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CLOSED IDEALS IN A CONVOLUTION ALGEBRA OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We consider the usual topological vector space $H(G)$ of all functions holomorphic in a region $G \subset \mathbb{C}$. If G satisfies certain conditions, it is possible to introduce the Hadamard product as multiplication in $H(G)$, and then $H(G)$ turns out to be a commutative topological algebra. In [5] we characterized the invertible elements in $H(G)$, and the aim of this paper is to study the closed ideals and some further questions.

Introduction. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ be power series with positive radii of convergence R_f and R_g, respectively. Then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n)z^n.$$

Note that the radius of convergence of the power series $f * g$ is at least $R_f R_g$, and thus positive. For a region $G \subset \hat{\mathbb{C}}$ let $H(G)$ denote the topological vector space of all functions which are holomorphic in G, where $H(G)$ carries the topology of locally uniform convergence. Throughout this paper we require that G satisfy the following conditions: $0 \in G$, $1 \notin G$ and $G^c \cdot G^c \subset G^c$, where G^c denotes the complement of G with respect to $\hat{\mathbb{C}}$ and $G^c \cdot G^c = \{z \cdot w : z, w \in G^c\}$. (Note that these conditions imply $G^c \cdot G^c = G^c$ and $\mathbb{D} \subset G$, where $\mathbb{D}$ denotes the unit disk.) We call such regions admissible. From the Hadamard multiplication theorem ([10], see also [2, pp. 21–22] or [7]) we obtain that in this case $H(G)$ with $*$ as multiplication is a complete metrizable locally convex topological algebra (a so-called B_0-algebra), which we denote by $H^*(G)$. Since $1 \notin G$, the algebra $H^*(G)$ has an identity γ given by

$$\gamma(z) = \frac{1}{1-z}.$$

In [5] we investigated the invertible elements of $H^*(G)$. The aim of this paper is to study the ideal structure in $H^*(G)$, where we are mainly interested in characterizing the closed ideals and closed maximal ideals. It will be seen that our problems are strongly related to several classical questions in function theory such as analytic continuation of lacunary series or zero distribution of entire functions. For motivation we recall some well-known results in the algebra $H(G)$ with the usual pointwise multiplication of functions, which can be found for example in [15, p. 121].

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THEOREM A. *For an ideal $I \subset H(G)$ the following statements are equivalent:*
i) I is closed.
ii) I is principal.
iii) I is finitely generated.

THEOREM B. *For an ideal $M \subset H(G)$ the following statements are equivalent:*
i) M is closed and maximal.
ii) $M = \{f \in H(G) : f(a) = 0\}$ for some $a \in G$.
iii) M is the kernel of some homomorphism $\phi \colon H(G) \longrightarrow \mathbb{C}$.

While in Theorems A and B the region G may be arbitrary, the results in $H^*(G)$ are much more complicated and strongly depend on the geometry of G. For the purpose of giving some examples of admissible regions let $\varrho \in [1, \infty]$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$,

$$\mathbb{D}_\varrho := \{z \in \mathbb{C} : |z| < \varrho\},$$
$$\Gamma_\alpha := \{t^{1+i\alpha} : t \in [1, \infty)\},$$

and

$$A_k := \{e^{2\pi ij/k} : j = 0, 1, \ldots, k-1\}.$$

(Note that Γ_α is part of a logarithmic α-spiral, and $\Gamma_0 = [1, \infty)$.) Then the regions $G = \mathbb{D} := \mathbb{D}_1$, $G = \mathbb{D}_\varrho \setminus \Gamma_\alpha$, $G = \mathbb{D}_\varrho \setminus A_k$ and $G = \hat{\mathbb{C}} \setminus A_k$ are admissible.

FIGURE 1: $\mathbb{D}_3 \setminus \Gamma_5$ and $\mathbb{D}_3 \setminus A_8$

The following lemma shows that in a certain sense these regions are typical. A region $G \subset \mathbb{C}$ with $0 \in G$ is called *α-starlike* (with respect to 0), if $G^c \cdot \Gamma_\alpha = G^c$. With this notation it is possible to divide the admissible regions into two classes.

LEMMA 1. *i) If G is admissible, and if 1 is not isolated in G^c, then G is α-starlike for some $\alpha \in \mathbb{R}$. In particular, G is simply connected and $G \subset \mathbb{D}_\varrho \setminus \Gamma_\alpha$ for some $\varrho \in [1, \infty]$.*

ii) If G is admissible, and if 1 is isolated in G^c, then $G^c \cap \mathbb{D}_\varrho = A_k$ for some $\varrho \in (1, \infty]$ and $k \in \mathbb{N}$. If, in addition, $\infty \in G$, then $G = \hat{\mathbb{C}} \setminus A_k$ for some $k \in \mathbb{N}$.

