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MORE LOCALIZED AUTOMORPHISMS OF THE CUNTZ ALGEBRAS

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Abstract We completely determine the localized automorphisms of the Cuntz algebras $\mathcal{O}_n$ corresponding to permutation matrices in $M_n \otimes M_n$ for $n = 3$ and $n = 4$. This result is obtained through a combination of general combinatorial techniques and large scale computer calculations. Our analysis proceeds according to the general scheme proposed in a previous paper, where we analysed in detail the case of $\mathcal{O}_2$ using labelled rooted trees. We also discuss those proper endomorphisms of these Cuntz algebras which restrict to automorphisms of their respective diagonals. In the case of $\mathcal{O}_3$ we compute the number of automorphisms of the diagonal induced by permutation matrices in $M_3 \otimes M_3 \otimes M_3$.

Keywords: Cuntz algebra; endomorphism; automorphism; permutation; tree

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1. Introduction and preliminaries

In [**9**], Cuntz noticed that the automorphism group of $\mathcal{O}_n$ [**8**] has a rich structure resembling that of semisimple Lie groups and suggested an intriguing definition of the Weyl group in this context. However, despite the fact that the Cuntz algebras $\mathcal{O}_n$ have been intensively studied over the past 30 years, to date precious little is known about the structure of these Weyl groups. In [**7**], we opened a new and promising line of investigation of this problem. We also discussed at length the case of $\mathcal{O}_2$ therein. In the present paper we follow it up with an analysis focused on the cases of $\mathcal{O}_3$ and $\mathcal{O}_4$, the main result being the complete classification of all the permutation automorphisms of $\mathcal{O}_n$ for $n = 3, 4$ arising at level 2 (i.e. induced by a permutation matrix in $M_n \otimes M_n$).

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Until now, only a few such automorphisms were known: for example, Archbold's flip-flop automorphism of $\mathcal{O}_2$ [**2**] and more generally Bogolubov (permutation) automorphisms of $\mathcal{O}_n$ [**1**, **10**–**12**, **17**]. The Matsumoto–Tomiyama automorphism of $\mathcal{O}_4$ [**14**] was somewhat more complicated; it was only recently recognized that it fits into a more general pattern (see [**16**] and § 3 of the present paper). However, all such known automorphisms were, in some sense, isolated examples and there was no general or systematic understanding of the overall situation. Finding all automorphisms through a case-by-case examination is unfeasible due to the exceedingly large scale of the problem, so an efficient reduction process is necessary. One could exploit the action of inner automorphisms and Bogolubov automorphisms in this process, but this is insufficient to significantly reduce the computation.

In [**7**] we discovered a powerful algorithm to construct those permutations leading to automorphisms; surprisingly, it relies on a certain combinatorial analysis of labelled rooted trees. This fact appears vaguely reminiscent of quantum field theory, although our set-up has nothing to do with perturbation theory. The aforementioned reduction is a result of purely theoretical analysis of the problem and has deep theoretical implications. However, in order to perform the subsequent massive computations that emerged, we employed the Magma [**3**] computational algebra system.

In particular, as a result of these computations, we have obtained a complete classification of automorphisms of $\mathcal{O}_n$ arising from permutations of the set of multi-indices $\{1,\dots,n\}^k$ for small values of n and k. As a by-product, by a similar method we can also access those endomorphisms of the Cuntz algebra that provide automorphisms of the diagonal.

We now briefly describe our notation and the set-up. For any integer $n \geqslant 2$, the Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by n isometries $S_1,\dots,S_n$ with mutually orthogonal ranges summing up to 1. One has the unital inclusions

$$\mathcal{O}_n \supset \mathcal{F}_n \supset \mathcal{D}_n,$$

where $\mathcal{F}_n$ is the uniformly hyperfinite algebra of type n^∞ and the diagonal $\mathcal{D}_n$ is maximal abelian in both $\mathcal{F}_n$ and $\mathcal{O}_n$. $\mathcal{F}_n$ is the closure in norm of the union $\bigcup_{k\in\mathbb{N}}\mathcal{F}_n^k$ of an increasing family of matrix algebras where, for each $k \in \mathbb{N}$, the C^*-subalgebra $\mathcal{F}_n^k$ is isomorphic to the algebra M_{n^k} of $n^k \times n^k$ complex matrices. Similarly, $\mathcal{D}_n$ is the norm-closure of the union of the increasing sequence of C^*-algebras $\mathcal{D}_n^k$, each isomorphic to the diagonal matrices in M_{n^k}.

