem. By the Faith–Utumi theorem (see [**3**, Section 4.7]), $xR\supseteq M_n(K)$ where K is a right Ore domain with quotient division ring D, and where the matrix units have been appropriately chosen. (Note, however, that K need not have an identity element.)

Consider the columns, $\nu_1,\ldots,\nu_n$ say, of the matrix $a\in M_n(D)$ as vectors in $D^{(n)}$. Suppose, after renumbering the columns if necessary, that $\nu_1,\ldots,\nu_m$ is a basis for the subspace of $D^{(n)}$ that they generate. Let $\omega_{m+1},\ldots,\omega_n$ be a further set of column vectors in $D^{(n)}$ chosen to extend $\nu_1,\ldots,\nu_m$ to a basis for $D^{(n)}$.

Since D is the right quotient ring of K, there exists nonzero $k\in K$ such that $\nu_ik\in K^{(n)}$, $\omega_jk\in K^{(n)}$ for $i\in\{1,\ldots,m\}$, $j\in\{m+1,\ldots,n\}$. Moreover, provided $K\neq\mathbb{Z}/(2)$, k can be chosen with $k\neq 1$. Suppose that is so for the moment. Consider the two sets of column vectors

$$\{\nu_1k, \nu_2k,\ldots,\nu_mk, \omega_{m+1}k,\ldots,\omega_nk\}$$

and

$$\{\nu_1(1-k), \nu_2(1-k),\ldots,\nu_m(1-k), \nu_{m+1}-\omega_{m+1}k,\ldots,\nu_n-\omega_nk\}.$$

The former is a subset of $K^{(n)}$, and each is clearly a basis for $D^{(n)}$. Thus, if b, c are the two matrices made up from these columns, reordered to match those of a, then b, c are units of $M_n(D)$. However $b, c \in R$, and so $b, c \in \mathscr{C}(0)$, and $b \in M_n(K) \subseteq xR$. Of course $a = b + c$ by construction.

It remains to consider the case when $K \simeq \mathbb{Z}/(2)$. Then $K = D \simeq \mathbb{Z}/(2)$, $R = Q \simeq M_n(\mathbb{Z}/(2))$, and $xR = R$. If a is singular, then a is equivalent to a super-diagonal matrix a', i.e.

$$a' = \begin{bmatrix} 010\ldots0 \\ 001\ldots0 \\ \ldots\ldots\ldots \\ 000\ldots0 \end{bmatrix} = (I + a') - I$$

with both terms evidently nonsingular. If a is nonsingular, then a is equivalent to the matrix

$$a' = \begin{bmatrix} 010\ldots0 \\ 001\ldots0 \\ \ldots\ldots\ldots \\ 000\ldots1 \\ 100\ldots0 \end{bmatrix} = \begin{bmatrix} 110\ldots00 \\ 011\ldots00 \\ \ldots\ldots\ldots \\ 000\ldots11 \\ 000\ldots01 \end{bmatrix} + \begin{bmatrix} 100\ldots00 \\ 010\ldots00 \\ \ldots\ldots\ldots \\ 000\ldots10 \\ 100\ldots01 \end{bmatrix}$$

with both terms nonsingular (since $n > 1$).

COROLLARY 2. *In a prime right Goldie ring each element is the sum of at most two regular elements.*

Proof. In $\mathbb{Z}/(2)$ this is clear, since $0 = 1 + 1$.

COROLLARY 3. *Let R be a semiprime right Goldie ring which does not have $\mathbb{Z}/(2)$ as a direct summand. Then each element of R is the sum of two regular elements.*

Proof. In this case the Faith–Utumi theorem provides a direct sum of matrix rings over integral domains. If one of these domains is $\mathbb{Z}/(2)$, then R has, as a direct summand, the appropriate size matrix ring over $\mathbb{Z}/(2)$. This means that $R = R_1 \oplus R_2$ with R_1 being a direct sum of matrix rings over $\mathbb{Z}/(2)$, and R_2 having no direct summands of this form. The proof of Theorem 1 is now easily extended to this case.