PROOF. i) From the assumptions and Lemma 2.2 in [1] we obtain that $\Gamma_\alpha \subset G^c$ for some $\alpha \in \mathbb{R}$, and therefore $G^c \subset G^c \cdot \Gamma_\alpha \subset G^c \cdot G^c = G^c$ which implies the assertion.

ii) Assume that there exists a sequence (z_n) in G^c such that $|z_n| > 1$ and $z_n \longrightarrow \zeta$ $(n \longrightarrow \infty)$ for some $\zeta \in \partial\mathbb{D} \cap G^c$. Then, for a given $\varepsilon > 0$ we can choose $m \in \mathbb{N}$ such that $|\zeta^m - 1| < \varepsilon/2$, and therefore $|z_n^m - 1| < \varepsilon$ for all n sufficiently large, which is a contradiction. Hence, we have $G^c \cap \mathbb{D}_\varrho \subset \partial\mathbb{D}$ for some $\varrho \in (1, \infty]$. Since G is admissible, this implies the assertion (see for example [15, p. 183]). ∎

The ideal structure of $H^*(\mathbb{D})$ has been investigated to a certain extent by Brooks ([4], see also [13, pp. 98–109]). But as far as we know there are no results for other admissible regions. For the characterization of closed ideals in $H^*(G)$ it turns out that the results are essentially different in the two cases indicated by Lemma 1, so that we will divide the paper into two parts.

1. Closed ideals in $H^*(G)$ for G such that 1 is not isolated in G^c.

1.1. *Characterization of closed ideals.* In this section let G always be an admissible region such that 1 is not isolated in G^c (unless otherwise stated), and therefore also α-starlike for some $\alpha \in \mathbb{R}$ by Lemma 1. For $f \in H^*(G)$ let

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad (z \in \mathbb{D}).$$

We may consider $\hat{f}$ as a function $\hat{f} \colon \mathbb{N}_0 \longrightarrow \mathbb{C}$ associated with f. For $B \subset \mathbb{N}_0$ we set

$$I_B := \{f \in H^*(G) : \hat{f}(n) = 0 \text{ for all } n \in B\}.$$

Obviously, I_B is an ideal in $H^*(G)$, and I_B is closed as is easily seen by the Cauchy integral formula. (Actually, this holds for arbitrary admissible regions $G \subset \hat{\mathbb{C}}$.) We will prove that there are no others.

THEOREM 1. *For an ideal $I \subset H^*(G)$ the following statements are equivalent:*

i) I is closed.

ii) $I = I_B$ for some $B \subset \mathbb{N}_0$.

iii) I is the closure of a principal ideal.

PROOF. Let I be closed. We define

$$B := \{n \in \mathbb{N}_0 : \hat{f}(n) = 0 \text{ for all } f \in I\}.$$

Then it is obvious that $I \subset I_B$. If $n \in \mathbb{N}_0 \setminus B$ then $\hat{f}(n) \neq 0$ for some $f \in I$, which implies $z^n = f(z) * z^n / \hat{f}(n) \in I$. Now we denote by Π_B the linear span of $\{z^n : n \in \mathbb{N}_0 \setminus B\}$ (*i.e.*, the space of all polynomials with gaps at every $n \in B$). Then we have $\overline{\Pi_B} \subset I \subset I_B$. Since G is α-starlike, a theorem of Arakelyan [1] shows that there exists a (universal) matrix summability method such that the power series around 0 of every $f \in H(G)$ is summable to f locally uniformly in G. For fixed $f \in H(G)$ we may achieve that this matrix is row finite by a suitable truncation (*cf.* [9, p. 10]). Therefore, if $f \in I_B$, we obtain a sequence of polynomials in Π_B converging to f locally uniformly in G, which means that $\overline{\Pi_B} = I_B$. This shows $I = I_B$.

To prove that I_B is the closure of a principal ideal we consider

$$f(z) := \sum_{n \notin B} \frac{z^n}{n!}.$$

Then f is entire, and therefore $f \in I_B$. For $n \in \mathbb{N}_0 \setminus B$ we have $z^n = f(z) * (n!\, z^n) \in (f)$ which implies $I_B = \overline{\Pi_B} \subset \overline{(f)}$. Since $\overline{(f)} \subset I_B$ is obvious, the theorem is proved. ∎

In view of Theorem A there arise several questions.

(1) Which principal ideals are closed?
(2) Which closed ideals are principal?
(3) Which finitely generated ideals are principal?

1.2. *Principal ideals which are closed.* We turn to the first question. Let $G \subset \mathbb{C}$ be admissible and let $f \in H^*(G)$ be given. We define $B(f) := \{n \in \mathbb{N}_0 : \hat{f}(n) = 0\}$ and formally

$$f_{-1}(z) := \sum_{n \notin B(f)} \frac{z^n}{\hat{f}(n)}.$$

Then it is obvious that (f) is closed in $H^*(G)$ if f_{-1} also defines a function in $H^*(G)$. For various results concerning the analytic continuation of f_{-1} we refer the reader to [5]. In particular, for $G = \mathbb{D}$ (and if f is not a polynomial) the condition

$$\lim_{\substack{n \to \infty \\ n \notin B(f)}} |\hat{f}(n)|^{1/n} = 1$$

is necessary and sufficient for f_{-1} to be in $H^*(\mathbb{D})$. The following result shows that a restriction on the absolute value of the nonvanishing coefficients is always necessary for (f) to be closed.

