Appendix K
Vortices in nuclei

In this appendix we follow the argument presented in Bertsch *et al*. (1988).

K.1 Simple estimates

Nuclei in their ground state can be viewed, in general, as a condensate of pairs of nucleons coupled to angular momentum zero. Evidence for the existence of multipole (non-zero J) pairing has also been found in a variety of nuclear properties (see Section 5.3). Empirically, d-state pairing correlation of a single pair is about half that of a monopole pair. The reduction is due to the decrease in phase space for valence pairs with higher J. This is shown schematically in Fig. K.1.

This situation may be rather different for rapidly rotating nuclei. In this case, large values of the angular momentum can be built by using a coupling scheme where both valence and core particles couple pairwise to angular momentum J. The lowest multipolarity different from zero to which pairs of particles can couple is $J = 1$. Under these circumstances, Galilean invariance allows one to redefine the phase space where dipole pairing acts, so that the resulting phase space is nearly the same as for $J = 0$ pairing. To be able to carry out analytically the different estimates we shall approximate the nucleus by a cylinder of the same radius as that of the nucleus, and a height such that the volume is conserved (see Fig. K.2). That is,

$$v = \frac{4\pi}{3} R^3 = \pi R^2 H, \tag{K.1}$$

leading to

$$H = \tfrac{4}{3} R. \tag{K.2}$$

In this way we also conserve density,

$$\rho_0 = \frac{mA}{v} = \frac{M}{v}, \tag{K.3}$$

where m is the nucleon mass and M the total mass of the system.

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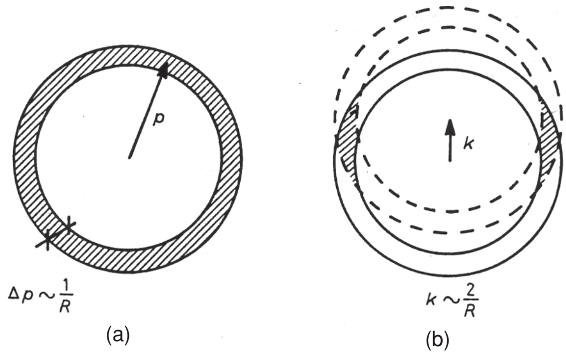

Figure K.1. Phase space for particles in paired wavefunctions. The available momenta for valence particles in a Fermi system are shown in (a). All momenta are allowed for a particle in a pair with total momentum zero. When the pair momentum is non-zero, the valence phase space is reduced as indicated in (b) (after Bertsch *et al.* (1988)). Copyright © Società italiana di Fisica.

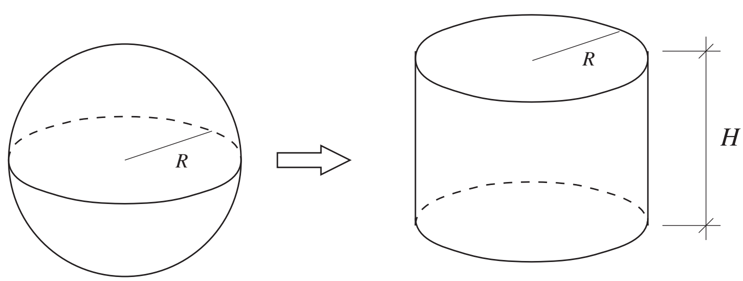

Figure K.2. Approximation used to describe vortex motion. The height H is defined such that $\pi R^2 H = (4/3)\pi R^3$.

Because of $J \neq 0$ superfluidity, a vortex forms with a cylindrical hole along the axis of rotation. The velocity field of the fluid in the vortex can be written as

$$V_0 = \frac{g}{r}, \tag{K.4}$$

where

$$g = \frac{\hbar}{2m}, \tag{K.5}$$

for $J = 1$ vorticity (i.e. each Cooper pair carries angular momentum $J = 1$). In this case the total angular momentum of the system is

$$I \approx \frac{A}{2}. \tag{K.6}$$

The energy of the vortex consists of a rotational part and a part associated with the surface created to generate the hole compatible with the velocity field given in equation

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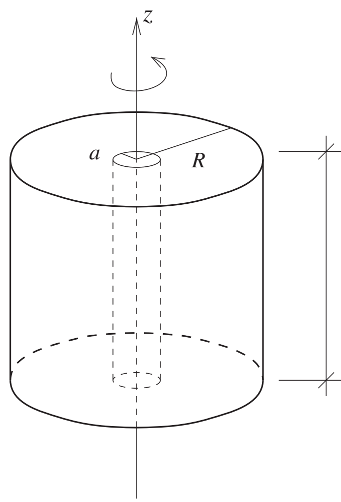

Figure K.3. Atomic nucleus with a vortex, i.e. in a condensed phase of pairs $a^{+}_{j,m}a^{+}_{j,-m+1}$, and $I_z = A/2$.