There is a well-known one-to-one correspondence, $u \mapsto \lambda_u$, between $\mathcal{U}(\mathcal{O}_n)$, the group of unitary elements in $\mathcal{O}_n$ and $\operatorname{End}(\mathcal{O}_n)$, the semigroup of unital $*$-endomorphisms of $\mathcal{O}_n$, where λ_u is uniquely determined by $\lambda_u(S_i) = u^*S_i$, $i = 1,\dots,n$ (here, we follow the convention in [**9**]). As in [**5**], endomorphisms λ_u corresponding to unitaries u in $\bigcup_{k\in\mathbb{N}}\mathcal{F}_n^k$ are called localized.

Cuntz showed that the automorphisms of $\mathcal{O}_n$ that restrict to automorphisms of the diagonal $\mathcal{D}_n$ are exactly the automorphisms induced by elements in the (unitary) normalizer

$$N_{\mathcal{O}_n}(\mathcal{D}_n) = \{z \in \mathcal{U}(\mathcal{O}_n) \mid z\mathcal{D}_n z^* = \mathcal{D}_n\}.$$

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Later, Power [**15**] described in detail the structure of such normalizers, showing that any element in $N_{\mathcal{O}_n}(\mathcal{D}_n)$ is the product of a unitary in $\mathcal{D}_n$ and a unitary that can be written as a finite sum of words in the S_is and their adjoints. In particular,

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) = \lambda(N_{\mathcal{O}_n}(\mathcal{D}_n))^{-1}$$

and

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \lambda(N_{\mathcal{F}_n}(\mathcal{D}_n))^{-1},$$

where for a subset $E \in \mathcal{U}(\mathcal{O}_n)$ we define

$$\lambda(E)^{-1} = \{\lambda_u \mid u \in E\} \cap \mathrm{Aut}(\mathcal{O}_n).$$

In this paper, and in [**7**], we are only concerned with the structure of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$, which gives rise (after taking the quotient) to the restricted Weyl group.

Let P_n^k be the group of permutations of the set $W_n^k = \{1, \dots, n\}^k$. Clearly, P_n^k is isomorphic to $\mathbb{P}_{n^k}$, the permutation group over n^k elements. To any $\sigma \in W_n^k$ one associates a unitary $u_\sigma \in \mathcal{F}_n^k$ by

$$u_\sigma = \sum_{\alpha \in W_n^k} S_{\sigma(\alpha)} S_\alpha^*.$$

Then $\sigma \mapsto u_\sigma$ is a group isomorphism of P_n^k onto its image, denoted $\mathcal{P}_n^k$, that can be further identified with the group of permutation matrices in M_{n^k}.

Now, it follows from the above that

$$N_{\mathcal{F}_n}(\mathcal{D}_n) = N_{\mathcal{O}_n}(\mathcal{D}_n) \cap \mathcal{F}_n = \mathcal{U}(\mathcal{D}_n) \cdot \mathcal{P}_n \simeq \mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{P}_n,$$

where $\mathcal{P}_n = \bigcup_k \mathcal{P}_n^k$ [**7**]. Thus, as Cuntz has already shown that every unitary in $\mathcal{U}(\mathcal{D}_n)$ induces an automorphism of $\mathcal{O}_n$, the problem that we are facing is to determine for which permutation matrices $w \in \mathcal{P}_n^k$, $k = 1, 2, 3, \dots$, one has $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$. This is exactly the point where (rooted labelled) trees come to the rescue. For a detailed discussion, see [**7**]; throughout the next section, we repeatedly use results from that paper.

For the reader's benefit we include the following elementary yet useful observation valid for all $n \geqslant 2$.

Proposition 1.1. *Let w be a unitary in $\mathcal{O}_n$.*

(a) *If $w \in \mathcal{F}_n$, then $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ if and only if $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$.*

(b) *If $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$, then $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ if and only if $w \in N_{\mathcal{O}_n}(\mathcal{D}_n)$.*

(c) *If $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, then λ_w is an irreducible endomorphism of $\mathcal{O}_n$, i.e. $\lambda_w(\mathcal{O}_n)' \cap \mathcal{O}_n = \mathbb{C}$.*

Proof. (a) Necessity has been proved in [**16**, Lemma 2]. On the other hand, $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$ implies that $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ since then $w^* \in \lambda_w(\mathcal{O}_n)$.