PROPOSITION 1. *Let $f \in H^*(G)$, f not a polynomial, be such that*

$$\liminf_{\substack{n \to \infty \\ n \notin B(f)}} |\hat{f}(n)|^{1/n} = 0,$$

or

$$\liminf_{\substack{n \to \infty \\ n \notin B(f)}} |\hat{f}(n)|^{1/n} < 1/\varrho,$$

if $G \subset \mathbb{D}_\varrho$ for some $\varrho \in [1, \infty)$. Then $I = (f)$ is not closed in $H^*(G)$. This holds in particular if f is a transcendental entire function.

PROOF. Suppose that f satisfies

$$\lim_{k \to \infty} |\hat{f}(n_k)|^{1/n_k} = 0$$

for some subsequence (n_k) of the nonnegative integers with $n_k \notin B(f)$. Now consider

$$g(z) := \sum_{k=0}^{\infty} \hat{g}(n_k) z^{n_k},$$

where $\hat{g}(n_k)^2 = \hat{f}(n_k)$ for $k \in \mathbb{N}$. It is clear that g is a (transcendental) entire function. Setting

$$P_N(z) := \sum_{k=0}^{N} \hat{g}(n_k) z^{n_k} \quad \text{and} \quad Q_N(z) := \sum_{k=0}^{N} z^{n_k} / \hat{g}(n_k)$$

we obtain that (P_N) converges to g in $H(\mathbb{C})$ and $P_N = Q_N * f \in I$, so that $g \in \bar{I}$. But $g \notin I$, since otherwise we would have $g = h * f$ for some $h \in H^*(G)$ which means $\hat{h}(n_k) = 1/\hat{g}(n_k)$ for all $k \in \mathbb{N}$. Thus $\limsup_{n\to\infty} |\hat{h}(n)|^{1/n} = \infty$ which is a contradiction.

If $G \subset \mathbb{D}_\varrho$ for some $\varrho \in [1, \infty)$ and

$$\lim_{k\to\infty} |\hat{f}(n_k)|^{1/n_k} < 1/\varrho$$

for some subsequence (n_k) of the nonnegative integers with $n_k \notin B(f)$, then

$$g(z) := \sum_{k=0}^{\infty} \hat{g}(n_k) z^{n_k} \in H(\mathbb{D}_\varrho) \subset H(G),$$

where $\hat{g}(n_k) = \hat{f}(n_k)^{1-\varepsilon}$ for $k \in \mathbb{N}$ and for a sufficiently small $\varepsilon > 0$. A similar argumentation as above shows $g \in \bar{I} \setminus I$. ∎

A question which obviously arises (but which seems to be hard to answer) is whether it is possible to characterize the closed principal ideals in $H^*(G)$ in the cases $G \neq \mathbb{D}$.

1.3. *Closed ideals which are principal.* Now we turn to the question of which of the closed ideals I_B are principal. For that purpose we consider the function

$$\gamma_B(z) := \sum_{n \notin B} z^n.$$

If $\gamma_B \in H(G)$, then it is obvious that γ_B generates the closed ideal I_B. For example, this is true, if

(i) $G = \mathbb{D}$ and $B \subset \mathbb{N}_0$ is arbitrary or
(ii) G is arbitrary and B or $\mathbb{N}_0 \setminus B$ is finite.

At first sight, for given $B \subset \mathbb{N}_0$ one could try to find more suitable functions $f \in H(G)$ such that $B(f) = B$ and $f_{-1} \in H(G)$. But for example, in the interesting case $G = \mathbb{C} \setminus \Gamma_\alpha$ this does not lead further than (ii) above since for every $f \in H(\mathbb{C} \setminus \Gamma_\alpha)$ with $f_{-1} \in H(\mathbb{C} \setminus \Gamma_\alpha)$ necessarily $B(f)$ is finite (see [5], Lemma 2). We will now give a more general answer for this case.

For that purpose we recall the notion of density. For a set $B \subset \mathbb{N}_0$ let $N(r)$ denote the number of all $n \in B \cap \mathbb{D}_r$ for $r > 0$. Then the quantities

$$\bar{d}(B) := \limsup_{r\to\infty} r^{-1} N(r)$$

and

$$\underline{d}(B) := \liminf_{r\to\infty} r^{-1} N(r)$$

are called the *upper and lower density* of B, respectively. The set B is said to have density $d(B)$, if the limit

$$d(B) := \lim_{r\longrightarrow\infty} r^{-1}N(r)$$

exists. Furthermore we assume that the reader is familiar with the definition and basic properties of functions of exponential type as it may be found in [3] or [11].

THEOREM 2. *Let $G = \mathbb{C} \setminus \Gamma_\alpha$ for some $\alpha \in \mathbb{R}$ and let $B \subset \mathbb{N}_0$.*
(i) If $\underline{d}(B) > 0$ and $\mathbb{N}_0 \setminus B$ is infinite, then I_B is not principal.
(ii) If $d(B) = 0$, then I_B is principal.