(K.4). The rotational energy is estimated as

$$E_{\text{vortex}} = \int d\tau \frac{\rho_0 V_0^2}{2} = \frac{\rho_0}{2}\int_0^{2\pi} d\phi \int_0^H dz \int_a^R V_0^2 r\, dr$$
$$= \frac{\rho_0}{2} 2\pi H g^2 \int_a^R d\ln r = \frac{1}{R^2 M} L^2 \ln\frac{R}{a}\ , \tag{K.7}$$

where

$$L = \int d\tau \rho_0 r V_0 = \rho_0 \int_0^{2\pi} d\phi \int_0^H dz \int_a^R V_0 r^2 dr$$
$$\approx \rho_0 2\pi H g \int_a^R r\, dr \approx \rho_0 \pi H g R^2 \tag{K.8}$$

is the angular momentum of the system.

Note that the above relation implies (see Fig. K.3) that

$$L = \frac{M}{\nu}\pi R^2 H \frac{\hbar}{2m} = \hbar\frac{A}{2}, \tag{K.9}$$

as assumed. The extra energy needed to create the hole of radius a is

$$E_{\text{surf}} = 2\pi a H \sigma, \tag{K.10}$$

where σ is the surface tension (cf. equation (7.32)).

To determine a we minimize the total energy

$$\frac{\partial}{\partial a}(E_{\text{vortex}} + E_{\text{surf}}) = -\frac{1}{R^2 M} L^2 \frac{1}{a} + 2\pi H \sigma = 0, \tag{K.11}$$

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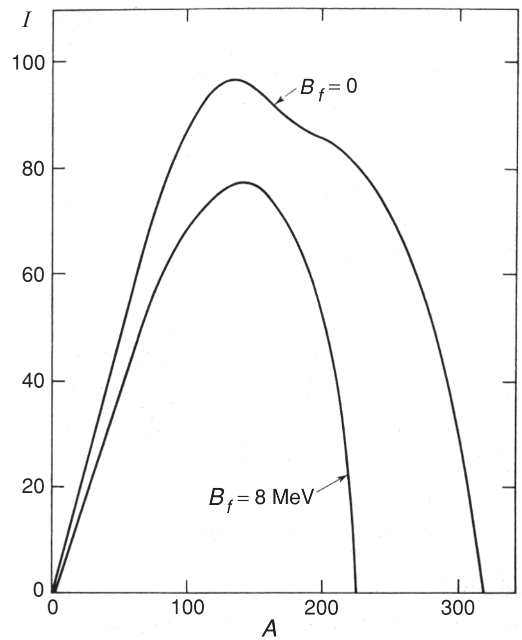

Figure K.4. Stability against fission for a rotating nucleus. The critical angular momentum I for which the nucleus becomes unstable against fission has been calculated in the liquid-drop model as a function of the mass number of the nucleus, and the corresponding curve is labelled $B_f = 0$. The curve labelled $B_f = 8$ MeV shows the angular momentum for which the fission barrier is found at an energy of 8 MeV above the ground state corresponding to the average neutron separation energy. The figure is based on Cohen *et al.* (1974) (see also Bohr and Mottelson (1974)). Reprinted from *Annals of Physics*, Vol. 82, Cohen *et al.*, 'Equilibrium configurations of rotating charged or gravitating liquid masses, II', page 557, Copyright 1974, with permission from Elsevier.

thus obtaining

$$a = \frac{1}{2\pi H\sigma}\frac{L^2}{R^2 M} = \frac{1}{8\sigma}\frac{A}{v}\frac{\hbar^2}{m}. \tag{K.12}$$

This now poses the following questions.