(b) This is part of the statement in [**9**, Proposition 1.5].

(c) Using the assumption and the fact that $\mathcal{D}_n$ is maximal abelian in $\mathcal{O}_n$, one obtains that

$$\lambda_w(\mathcal{O}_n)' \cap \mathcal{O}_n \subset \lambda_w(\mathcal{D}_n)' \cap \mathcal{O}_n = \mathcal{D}_n' \cap \mathcal{O}_n \subset \mathcal{D}_n = \lambda_w(\mathcal{D}_n) \subset \lambda_w(\mathcal{O}_n)$$

and the conclusion readily follows from $\mathcal{O}_n$ being simple. □

As the endomorphisms of $\mathcal{O}_n$ (with $n \leqslant 4$) considered in this paper and in [**7**] are all induced by unitaries w in $\bigcup_k \mathcal{P}_n^k \subset N_{\mathcal{F}_n}(\mathcal{D}_n) = N_{\mathcal{O}_n}(\mathcal{D}_n) \cap \mathcal{F}_n$, when they are automorphisms they also provide, by restriction, automorphisms of $\mathcal{D}_n$ and $\mathcal{F}_n$; when they only satisfy the weaker condition $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ they still act irreducibly on $\mathcal{O}_n$. For example, there are four such irreducible endomorphisms of $\mathcal{O}_2$ corresponding to permutations in P_2^2 [**6**, **13**].

2. Classification results

According to the analysis in [**7**], the search for automorphisms of $\mathcal{O}_n$ induced by permutations in P_n^k involves the following two main steps:

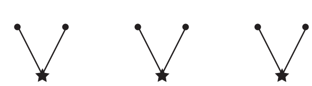
(b) finding n-tuples of rooted trees with vertices suitably labelled by elements of W_n^{k-1} that satisfy [**7**, Lemma 4.5] (or equivalently [**7**, Proposition 4.7]);

(d) verifying which of the n-tuples satisfying (b) above also satisfy [**7**, Lemma 4.10] (or equivalently [**7**, Proposition 4.11]).

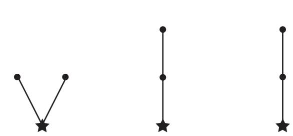
In turn, the solutions to condition (b) alone provide by restriction automorphisms of the diagonal $\mathcal{D}_n$.

2.1. The case of $\mathcal{P}_3^2$

In this case, there are only two rooted trees with three vertices. Condition (b) can only be satisfied for the following 3-tuples of unlabelled trees:



and the three distinct 3-tuples arising by permuting



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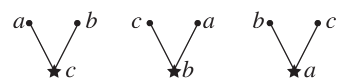
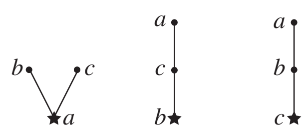
Thus, we have four different 3-tuples. For each such 3-tuple, there are precisely $3!(3!^3) = 6 \cdot 216$ permutations in P_3^2 satisfying condition (b) and among them $3!24$ permutations also satisfying condition (d). These figures have been obtained through computer computations.

The corresponding labelled trees are of the form



and



where a, b and c are distinct elements in $\{1, 2, 3\}$.

In particular, for each fixed set of labels on a 3-tuple (and there are 3! of them) there are 24 permutations satisfying both the conditions (b) and (d).

Example 2.1. Bogolubov automorphisms always give rise to 3-tuples of the first type [**7**, Example 4.4]. An example of a 3-tuple of the second type (with labels $a = 1$, $b = 2$, $c = 3$) is provided by the transposition $(2, 3)$, where we identify elements of $W_2^2 = \{1, 2, 3\}^2 = \{11, 21, 31, 12, 22, 32, 13, 23, 33\}$ with $\{1, 2, \ldots, 9\}$.

All in all, we see that the numbers of automorphisms arising from P_2^3 are as follows.