PROOF. (i) Since $\underline{d}(B) > 0$ we have $\bar{d}(\mathbb{N}_0 \setminus B) < 1$. Therefore, a theorem of Pólya ([12, p. 772], see also [2, p. 76]) implies that every function $f \in I_B$ is entire. Now the assertion follows from Proposition 1 (note that I_B cannot be generated by a polynomial).

(ii) We restrict ourselves to the special case $\alpha = 0$ (*i.e.*, $G = \mathbb{C} \setminus [1,\infty)$), because the general case may be proved in a similar way with only some slight changes in technical details.

We set

$$F(w) := \prod_{n\in B}\Big(1 - \frac{w^2}{n^2}\Big), \quad \text{if } 0 \notin B$$

and

$$F(w) := w \prod_{n\in B\setminus\{0\}}\Big(1 - \frac{w^2}{n^2}\Big), \quad \text{if } 0 \in B.$$

Then F is an entire function of zero exponential type [11, p. 595]. By a theorem of Wigert [2, p. 8] the power series

$$f(z) := \sum_{n=0}^{\infty} F(n)z^n$$

defines a function in $H(\hat{\mathbb{C}} \setminus \{1\})$, and thus in I_B. We will show that $I_B = (f)$.

For that purpose let $g \in I_B$. By a theorem of Arakelyan [1] there exists a function G holomorphic and of inner exponential type zero in the half plane $\Pi := \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\}$ (inner exponential type means that G is of exponential type in every closed sector $\{w \in \mathbb{C} : |\arg w| \leq \beta\}$ with $\beta < \pi/2$) such that

$$g(z) = \sum_{n=0}^{\infty} G(n)z^n.$$

Setting $H := G/F$, it is clear that H is holomorphic in Π. If H is of inner exponential type zero, then the above mentioned theorem of Arakelyan implies that the power series

$$h(z) := \sum_{n=0}^{\infty} H(n)z^n$$

defines a function in $H\big(\mathbb{C} \setminus [1,\infty)\big)$, and thus $g = f * h$. In fact, we will prove that

$$\limsup_{\substack{|w|\longrightarrow\infty \\ |\arg w|\leq\beta}} |w|^{-1}\log|H(w)| = 0$$

for all $\beta < \pi/2$.

We have [3, p. 137]

$$\lim_{r\to\infty} r^{-1} \log |F(re^{\pm i\beta})| = 0.$$

Let $\varepsilon > 0$. Then we obtain for $r \geq r_0$ sufficiently large $|F(re^{\pm i\beta})| \geq e^{-\varepsilon r}$. Furthermore, there exists a sequence (r_n) such that $r_n \longrightarrow \infty$ $(n \longrightarrow \infty)$ and $|F(w)| \geq e^{-\varepsilon r_n}$ for $|w| = r_n$ (see [3, p. 52]). Since G is of inner exponential type zero in Π we have $|G(w)| \leq M_1 e^{\varepsilon|w|}$ for $|\arg w| \leq \beta$ and some constant $M_1 > 0$. Setting $c := 2/\cos\beta$ we get $|H(w)e^{-\varepsilon cw}| \leq (|G(w)|e^{-\varepsilon|w|})(|F(w)|^{-1}e^{-\varepsilon|w|}) \leq M := \max\{M_1, 1\}$ for $w = re^{\pm i\beta}, r \geq r_0$ and $|w| = r_n$. Using the Phragmén-Lindelöf theorem [3, p. 4] we arrive at $|H(w)| \leq M|e^{\varepsilon cw}| \leq Me^{2\varepsilon|w|}$ for $|\arg w| \leq \beta$ which easily implies the assertion. ∎

1.4. *Finitely generated ideals.* Concerning the question whether every finitely generated ideal in $H^*(G)$ is principal, the answer is yes in the special case $G = \mathbb{D}$ but it remains open for arbitrary simply connected admissible regions G. The following results may be proved along the lines of the work of von Renteln [17], so that we only state the results and omit the proofs.

PROPOSITION 2. *Let $f, g \in H^*(\mathbb{D})$. Then f divides g if and only if to every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_0$ such that $|\hat{g}(n)| \leq (1+\varepsilon)^n |\hat{f}(n)|$ for all $n \geq n_0$.*

PROPOSITION 3. *Every finite number of functions $f_1, \ldots, f_m \in H^*(\mathbb{D})$ has a greatest common divisor d which is given (up to invertible elements) by*

$$d(z) = \sum_{n=0}^{\infty} d_n z^n \quad \text{and} \quad d_n = \max\{|\hat{f}_j(n)| : j = 1, \ldots, m\}.$$

PROPOSITION 4. *Let $f, f_1, \ldots, f_m \in H^*(\mathbb{D})$. Then f belongs to the finitely generated ideal $(f_1, \ldots, f_m)$ if and only if to every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_0$ such that*

$$|\hat{f}(n)| \leq (1+\varepsilon)^n \sum_{j=1}^{m} |\hat{f}_j(n)| \quad \text{for all } n \geq n_0.$$