1. Does the nucleus allows spins as high as $I \sim A/2$?
2. How does the energy cost in forming a vortex compare with the energy gain of pairing?

Making use of the liquid-drop model with a surface tension $\sigma(= 1\,\text{MeV fm}^{-1})$, one obtains the curves given in Fig. K.4. Thus, a nucleus of $A \approx 150$ can, in principle, sustain about 80 units of angular momentum, i.e. of the order of $A/2 \approx 75$ as required to make a vortex.

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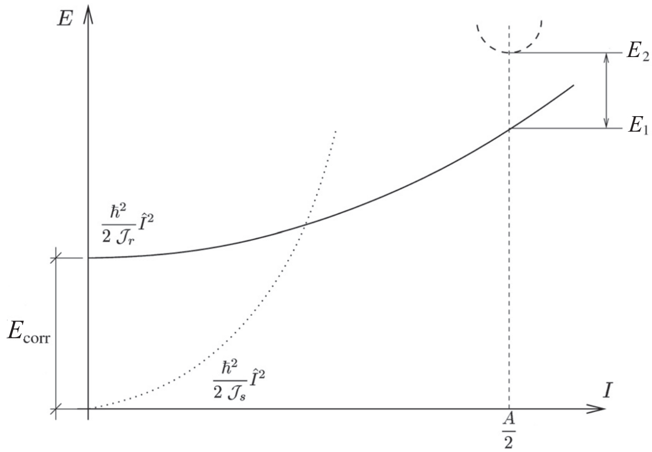

Figure K.5. Schematic representation of rotational bands with superfluid moment of inertia ($\mathcal{J}_s$, dotted curve) and rigid moment of inertia ($\mathcal{J}_r$, continuous curve) as well as roton minimum (dashed curve). Also shown is the summed pairing correlation energy (protons plus neutrons) $\approx \Delta^2/d$ (see Section 3.5).

The answer to question 2 is schematically given in Fig. K.5, where

$$E_1 = E_{\text{rigid}}(L = 75) + |E_{\text{corr}}|, \tag{K.13}$$

and

$$E_2 = E_{\text{vortex}} + E_{\text{surf}}. \tag{K.14}$$

The quantity E_{corr} is the pairing correlation energy. In other words, question 2 is equivalent to asking whether E_1 is smaller or larger than E_2.

The pairing correlation energy is given by (see equation (3.60))

$$E_{\text{corr}} = -\frac{\Delta_n^2 + \Delta_p^2}{2d}, \tag{K.15}$$

with

$$\Delta_n \approx \Delta_p \approx \frac{12}{\sqrt{A}}\,\text{MeV} \tag{K.16}$$

and

$$d \approx 0.4\,\text{MeV}. \tag{K.17}$$

Thus

$$E_{\text{corr}} \approx -\frac{\Delta^2}{d} \approx -\frac{360}{A}\,\text{MeV}\ . \tag{K.18}$$

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Making use of the fact that

$$E_{\rm rigid} = \frac{L^2}{MR^2}, \tag{K.19}$$

one obtains for the energy of the vortex measured with respect to $E_{\rm rigid} + E_{\rm corr}$

$$\delta E = E_2 - E_1 = \frac{L^2}{MR^2}\left(\ln\frac{R}{a} - 1\right) + 2\pi H\sigma a - \frac{360\,{\rm MeV}}{A}. \tag{K.20}$$

Assuming $A = 150$ and making use of the parameters

$$R = 1.2A^{1/3}\ {\rm fm} = 6.4\ {\rm fm}, \tag{K.21}$$

$$H = \tfrac{4}{3}R \approx 8.5\ {\rm fm}, \tag{K.22}$$

$$a = \frac{1}{8\times\frac{1\,{\rm MeV}}{{\rm fm}^2}} \times \frac{150}{\pi(6.4\ {\rm fm})^2 8.5\ {\rm fm}} 40\ {\rm MeV\ fm}^2 \approx 0.7\ {\rm fm} \tag{K.23}$$

and

$$\begin{aligned}\frac{L^2}{MR^2} &= \left(\frac{A}{2}\right)^2\frac{\hbar^2}{Am}\frac{1}{R^2} = (75)^2 \times \frac{40\,{\rm MeV\ fm}^2}{150\times(6.4\ {\rm fm})^2}\\ &= 36.6\,{\rm MeV},\end{aligned} \tag{K.24}$$