Theorem 2.2. *One has*

$$\#\{\sigma \in P_3^2 : \lambda_{u_\sigma}|_{\mathcal{D}_3} \in \mathrm{Aut}(\mathcal{D}_3)\} = 4 \cdot 3! \cdot 216 = 5184,$$
$$\#\{\sigma \in P_3^2 : \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_3)\} = 4 \cdot 3! \cdot 24 = 576.$$

In particular, there are $4 \cdot 24 = 96$ *distinct classes of automorphisms in* $\mathrm{Out}(\mathcal{O}_3)$ *corresponding to permutations in* P_3^2.

The latter number has been independently verified by solving the equations in [**7**, § 6.1].

2.2. The case of $\mathcal{P}_3^3$

In this case, there are 286 rooted trees with $n^{k-1} = 9$ vertices, of which 171 satisfy our basic conditions: each vertex has in-degree at most $n = 3$ (recall that there is a loop at the root, adding 1 to its in-degree). Let us define the *in-degree type* of a rooted tree to be the multiset of the in-degrees of its vertices.

We list the 171 rooted trees in Figure 1; they are classified by the 11 in-degree types $\{A \ldots K\}$ listed in Table 1.

Table 1. *The in-degree types for $\mathcal{P}_3^3$.*

type	multiplicities in-degree: 0	1	2	3	number of trees
A	6	0	0	3	2
B	5	1	1	2	18
C	5	0	3	1	8
D	4	3	0	2	14
E	4	2	2	1	46
F	4	1	4	0	9
G	3	4	1	1	33
H	3	3	3	0	24
I	2	6	0	1	4
J	2	5	2	0	12
K	1	7	1	0	1

We wish to find 3-tuples f of labelled trees such that

$$\text{for all } j \in \{1 \dots 9\}, \quad f_1(j) + f_2(j) + f_3(j) = 3.$$

For such an f we define the in-degree *alignment* matrix M, where M_{ij} is the in-degree of the vertex labelled j in the tree f_i. Every in-degree alignment matrix has each row adding to $n^{k-1} = 9$ and each column adding to $n = 3$. In order to find all the required f we first determine the possible in-degree alignments of our 11 types.

Now the number of size-3 multisets with elements chosen from a set of 11 is 286 (the eleventh tetrahedral number*). Of the 286 size-3 multisets of in-degree types, we compute that 100 have at least one alignment. The number of alignments (up to consistent relabelling) is 133.

After about 200 processor days we report that condition (b) is satisfied for a set $\mathcal{F}$ of 7390 3-tuples of labelled trees, up to permutation of tree position (action of S_3) and consistent relabelling of all trees (action of S_9). Only 110 of the 171 unlabelled trees appear in $\mathcal{F}$ (those which do not appear in $\mathcal{F}$ are marked in Figure 1 with a dotted backslash); they have the first eight, $\{A \dots H\}$, of the 11 in-degree types. In these $\mathcal{F}$ there occur 474 multisets of three unlabelled trees; they have the six distinct three-element multisets of in-degree types listed in the first column of Table 2. The second column contains the number of three-element multisets of unlabelled trees having the respective types; the third column, $\mathcal{F}_{\text{types}}$, is the partition of $\mathcal{F}$ according to the respective types; the fourth column of each row in Table 2, $\#f$ covered, is the inner product of the last two columns of the corresponding table in the appendix to this article [**4**].

* Is it just a combinatorial accident that the eleventh tetrahedral number is the same as the number of all (unordered) rooted trees on 9-vertices?