THEOREM 3. *Every finitely generated ideal in $H^*(\mathbb{D})$ is principal.*

1.5. *Closed maximal ideals.* The final aim of this section is to characterize the closed maximal ideals in $H^*(G)$ and the spectrum

$$\mathcal{M}\big(H^*(G)\big) := \{\phi : H^*(G) \longrightarrow \mathbb{C} : \phi \text{ is a nonzero continuous algebra homomorphism}\}$$

of $H^*(G)$. By the Cauchy integral formula we see that $\phi_n \in \mathcal{M}\big(H^*(G)\big)$ for every $n \in \mathbb{N}_0$, where

$$\phi_n(f) := \hat{f}(n) \quad \big(f \in H^*(G)\big).$$

(This also holds for arbitrary admissible regions G.) The following analogue of Theorem B, which is an easy consequence of Theorem 1 and the fact that a complex homomorphism is uniquely determined by its kernel, shows in particular that $\mathcal{M}\big(H^*(G)\big) = \{\phi_n : n \in \mathbb{N}_0\}$.

COROLLARY 1. *For an ideal $M \subset H^*(G)$ the following statements are equivalent:*
i) M is closed and maximal.
ii) $M = I_{\{n\}} = \{f \in H^*(G) : \hat{f}(n) = 0\}$ for some $n \in \mathbb{N}_0$.
iii) M is the kernel of a unique $\phi \in \mathcal{M}\big(H^*(G)\big)$.

From the above results it follows that every closed maximal ideal in $H^*(G)$ is principal, and every closed ideal is an intersection of closed maximal ideals. Furthermore, there is a one-to-one correspondence between the closed maximal ideals and the elements of $\mathcal{M}\big(H^*(G)\big)$. We do not know whether every complex homomorphism is automatically continuous. In [4] (see also [13, p. 103]) Brooks proved that there exists a one-to-one correspondence between all the maximal ideals in $H^*(\mathbb{D})$ and the Stone-Čech compactification of $\mathbb{N}_0$. It would be of interest whether such a result also holds for $H^*(G)$.

2. Closed ideals in $H^*(G)$ for G such that 1 is isolated in G^c.

2.1. *The special case $G = \hat{\mathbb{C}} \setminus \{1\}$.* At first we consider the special algebra

$$H_0^* := \{f \in H^*(\hat{\mathbb{C}} \setminus \{1\}) : f(\infty) = 0\},$$

because the situation here is very lucid. Furthermore, let E_0 denote the algebra of all entire functions of zero exponential type with the usual pointwise multiplication of functions. By a theorem of Wigert [2, p. 8] the power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

defines a function in H_0^* if and only if there exists a function $F \in E_0$ such that $F(n) = \hat{f}(n)$ for all $n \in \mathbb{N}_0$, and F is uniquely determined by Carlson's theorem ([6], see also [3, p. 153]). In the following we write $\hat{f}$ instead of F. This means that $T: H_0^* \longrightarrow E_0$ defined by $T(f) := \hat{f}$ is an algebra isomorphism. For $\varepsilon > 0$ and $F \in E_0$ we set

$$\|F\|_\varepsilon := \inf\{C > 0 : |F(w)| \le Ce^{\varepsilon|w|} \text{ for all } w \in \mathbb{C}\}.$$

Then $\{\|\cdot\|_\varepsilon : \varepsilon > 0\}$ is a system of norms on E_0 which defines a locally convex topology on E_0 that is completely metrizable, *i.e.*, E_0 with this topology is a Fréchet space in which also the multiplication is continuous (hence a B_0-algebra). This topology was introduced by Raševskiĭ [14]. With the closed graph theorem (see for example [18, p. 50]) it is easy to see that $T: H_0^* \longrightarrow E_0$ is continuous, and therefore the open mapping theorem (see for example [18, p. 47]) implies that T is a homeomorphism. Thus, we have found that

$$H_0^* \text{ and } E_0 \text{ are algebraically and topologically isomorphic.}$$

Now we are able to apply results concerning E_0 to our problem. For that purpose let (a_n) be a sequence in $\mathbb{C}$ such that either (a_n) is finite or $(|a_n|)$ is monotonically increasing and $|a_n| \longrightarrow \infty$ $(n \longrightarrow \infty)$. Then we set

$$I(a_n) := \{f \in H_0^* : \hat{f}(a_n) = 0 \text{ for all } n\},$$

where a_n is an m-fold zero of $\hat{f}$ whenever a_n occurs in the sequence (a_n) m times. Obviously, $I(a_n)$ is a closed ideal in H_0^*. It may occur that $I(a_n)$ is trivial.

