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On the Morse-Smale characteristic of a differentiable manifold

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In this note, the Morse-Smale characteristic of a differentiable manifold is defined and certain of its properties are studied.

Let M be a closed (that is, compact without boundary) C^∞ differentiable manifold and $f : M \to R$ be a C^∞ differentiable function on M. A point $p \in M$ is *a critical point* of f, if and only if, the induced map

$$f_* : TM_p \to TR_{f(p)}$$

is zero, where TM_p is the tangent space of M at p. A real number $a \in R$ is *a critical value* of f if $f^{-1}(a)$ contains a critical point of f. If $f^{-1}(a)$ contains no critical points, then a is *a regular value* of f. A critical point p of f is said to be *non-degenerate* if and only if the matrix

$$\left(\left(\partial^2 f/\partial x_i \partial x_j\right)(p)\right)$$

is non-singular. This matrix defines a symmetric bilinear form on the tangent space TM_p. This bilinear form is the *Hessian* of f at p. *The index* of a critical point p of f is the maximal dimension of a subspace of TM_p on which the Hessian of f is negative definite.

A C^∞ differentiable function $f : M \to R$ is said to be *a Morse function* if f has only non-degenerate critical points.

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DEFINITION. Let M be a closed C^∞ differentiable manifold of dimension n. The *Morse-Smale characteristic* of M, denoted by $\mu(M)$, is

$$\mu(M) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M, f) ,$$

where Ω is the space of Morse functions on M, and $c_i(M, f)$ is the number of critical points of index i of f in Ω.

THEOREM. *Let M be a closed C^∞ differentiable manifold,* $\dim M = n < 4$. *Then*

$$\mu(M) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M, f) .$$

Proof. If $n = 1$ it is obviously true. If $n = 2$, it can be also proved that the equality holds.

If $n = 3$, then by Smale [1], [2], there exists a Morse function f on M^3 having a single critical point of index 0, and a single one of index 3; that is,

$$c_0(M^3, f) = c_3(M^3, f) = 1 .$$

However, the Euler characteristic of M^3 equals zero. Thus $c_1(M^3, f) = c_2(M^3, f)$ because of the last (equality) of the Morse inequalities. Hence

$$\min_{f \in \Omega} \sum_{i=0}^{3} c_i(M^3, f) = \min_{f \in \Omega} \left(2+2c_1(M^3, f)\right) = 2 + 2 \min_{f \in \Omega} c_1(M^3, f)$$

$$= \sum_{i=0}^{3} \min_{f \in \Omega} c_i(M^3, f) .$$

PROBLEM. Find necessary and sufficient conditions on M^n, $n \geq 4$, so that

$$\mu(M^n) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f) .$$

Using an argument similar to that of the previous theorem, in particular the fact that the Euler characteristic of any closed odd-dimensional manifold equals zero, the following proposition can be proved.

PROPOSITION. *The Morse-Smale characteristic of any closed* C^∞ *differentiable odd-dimensional manifold is an even integer greater than or equal to* 2 .

REMARK. It is not known yet for which closed C^∞ differentiable manifolds M, N ,

$$\mu(M \times N) = \mu(M) \cdot \mu(N) .$$

(Of course, $\mu(M \times N) \leq \mu(M) \cdot \mu(N)$.) If this is the case, then $\mu(H^3) = 2$ where H^3 is any homotopy 3-sphere and so the Poincaré conjecture would be true since $\mu(H^3 \times H^3) = 4$. In particular, the Poincaré conjecture is true, if and only if, $\mu(H^3) \leq \mu(H^3 \times H^3)$.

References

[1] Stephen Smale, "Generalized Poincaré's conjecture in dimensions greater than four", *Ann. of Math.* (2) 74 (1961), 391-406.

[2] S. Smale, "On the structure of manifolds", *Amer. J. Math.* 84 (1962), 387-399.

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