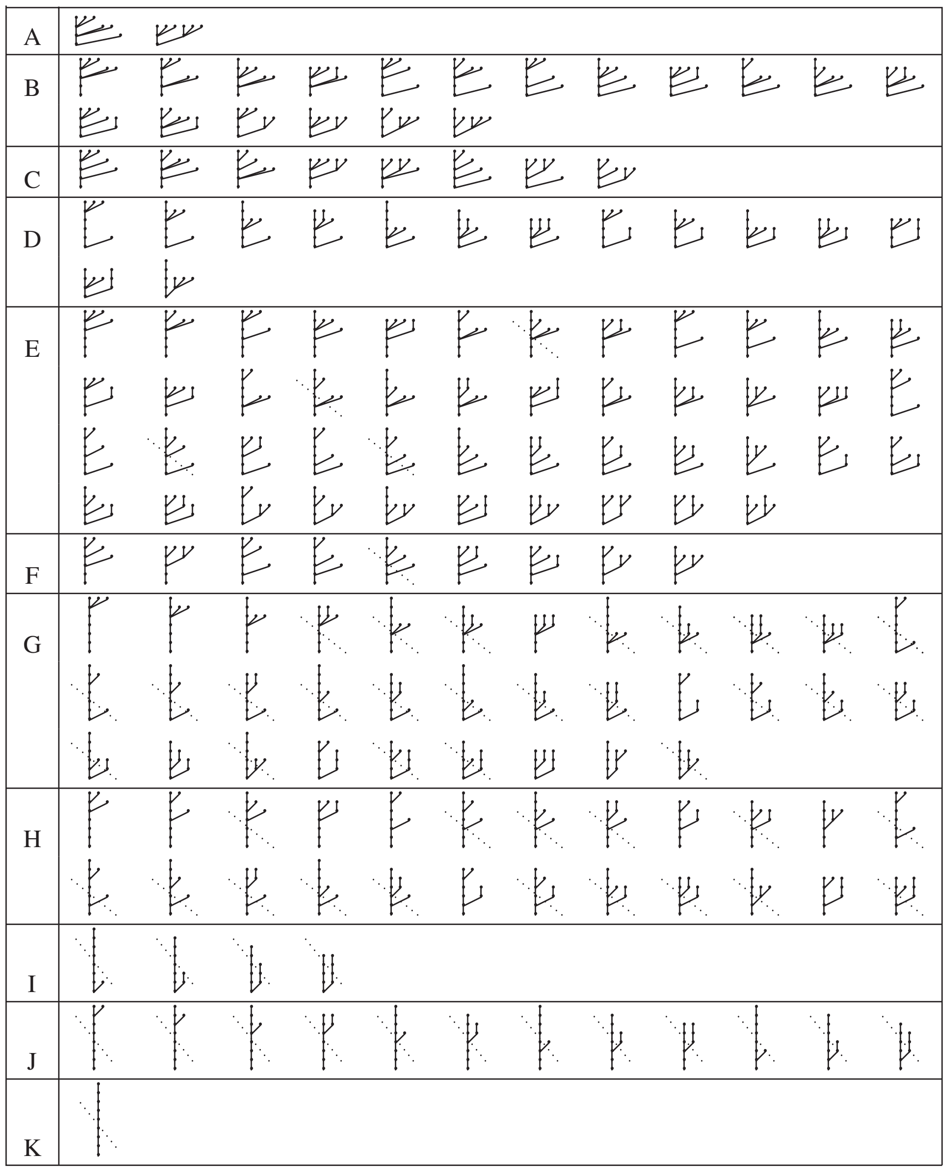


Figure 1. In-degree types and trees.

In total, we have

$$44\,172 \times 9! = 16\,029\,135\,360$$

3-tuples of labelled trees satisfying condition (b). Therefore, we have

$$44\,172 \times 9! \times n!^{n^{k-1}} = 44\,172 \times 9! \times 6^9 = 161\,536\,753\,300\,930\,560$$

permutations $\sigma \in P_3^3$ satisfying condition (b).

Table 2. *Three-element multisets of in-degree types.*

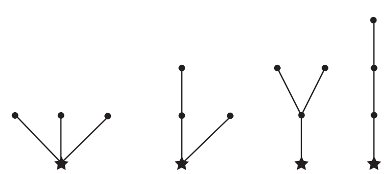
ID types			# tree triples	$\#\mathcal{F}_{\text{types}}$	$\#f$ covered
A	A	A	4	2 168	$12\,924 \cdot 9!$
A	B	B	176	2 782	$16\,650 \cdot 9!$
A	C	D	75	950	$5\,700 \cdot 9!$
A	E	E	180	1 072	$6\,396 \cdot 9!$
A	F	G	31	392	$2\,352 \cdot 9!$
A	H	H	8	26	$150 \cdot 9!$
total			474	7 390	$44\,172 \cdot 9!$

Unfortunately, at this stage we cannot provide the precise number of permutations satisfying condition (d) as this job exceeds our computational resources: it would take about 32 processor years to compute.