According to Hadamard's factorization theorem [3, p. 22] and Lindelöf's theorem [3, p. 27], it is easy to see that for $\hat{f}, \hat{g} \in E_0$ such that $\hat{f}/\hat{g}$ is an entire function, we also have $\hat{f}/\hat{g} \in E_0$. Thus, $\hat{g}$ divides $\hat{f}$ in E_0 if and only if every zero of $\hat{g}$ is also a zero of $\hat{f}$ (counting multiplicity). It follows that every principal ideal $I \subset H_0^*$ is of the form $I = I(a_n)$. In fact, again referring to Lindelöf's theorem, an ideal $I \subset H_0^*$ is principal and nontrivial if and only if $I = I(a_n)$ for some sequence (a_n) such that (a_n) is finite or $|a_n|/n \longrightarrow \infty$ $(n \longrightarrow \infty)$ and $\sum_n 1/a_n$ is convergent, where the sum ranges over all n such that $a_n \neq 0$.

An application of Raševskiĭ's theorem [14] yields

THEOREM 4. *An ideal $I \subset H_0^*$ is closed and non-trivial if and only if $I = I(a_n)$ for some sequence (a_n) such that (a_n) is finite or $|a_n|/n \longrightarrow \infty$ $(n \longrightarrow \infty)$.*

COROLLARY 2. *There exist closed ideals in H_0^* which are not principal.*

In view of Corollary 2 the question arises whether every closed ideal in H_0^* is finitely generated. We do not know the answer but we prove

THEOREM 5. *Every closed ideal in H_0^* is the closure of a two-fold generated ideal.*

PROOF. According to Theorem 4 we may assume that $I = I(a_n)$ is such that (a_n) is infinite and $a_1 \neq 0$. We define sequences (b_n) and (c_n) by

$$b_{2n-1} := a_n,\ b_{2n} := -a_n \quad (n \in \mathbb{N}),$$

and

$$c_{2n-1} := a_n,\ c_{2n} := -a_n - \delta_n \quad (n \in \mathbb{N}),$$

where

$$\delta_n := \min\{\tilde{\delta}_n/2, 1, |a_n|\},$$

and

$$\tilde{\delta}_n := \min\{|a_n \pm a_m| : m \in \mathbb{N}, a_n \neq \pm a_m\}.$$

Then it is obvious that $\sum_{n=1}^{\infty} 1/b_n = 0$, and an easy estimate shows that $\sum_{n=1}^{\infty} 1/c_n$ converges to $c \in \mathbb{C}$, say. Now by Lindelöf's theorem [3, p. 27] the infinite products

$$F_1(w) := \prod_{n=1}^{\infty} \left(1 - \frac{w}{b_n}\right)$$

and

$$F_2(w) := e^{-cw} \prod_{n=1}^{\infty} \left(1 - \frac{w}{c_n}\right) e^{w/c_n}$$

define functions in E_0, so that $f_1 := T^{-1}(F_1)$ and $f_2 := T^{-1}(F_2)$ are in I. Setting $\tilde{I} := \overline{(f_1, f_2)}$ we see that $\tilde{I}$ is a closed ideal in H_0^* and thus $\tilde{I}$ must be of the form $I(\tilde{a}_n)$. But by construction of (b_n) and (c_n) we have $(\tilde{a}_n) = (a_n)$ which completes the proof. ∎

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The assertion of Theorem 3 is false in H_0^*. According to an example of von Renteln [16, p. 10] there exist functions $F_1, F_2 \in E_0$ such that F_1 and F_2 have no common zeros and $(F_1, F_2) \neq E_0$. Setting $I = \big(T^{-1}(F_1), T^{-1}(F_2)\big)$ we see that $I \neq H_0^*$, and thus I is neither principal nor closed.

In view of the closed maximal ideals in H_0^* and the spectrum $\mathcal{M}(H_0^*)$ we have an analogue of Corollary 1, but here $\mathcal{M}(H_0^*) = \{\phi_a : a \in \mathbb{C}\}$, where $\phi_a(f) := \hat{f}(a)$ for $f \in H_0^*$, is uncountable.

COROLLARY 3. *For an ideal $M \subset H_0^*$ the following statements are equivalent:*

i) M is closed and maximal.

ii) $M = I(a) = \{f \in H_0^* : \hat{f}(a) = 0\}$ for some $a \in \mathbb{C}$.

iii) M is the kernel of a unique $\phi \in \mathcal{M}(H_0^*)$.

2.2. *The case $G = \hat{\mathbb{C}} \setminus A_k$.* Now we turn to the more general case of the algebras

$$H_{0,k}^* := \{f \in H^*(\hat{\mathbb{C}} \setminus A_k) : f(\infty) = 0\} \quad (k \in \mathbb{N}),$$

which is essentially reducible to the special case $H_0^* = H_{0,1}^*$. At first we introduce some notation. For $k \in \mathbb{N}$ and $j \in \{0, \dots, k-1\}$ we define

$$\gamma_{jk}(z) := \gamma(z/e^{2\pi ij/k}) \quad (z \neq e^{2\pi ij/k})$$

and $\gamma_{jk} * H_0^* := \{\gamma_{jk} * f : f \in H_0^*\} = \{f \in H(\hat{\mathbb{C}} \setminus \{e^{2\pi ij/k}\}) : f(\infty) = 0\}$.