The assumption that R has not got $\mathbb{Z}/(2)$ as a direct summand is necessary—the element $(e_{11}, 1)$ in $M_2(\mathbb{Z}) \oplus \mathbb{Z}/(2)$ is not the sum of two regular elements. The argument above can be extended to yield a result of Chatters and Ginn [**1**, Corollary 2.9], namely that, if R has at most one summand isomorphic to $\mathbb{Z}/(2)$ then each element of R is the sum of at most 3 regular elements.

COROLLARY 4. *Let R be a right order in a right Artinian ring, N the prime radical of R and suppose that R/N does not have $\mathbb{Z}/(2)$ as a direct summand. Then each element of R is the sum of two regular elements.*

Proof. Let $a \in R$. By Corollary 3, $a = b + c$ with $b, c \in \mathscr{C}(N)$. But $\mathscr{C}(N) = \mathscr{C}(0)$.

The next pair of results extend and elaborate on further work of Chatters and Ginn [**1**, Theorem 2.6 and Corollary 2.5].

THEOREM 5. *Let $P_1, \ldots, P_t$ be prime ideals of a ring R such that* (i) *R/P_i is right Goldie for each i, and* (ii) *no R/P_i is isomorphic to $\mathbb{Z}/(2)$. Let $a \in R$. Then there exist $b, c \in R$ such that $a = b + c$ and with $b, c \in \mathscr{C}(P_i)$ for $i = 1, \ldots, t$.*

Proof. First renumber the prime ideals P_i so that those maximal within this set are, say, $P_1, \ldots, P_m$, and so that $i < j$ implies $P_i \nsubseteq P_j$. Note, by Corollary 3 applied to $R/P_1 \cap \ldots \cap P_m$, that $a = b_m + c_m$ with b_m, $c_m \in \mathscr{C}(P_1 \cap \ldots \cap P_m) = \mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_m)$. If $m = t$, the result is proved. Otherwise, let $m \leq i < t$ and assume, by induction, that $a = b_i + c_i$ with $b_i, c_i \in \mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_i)$. It is enough to extend this to $i+1$.

Let R^* denote R/P_{i+1} and let * denote image in R^*. Set $X = P_1 \cap \ldots \cap P_i$; then X^* is a nonzero ideal of R^*, and so contains a regular element. Moreover, since P_{i+1} is not maximal amongst $P_1, \ldots, P_t$, then $R/P_{i+1} \neq M_n(\mathbb{Z}/(2))$ for any n.

It is clear from Theorem 1, applied to R/P_{i+1}, that there exists $y \in X$ such that $b_i + y \in \mathscr{C}(P_{i+1})$. Let $b_i' = b_i + y$ and $c_i' = c_i - y$. Then $a = b_i' + c_i'$ and, since $y \in X$, it is still true that $b_i', c_i' \in \mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_i)$. In other words, without loss of generality, it can be supposed that $b_i \in \mathscr{C}(P_{i+1})$.