Some examples of 3-tuples of labelled trees satisfying condition (b) are listed in the first column of Figure 2; the second column contains the size of the orbit of the combined actions of S_3 and S_9 on the first entry of each row; the third column is a count of the number of permutations corresponding to f that satisfy condition (d); the last column contains (when one exists) an example permutation satisfying condition (d) with labels $(a, b, c, \dots, i)$ chosen to be $(1, 2, 3, \dots, 9) = ((1, 1), (2, 1), (3, 1), \dots, (3, 3))$.

2.3. The case of $\mathcal{P}_4^2$

In this case, there are four (unlabelled) rooted trees with four vertices, namely



One verifies that only eight types of 4-tuples of such trees admit labellings satisfying condition (b). By a type we mean an unordered set of four trees making up a 4-tuple (two different 4-tuples belong to the same type if one can be obtained from the other by a permutation of the unlabelled trees). These types are listed in the first column of Table 3. The second column of this table gives the number of distinct labellings satisfying condition (b) and corresponding to each type. These numbers are factorized as $X \cdot Y \cdot Z$, where X is the number of distinct 4-tuples of unlabelled trees of the given type, $Y = 4!$ is the number of permutations of labels (it corresponds to action of inner automorphisms arising from P_4^1 [**7**, § 4.2]) and Z is the number of orbits under this action. The last column contains the number of all permutations in P_4^2 satisfying both conditions (b) and (d) whose corresponding trees are of the given type.

The number of permutations satisfying condition (d) depends both on the type of the corresponding 4-tuple of trees and on the specific labelling. However, as it turns

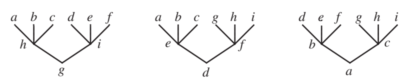
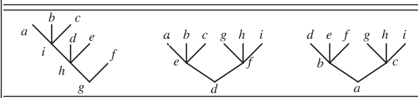
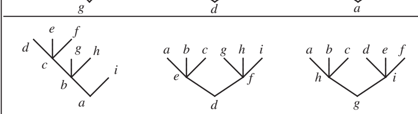
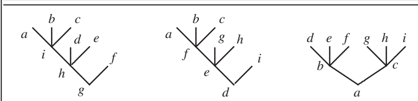
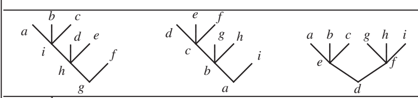
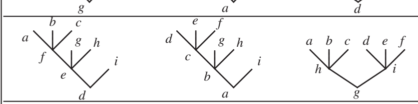
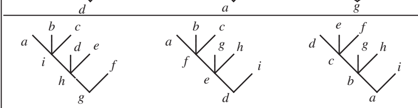
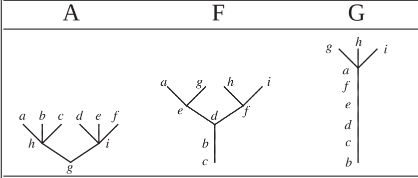
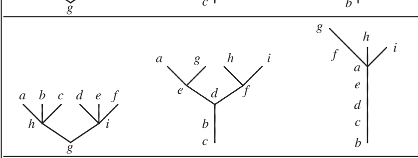
labelled trees	# (b)	# (d)	example
A A A			
	1·9!	312	(1,6,26,7,22,17) (2,12,24,20,18,13,14) (3,27,16,25,19,9,10) (4,11,15,23,8)
	6·9!	0	
	6·9!	240	(1,25,24,23,2,19) (3,16,27,15,26) (4,17,9,18,12,10) (6,20,22,14,8) (7,21,13,11)
	3·9!	216	(1,3,27,4,26,10,9) (2,18,7,16,19,6) (5,20,12,21,24) (8,25,22,11,15) (13,14,17)
	6·9!	0	
	6·9!	0	
	6·9!	0	
A F G			
	6·9!	0	
	6·9!	0	

Figure 2. Examples for $\mathcal{P}_3^3$.

out, it does not depend on the permutation of unlabelled trees within the type. Precise information to this effect is provided in Figure 3. There is a natural action of $S_4 \times S_4$ on 4-tuples of labelled trees with four vertices, by permutation of the labels (simultaneously on all four trees) and permutation of the four trees. Labellings satisfying condition (b) give rise to 19 orbits for this action, and representatives of these 19 orbits are listed in the first column. They are further grouped according to their types. The second column