Then we have

$$H_{0,k}^* = \bigoplus_{j=0}^{k-1} \gamma_{jk} * H_0^*,$$

where $\oplus$ denotes a topological direct sum. By Laurent expansion we see that every $f \in H_{0,k}^*$ has a unique decomposition

$$f = \sum_{j=0}^{k-1} \gamma_{jk} * f_j$$

with $f_j \in H_0^*$ for $j = 0, \dots, k-1$, and a simple application of the maximum principle shows that the subspaces $\gamma_{jk} * H_0^*$ are closed in $H_{0,k}^*$.

The key for "lifting" the results from H_0^* to $H_{0,k}^*$ lies in the knowledge of the homomorphisms from $H_{0,k}^*$ into H_0^*.

LEMMA 2. *For $k \in \mathbb{N}$ the following statements are equivalent:*

i) T is an algebra homomorphism from $H_{0,k}^*$ into H_0^*.

ii) There exists an algebra endomorphism t on H_0^* and a k-th root of unity ζ such that

$$T(f) = T\Big(\sum_{j=0}^{k-1} \gamma_{jk} * f_j\Big) = t\Big(\sum_{j=0}^{k-1} \zeta^j f_j\Big) \quad (f \in H_{0,k}^*).$$

In this case we have $t = T_{|H_0^*}$. Moreover, T is continuous if and only if t is continuous.

PROOF. 1. Let T be a homomorphism from $H^*_{0,k}$ into H^*_0, and let $f = \sum_{j=0}^{k-1} \gamma_{jk} * f_j \in H^*_{0,k}$ be given. Then we have

$$T(f) = \sum_{j=0}^{k-1} T(\gamma_{jk}) * T(f_j) = \sum_{j=0}^{k-1} T(\gamma_{jk}) * t(f_j)$$

where $t := T_{|H^*_0}$ is an endomorphism on H^*_0. It remains to show that there exists a ζ such that $\zeta^k = 1$ and $T(\gamma_{jk}) = \zeta^j\gamma$. Let $T(\gamma) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$. Since $T(\gamma) = T(\gamma) * T(\gamma)$ we have $\hat{g}(n) \in \{0, 1\}$ for every $n \in \mathbb{N}_0$. Now it follows from results on the distribution of zeros of entire functions of exponential type zero that $\hat{g} \equiv 0$ or $\hat{g} \equiv 1$ (*cf.* [3, p. 166]). If $T(\gamma) = 0$, then $t = 0$ and $T = 0$ and we are done. So let $T(\gamma) = \gamma$. Since $\big(\widehat{T(\gamma_{1k})}\big)^k = \widehat{T(\gamma)} = 1$ in E_0 we find that $\widehat{T(\gamma_{1k})} \in E_0$ has no zeros and so $\widehat{T(\gamma_{1k})}$ is necessarily a constant ζ, say, which satisfies $\zeta^k = 1$. It follows more generally that $\widehat{T(\gamma_{jk})} = \big(\widehat{T(\gamma_{1k})}\big)^j = \zeta^j$ for every $j \in \{0, \ldots, k-1\}$ and thus $T(\gamma_{jk}) = \zeta^j\gamma$ for every $j \in \{0, \ldots, k-1\}$.

2. If T is as in (ii), then the linearity of T follows from the linearity of t. Furthermore, for $f, g \in H^*_{0,k}$ one computes

$$f * g = \sum_{m=0}^{k-1} \gamma_{mk} * \sum_{j+\ell=m \pmod k} f_j * g_\ell$$

and therefore

$$\begin{aligned} T(f * g) &= t\Big(\sum_{m=0}^{k-1} \zeta^m \sum_{j+\ell=m \pmod k} f_j * g_\ell\Big) = t\Big(\Big(\sum_{j=0}^{k-1} \zeta^j f_j\Big) * \Big(\sum_{\ell=0}^{k-1} \zeta^\ell g_\ell\Big)\Big) \\ &= T(f) * T(g). \end{aligned}$$

3. The above considerations show that $t = T_{|H^*_0}$ and so continuity of T implies continuity of t. On the other hand, if t is continuous, then the continuity of T follows from the fact that the mappings $f \longrightarrow f_j$ are continuous by the maximum principle. ∎

Let T_ζ denote the (continuous and surjective) homomorphism from Lemma 2 corresponding to ζ and (for our purposes without loss of generality) the identity t, let K_ζ denote the kernel of T_ζ and set

$$\mathcal{J}_\zeta := \{J \subset H^*_{0,k} : J \text{ is an ideal such that } K_\zeta \subset J\}.$$

Then we have

THEOREM 6. *Let $k \in \mathbb{N}$ and a k-th root of unity ζ be given. Then*

$$I \longmapsto (T_\zeta)^{-1}(I)$$

defines a one-to-one mapping from the set of all ideals in H^*_0 onto $\mathcal{J}_\zeta$. Moreover, if I is closed, then also $(T_\zeta)^{-1}(I)$ is closed, and if $I = (f_1, \ldots, f_n)$ and $K_\zeta = (g_1, \ldots, g_m)$, then

$$(T_\zeta)^{-1}(I) = (f_1, \ldots, f_n, g_1, \ldots, g_m).$$

PROOF. The first statement follows directly from a well-known general result on homomorphic images of rings (see for example [8, p. 87]), and the second follows directly from the fact that T_ζ is continuous.