Now X^* contains $M_n(K)$, with K being a right Ore domain with quotient division ring D, and X^* and R^* and $M_n(K)$ all having $M_n(D)$ as right quotient ring. View b_i^*, c_i^* as matrices in $M_n(D)$; let their columns be $\nu_1, \ldots, \nu_n$ and $\omega_1, \ldots, \omega_n$ respectively. Since $b_i \in \mathscr{C}(P_{i+1})$, then $\nu_1, \ldots, \nu_n$ are D-independent. Suppose that $\omega_1, \ldots, \omega_r$ are D-independent, but ω_{r+1} is a D-linear combination of $\omega_1, \ldots, \omega_r$. Some ν_p is independent of $\omega_1, \ldots, \omega_r$. Pick $k \in K$ such that each $\nu_i k \in K^{(n)}$, with $k \neq 0$, 1. Then replacing ω_{r+1} by $\omega_{r+1} + \nu_p k$ and ν_{r+1} by $\nu_{r+1} - \nu_p k$ leaves the set $\{\nu_i\}$ independent, yet makes $\omega_1, \ldots, \omega_{r+1}$ independent. Repeating this process, one can arrange that $\omega_1, \ldots, \omega_n$ become independent. Since $\nu_p k \in K^{(n)}$ and $M_n(K) \subseteq X^*$, this means one can choose $z \in X$ such that $b_i + z \in \mathscr{C}(P_{i+1})$ and $c_i - z \in \mathscr{C}(P_{i+1})$. Of course, since $z \in X$, then $b_i + z$, $c_i - z \in \mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_i) \cap \mathscr{C}(P_{i+1})$. Setting $b_{i+1} = b_i + z$, $c_{i+1} = c_i - z$ completes the proof.

COROLLARY 6. *Let R be a Noetherian ring with no nonzero Artinian right ideals. Then each element of R is the sum of two regular elements.*

Proof. It follows from [**5**, Section 2] that $\mathscr{C}(0) = \mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_t)$ for a certain set of prime ideals P_i, and that no R/P_i is Artinian.

The arguments here have all depended upon the existence of matrix rings related to the ring R. When R is itself a matrix ring over an integral domain, the chain conditions on R can be removed, as the next few results demonstrate.

LEMMA 7. *Let K be any ring, let α be a regular element of $M_{n-1}(K)$ and δ a regular element of K. Then the matrices*

$$\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix} \in M_n(K)$$

are regular for any vectors β, γ of appropriate size.

Proof. This is easily verified.

THEOREM 8. *If $t \geq 2$ and K is a ring in which each element is the sum of t regular elements, then $M_n(K)$ has the same property.*

Proof. Given $a \in M_n(K)$, partition a in the form $a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $\alpha \in M_{n-1}(K)$, $\delta \in K$, and β, γ appropriate sizes. By induction, we may suppose $\alpha = \sum_{i=1}^{t} \alpha_i$ and $\delta = \sum_{i=1}^{t} \delta_i$ with all the summands being regular. Then

$$a = \begin{bmatrix} \alpha_1 & \beta \\ 0 & \delta_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ \gamma & \delta_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 & 0 \\ 0 & \delta_3 \end{bmatrix} + \ldots + \begin{bmatrix} \alpha_t & 0 \\ 0 & \delta_t \end{bmatrix},$$

and, by Lemma 7, each summand is regular.

COROLLARY 9. *If K is an integral domain, then each element of $M_n(K)$ is the sum of two regular elements unless $n = 1$ and $K \simeq \mathbb{Z}/(2)$.*

Proof. If $K \not\simeq \mathbb{Z}/(2)$, then each element of K is the sum of two regular elements and so Theorem 8 applies. If $K \simeq \mathbb{Z}/(2)$ and $n > 1$, then Theorem 1 applies.

This result should be compared with the results of Henriksen [**2**]. There it is shown that, for any ring K with 1 and for $n > 1$, each element of $M_n(K)$ is the sum of three units, and examples show that elements need not be the sum of two units, even when K is a commutative Noetherian integral domain.

Finally, we note the following result.

COROLLARY 10. *Suppose that K is either* (i) *an integral domain, or* (ii) *a Noetherian ring, or* (iii) *a right order in a right Artinian ring, and suppose that $R = M_n(K)$ with $n \geq 2$. Then each element of R is the sum of two regular elements.*

Proof. Note that, given any prime ideal P of R, there is a prime ideal P' of K such that $R/P \simeq M_n(K/P')$. Thus $R/P \not\simeq \mathbb{Z}/(2)$, and the result follows from Corollary 4, the proof of Corollary 6, and Corollary 9.

This does raise the question of whether the result is valid for all rings K.

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