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Table 3. *Tree types for* P_4^2.

type	# (b)	#σ (d)
$\alpha\alpha\alpha\alpha$	$24 = 1 \cdot 24 \cdot 1$	51 840
$\alpha\alpha\beta\beta$	$576 = 6 \cdot 24 \cdot 4$	787 968
$\alpha\alpha\gamma\gamma$	$288 = 6 \cdot 24 \cdot 2$	311 040
$\alpha\beta\beta\beta$	$768 = 4 \cdot 24 \cdot 8$	746 496
$\alpha\beta\beta\delta$	$1152 = 12 \cdot 24 \cdot 4$	1 575 936
$\alpha\beta\gamma\delta$	$1152 = 24 \cdot 24 \cdot 2$	1 244 160
$\beta\beta\gamma\gamma$	$1152 = 6 \cdot 24 \cdot 8$	787 968
$\gamma\gamma\gamma\gamma$	$288 = 1 \cdot 24 \cdot 12$	266 112
total	5 400	5 771 520

describes the partition of each orbit of the $S_4 \times S_4$-action into orbits of an action of S_4 by permutation of labels. For example, $144 = 6 \times 4!$ indicates that the corresponding $S_4 \times S_4$-orbit has 144 elements, partitioned into six S_4-orbits with $4! = 24$ elements each. The total number of permutations satisfying condition (b) corresponding to each row is thus obtained by multiplying the number in the second column by the combinatorial factor $4!^4 = 331\,776$ [**7**, § 4.2]. The third column contains the number of permutations satisfying condition (d) for each element in the $S_4 \times S_4$-orbit. The total number of permutations satisfying condition (d) corresponding to a given row is thus obtained by multiplying the numbers in the second and third columns. The last column of Figure 3 contains an example of a permutation satisfying condition (d) (if it exists) with the choice of labels $a = 1$, $b = 2$, $c = 3$, $d = 4$ and order of trees as given in Table 3.

As a consequence of the above, we obtain the following result.

Theorem 2.3. *One has*

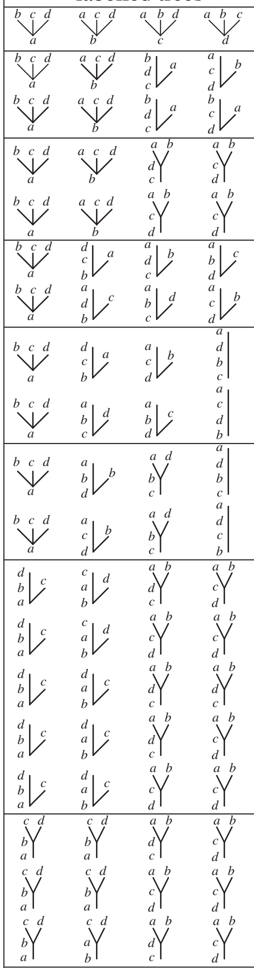
$$\#\{\sigma \in P_4^2 : \lambda_{u_\sigma}|_{\mathcal{D}_4} \in \mathrm{Aut}(\mathcal{D}_4)\} = 5400 \cdot 4!^4 = 1\,791\,590\,400,$$
$$\#\{\sigma \in P_4^2 : \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_4)\} = 5\,771\,520.$$

In particular, there are 240 480 *distinct classes of automorphisms in* $\mathrm{Out}(\mathcal{O}_4)$ *corresponding to permutations in* P_4^2.

3. Additional examples

We wish to relate the above analysis to the automorphisms constructed in [**16**], namely Examples 8 and 9 therein.