If $I = (f_1, \ldots, f_n)$ and $K_\zeta = (g_1, \ldots, g_m)$, then obviously

$$(T_\zeta)^{-1}(I) \supset (f_1, \ldots, f_n, g_1, \ldots, g_m).$$

On other hand, let $h \in (T_\zeta)^{-1}(I)$ be given. Then there exist functions $g_j \in H_0^*$ with $T_\zeta(h) = \sum_{j=1}^n g_j * f_j$. Since T_ζ is surjective, we have $g_j = T_\zeta(h_j)$ for some $h_j \in H_{0,k}^*$ and so

$$T_\zeta(h) = T_\zeta\Big(\sum_{j=1}^n h_j * f_j\Big).$$

Therefore, $h = \sum_{j=1}^n h_j * f_j + h_0$ with some $h_0 \in K_\zeta = (g_1, \ldots, g_m)$, which shows $h \in (f_1, \ldots, f_n, g_1, \ldots, g_m)$. ■

According to Lemma 2 we see that for every $a \in \mathbb{C}$ and every k-th root of unity ζ the composition $\phi_{a,\zeta} := \phi_a \circ T_\zeta \colon H_{0,k}^* \longrightarrow \mathbb{C}$ is a continuous homomorphism. Actually, it turns out that

$$\mathcal{M}(H_{0,k}^*) = \{\phi_{a,\zeta} : a \in \mathbb{C}, \zeta \text{ is a } k\text{-th root of unity}\}.$$

Let $\phi \in \mathcal{M}(H_{0,k}^*)$ be given. First, since $\phi_{|H_0^*} \in \mathcal{M}(H_0^*)$, by Corollary 3 there exists a constant $a \in \mathbb{C}$ with $\phi_{|H_0^*} = \phi_a$. Now, for $f = \sum_{j=0}^{k-1} \gamma_{jk} * f_j \in H_{0,k}^*$ we find

$$\phi(f) = \sum_{j=0}^{k-1} \phi(\gamma_{jk}) \cdot \phi(f_j) = \sum_{j=0}^{k-1} \phi(\gamma_{jk}) \cdot \phi_a(f_j).$$

Since $\phi(\gamma)^2 = \phi(\gamma)$ and $\phi \not\equiv 0$, we have $\phi(\gamma) = 1$. Moreover, $\phi(\gamma_{1k})^k = \phi(\gamma) = 1$, thus $\phi(\gamma_{1k}) = \zeta$ for some k-th root of unity ζ, and finally $\phi(\gamma_{jk}) = \phi(\gamma_{1k})^j = \zeta^j$ for $j = 1, \ldots, k-1$.

Clearly, for every a and ζ as above

$$M_{a,\zeta} := \phi_{a,\zeta}^{-1}(\{0\}) = \Big\{f \in H_{0,k}^* : \sum_{j=0}^{k-1} \zeta^j \hat{f}_j(a) = 0\Big\} = (T_\zeta)^{-1}\big(I(a)\big)$$

is a closed maximal ideal in $H_{0,k}^*$ such that $K_\zeta \subset J$. According to Corollary 3 it would be of interest whether every closed maximal ideal is obtained in this way.

2.3. *The general case.* Finally, we consider briefly the general case of an admissible region G such that 1 is isolated in G^c and $G \neq \hat{\mathbb{C}} \setminus A_k$ for every $k \in \mathbb{N}$. In this case, by Lemma 1, we have $\mathbb{D}_\varrho \subset G \cup A_k \subset \mathbb{C}$ for some $\varrho > 1$. Much as in Section 2.2, we see by separation of singularities that

$$H^*(G) = H^*_{0,k} \oplus H(G \cup A_k)$$

is a topological direct sum. Moreover, let T denote the (continuous) projection from $H^*(G)$ onto $H^*_{0,k}$, *i.e.*

$$T(f) = f_1 \quad \big(f \in H(G)\big),$$

where $f = f_1 + f_2$ is such that $f_1 \in H^*_{0,k}$ and $f_2 \in H(G \cup A_k)$. Since by the Hadamard multiplication theorem $H(G \cup A_k)$ is an ideal in $H(G)$, we find that T is a homomorphism. According to [8, p. 87], this implies

PROPOSITION 5. *Let G be an admissible region such that* 1 *is isolated in G^c and $G \neq \hat{\mathbb{C}} \setminus A_k$ for all $k \in \mathbb{N}$. Then*

$$I \longmapsto T^{-1}(I) = I \oplus H(G \cup A_k)$$

defines a one-to-one mapping between the set of all ideals in $H^*_{0,k}$ and the set of all ideals in $H^*(G)$ containing $H(G \cup A_k)$. Moreover, I is closed if and only if $I \oplus H(G \cup A_k)$ is closed.

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