$$\begin{aligned}\delta E &= 36.6\ {\rm MeV}\left(\ln\frac{6.4}{0.7} - 1\right) + 2\pi 8.5\ {\rm fm}\frac{1\ {\rm MeV}}{{\rm fm}^2}0.7\ {\rm fm} - \frac{360}{150}\ {\rm MeV}\\ &= 36.6\ {\rm MeV}\times 1.2 + 37.4\ {\rm MeV} - 2.4\ {\rm MeV}\\ &\approx 79\ {\rm MeV}.\end{aligned} \tag{K.25}$$

Thus

$$E_2 > E_1\ . \tag{K.26}$$

Consequently, a vortex can, in principle, exist in an atomic nucleus. However, its statistical weight is likely to be too small to be observed, because of its high excitation energy above the yrast state with the same angular momentum (see, however Section 3.10.1). One reason for this is that the vortex kinetic energy is about twice the kinetic energy of rigid rotation ($E_{\rm vortex} \approx 2.2E_{\rm rigid}$). The other is the large surface energy of the vortex core ($E_{\rm surf} \approx 37\,{\rm MeV}$).

K.2 Critical velocity for the excitation of rotons

From the value of the vortex angular momentum

$$L = p_0R = \hbar k_0 R = \hbar I, \tag{K.27}$$

$$I \approx \frac{A}{2}, \tag{K.28}$$

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one can determine the associated momentum

$$k_0 = \frac{A}{2R} \approx \frac{A^{2/3}}{2.4 \times \text{fm}}, \tag{K.29}$$

$$k_0 \approx 0.4 \times A^{2/3} \text{fm}^{-1}. \tag{K.30}$$

Making use of the excitation energy of the roton (see equations (1.6), (K.25) as well as Figs. 1.6 and K.5)

$$\Delta = \delta E \approx 79 \text{ MeV}, \tag{K.31}$$

$$(V_{\text{cr}})_{\text{vortex}} = \frac{\Delta}{\hbar k_0} = \frac{79 \text{ MeV}}{(\hbar c) \times 0.4 A^{2/3} \text{ fm}^{-1}} c \tag{K.32}$$

$$\approx \frac{79 \text{ MeV}}{200 \text{ MeV fm} \times 0.4 A^{2/3} \text{ fm}^{-1}} c \approx \frac{c}{A^{2/3}}, \tag{K.33}$$

consequently,

$$(V_{\text{cr}})_{\text{vortex}} \approx \frac{c}{A^{2/3}} \approx \frac{c}{25} \approx 12 \times 10^6 \text{ m s}^{-1} . \tag{K.34}$$

which is the lowest velocity needed to excite a vortex, i.e. one of the elementary modes of excitation of the system.

K.3 Critical velocity for superfluidity

$$(V_{\text{cr}})_{\text{sup}} \approx \frac{\Delta}{\hbar k_F} = \frac{12}{\sqrt{A}} \text{MeV} \times \left(\frac{1}{200 \text{ MeV fm} \times 1.36 \text{ fm}^{-1}} \right) c$$
$$\approx \frac{4 \times 10^{-2}}{\sqrt{A}} c , \tag{K.35}$$

where c is the velocity of light. Thus, when $A = 150$,

$$(V_c)_{\text{sup}} \approx 1.1 \times 10^6 \text{ ms}^{-1} , \tag{K.36}$$

which is the critical velocity to excite quasiparticles, in other words, the lowest velocity needed to excite one of the elementary modes of the system.

As already stated above, we note that in equation (K.32) Δ is the gap at the roton minimum ($= \delta E$, equation (K.31)) while in equation (K.35) it is the BCS superfluid pairing gap of a nucleus (see equation (1.17) as well as (1.21)) (see also the last paragraph of Section 1.5).

While one does not expect supercurrents to take place in nuclei, the phenomenon may be realized in neutron stars (see Sections 1.10 and 10.5). In any case, the estimates given in equations (K.34) and (K.36) can be viewed as an excercise concerning orders of magnitude within the framework of the discussion carried out in Sections 1.4 and 1.5.