Example 3.1. Consider a non-trivial partition $W_n^1 = R_1 \cup \cdots \cup R_r$ of W_n^1 into a union of r disjoint subsets, $1 < r \leqslant n$. Let $\sigma_i \in P_n^1$, $i = 1, \ldots, r$, be permutations of W_n^1 such that $\sigma_i \sigma_j^{-1}(R_m) = R_m$ for all $i, j, m \in \{1, \ldots, r\}$. We define $\psi \in P_n^2$ as



labelled trees	# (b)	# (d)	example
	24 = 1·4!	2160	Id
	288 = 12·4!	576	(4,7)
	288 = 12·4!	2160	(7,8)
	144 = 6·4!	2160	(3,4) (7,8)
	144 = 6·4!	0	
	576 = 24·4!	576	(3,7,14,10,6,4) (5,13,9) (8,16,12) (11,15)
	192 = 8·4!	2160	(2,3,4)
	576 = 24·4!	2160	(7,8) (11,12) (14,16,15)
	576 = 24·4!	576	(3,4,7,10,8,11,6) (5,9) (14,16,15)
	576 = 24·4!	0	
	576 = 24·4!	2160	(2,3) (6,8) (10,12) (14,15,16)
	288 = 12·4!	576	(3,4) (7,8) (10,13)
	288 = 12·4!	0	
	144 = 6·4!	0	
	288 = 12·4!	2160	(3,4) (7,8) (13,14)
	288 = 12·4!	0	
	144 = 6·4!	0	
	72 = 3·4!	1536	(2,9,5) (4,11,7)(6,10,13) (8,12,15)
	72 = 3·4!	2160	(3,4) (7,8) (9,10) (13,14)

Figure 3. Labelled trees for P_4^2.

$\psi(\alpha,\beta)=(\alpha,\sigma_i(\beta))$ for $\alpha\in R_i$, $\beta\in W_n^1$. So constructed, λ_ψ is invertible, with inverse $\lambda_{\bar\psi}$, where $\bar\psi\in P_n^3$ is given by

$$\bar\psi(\alpha,\beta,\gamma)=(\alpha,\sigma_i^{-1}(\beta),\sigma_j\sigma_k^{-1}(\gamma))$$

for $\alpha\in R_i$, $\beta\in R_k$, $\sigma_i^{-1}(\beta)\in R_j$. Moreover, it is easy to see that $\lambda_\psi\in\mathrm{Inn}(\mathcal{O}_n)$ if and only if $\psi=\mathrm{id}$.

If $n=4$, $r=2$, $R_1=\{1,2\}$, $R_2=\{3,4\}$, $\sigma_1=(23)(=\sigma_1^{-1})$, $\sigma_2=(1243)$, ψ is constructed from these data as above and $w=S_1S_1^*+S_3S_2^*+S_2S_3^*+S_4S_4^*\sim\sigma_1$, then $\mathrm{Ad}(w)\lambda_\psi$ is the outer automorphism of $\mathcal{O}_4$ constructed and discussed in [**14**]. For this specific example, it is not difficult to verify that the corresponding 4-tuple of rooted trees is of type $\alpha\alpha\alpha\alpha$ according to Table 3.

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More generally, for any $\lambda_\psi \in \mathrm{Aut}(\mathcal{O}_n)$ constructed as in Example 3.1, the corresponding n-tuple of rooted trees can easily be described as follows. Each tree has n vertices labelled by the elements in $W_n^1 = \{1, \dots, n\}$, and the ith tree has root i and all the other vertices are connected to the root. This readily follows from the fact that the defining relation [**7**, § 4.1]

$$(i, \alpha) = \psi(\beta, m), \quad \alpha, \beta \in W_n^1,$$

for some $m \in \{i, \dots, n\}$ forces $\beta = i$ and then it can be solved for all α.

Example 3.2. Let $n \geqslant 3$, $\phi = (123) \in P_n^1$ and let $\psi \in P_n^2$ be constructed as in Example 3.1 with $r = 2$ from the data: $R_1 = \{1, 2\}$, $R_2 = \{3, \dots, n\}$, $\sigma_1 = \mathrm{id}$, $\sigma_2 = (12)$. Then one checks that λ_ϕ and λ_ψ are outer automorphisms of $\mathcal{O}_n$ of order 3 and 2, respectively, and the group generated by λ_ϕ and λ_ψ in $\mathrm{Out}(\mathcal{O}_n)$ is $\mathbb{Z}_3 * \mathbb{Z}_2$ [**16**].

Since λ_ϕ is a Bogolubov automorphism, the trees associated to ϕ (which can be thought of as an element in P_n^k for $k > 1$) are computed in [**7**, Example 4.4]. Also, as discussed above, the n trees corresponding to $\psi \in P_n^2$ are also all identical, with the root receiving $n - 1$ edges from the other vertices.

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