POLYNOMIALS IN A HERMITIAN ELEMENT

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For an element a of a unital Banach algebra A with dual space A', we define the numerical range $V(a)=\{f(a):f\in A', \|f\|=f(1)=1\}$, and the numerical radius $v(a)=\sup\{|z|:z\in V(a)\}$. An element a is said to be Hermitian if $V(a)\subseteq\mathbb{R}$, equivalently $\|\exp(ita)\|=1$ $(t\in\mathbb{R})$. Under the condition $V(h)\subseteq[-1,1]$, any polynomial in h attains its greatest norm in the algebra $Ea[-1,1]$, generated by an element h with $V(h)=[-1,1]$.

In [**3**] we proved that in $Ea[-1,1]$ all elements $a=(ih)^m+\xi_0$, $\xi_0\in\mathbb{R}$, have $v(a)=\|a\|$: on pages 39, 44 of [**3**] we find $\zeta\in V(h^m)$ such that $|\zeta-\tau'|=\|h^m-\tau'\|$. Here we extend this to any element

$$a=\xi_0+\xi_1 ih+\ldots+\xi_{m-1}(ih)^{m-1}+(ih)^m \tag{1}$$

where $m\geqslant 1$ and $\xi_i\in\mathbb{R}$ $(i=0,1,\ldots,m-1)$. As in [**3, 6**], we represent $Ea[-1,1]$ as a subalgebra of the bounded linear operators on the Banach space X of entire functions f such that $\|f\|=\sup\{|f(z)|/\exp(|\operatorname{Im} z|):z\in\mathbb{C}\}$ exists. If f is entire with $f(z)/\exp(|z|)$ bounded, and $f(x)$ is bounded for $x\in\mathbb{R}$, then $f\in X$ and $\|f\|=\sup\{|f(x)|:x\in\mathbb{R}\}$. Then $Ea[-1,1]$ is generated by $h=-iD$, where $Df(z)=f'(z)$. We show that there is a function e in X corresponding to a support functional Φ of the element a of (1) such that $e'^2/(1-e^2)$ is rational. We have $\Phi(b)=(be)(0)(b\in Ea[-1,1])$, and $e(z)=\Phi(\exp(izh))$, which indicates how we can identify $Ea[-1,1]'$ with X.

We can also consider $Ea[-1,1]$ as the algebra of functions f on $[-1,1]$ given by $f(t)=\sum_{k=1}^{\infty} c_k\exp(id_kt)$, $c_k\in\mathbb{C}$, $d_k\in\mathbb{R}$, $\sum|c_k|$ finite, with $\|f\|=\inf\sum|c_k|$ over such representations, i.e. a quotient of $l^1(\mathbb{R})$ ([**5**]). The function $h(t)=t$ corresponds to the element h. The element a of (1) has a representation as above $\sum c_k\exp(id_kh)$ with $\sum|c_k|=\|a\|$, where $e(d_k)=\pm1$ for all k. This follows from (11), and is valid for this polynomial in a Hermitian element of norm at most 1 in any Banach algebra ([**1, 4**]).

Note that $f\in X_1\Rightarrow f'\in X_1$—Bernstein's inequality, or equivalently, $\|h\|=1$. Define $T\in X'$ by $Tf=(af)(0)$, i.e.

$$Tf=\xi_0 f(0)+\xi_1 f'(0)+\ldots+f^{(m)}(0). \tag{2}$$

By Lemma 4 of [**3**], $\|a\|=\|T\|=\sup\{|Tf|:f\in X_1\}$, where $X_1=\{f\in X:\|f\|\leqslant 1\}$. This supremum is attained by an extremal function. Hence finding $\|a\|$ is equivalent to maximizing (2) over $f\in X_1$. R. Boas [**2**, Section 11.4] considers this—his method gives the extremal function when it is a translate of sin z. In [**3**] we proved that translates of $\cos\sqrt{z^2+\theta^2}$, $0\leqslant\theta<\pi/2$, were enough for elements $(ih)^m+\xi_0$ ([**3**], page 39 for m even, page 41 for m odd). Here we prove the following theorem.

THEOREM 1. *Let a, T be as in* (1), (2). *Then $\|a\|=\|T\|=Te$ for a certain $e\in X_1$*

which is real on $\mathbb{R}$, *and such that there exist* $\mu \leq m-1$ *and* a_j, $b_j \in \mathbb{C}$, $a_j \notin \mathbb{R}$, a_j, $\bar{a}_j \neq b_k$ $(j, k = 1, 2, \ldots, \mu)$ *such that for all* $z \in \mathbb{C}$,

$$e'^2(z) \prod_{j=1}^{\mu} (z-a_j)(z-\bar{a}_j) = (1-e^2(z)) \prod_{j=1}^{\mu} (z-b_j)^2. \tag{3}$$

(*Allow* $\mu = 0$, *i.e.* $e'^2 = 1 - e^2$.). *Further,* $v(a) = \|a\|$.

We expect that the functional T has an extremal function e in X_1 which oscillates on $\mathbb{R}$ between ± 1, apart from at a finite number of turning points. A theorem of Sonin–Polya [**7**, p. 164] pointed out by J. Duncan suggests that $e^2 + \varphi^{-1} e'^2 = 1$, where the function φ is positive on the intervals of $\mathbb{R}$ of constant oscillation. Lemma 2 gives a sequence p_n which converges to e: p_n is the extremal of T restricted to a class of trigonometric polynomials. Then $p_n'^2/(1-p_n^2)$ is a trigonometric rational function, and a variational argument puts a bound on the number of its factors in lowest form. The Hadamard factorisations in X show that the limit of this sequence, $e'^2/(1-e^2) = \varphi$, is a rational function multiplied by an exponential. Using the fact that Te is extremal, we prove that φ is rational. Finally we construct $g = e + if \in X_1$ with $f(\mathbb{R}) \subseteq \mathbb{R}$ and $|g(0)| = 1$, which is enough to give $v(a) = \|a\|$.

For $n \in \mathbb{N}$, let $P_n \subseteq X$ be the set of functions $z \to \sum_{k=-n}^{n} \alpha_k \exp(ikz/n)$ where $\alpha_k \in \mathbb{C}$, and let $P = \bigcup_{n \in \mathbb{N}} P_n$.

LEMMA 2. *Let* $f \in X_1$. *Then there exists a sequence* $(f_n)_{n=1}^{\infty} \subseteq P \cap X_1$ *such that* $f_n \to f$ *as* $n \to \infty$ *uniformly on compact sets.*

Proof. For $0 < \delta < \frac{1}{2}$, define f_δ, $g_\delta \in X$ by

$$f_\delta(z) = f((1-2\delta)z)(\delta z)^{-2} \sin^2 \delta z = g_\delta(z)/z^2.$$

Then $f_\delta \to f$ as $\delta \to 0$ uniformly on compact sets. Since $|f_\delta(x)| \leq 1$ $(x \in \mathbb{R})$, we have $f_\delta \in X_1$. Thus it is enough to prove the lemma for a function $f \in X_1$ given by $f(z) = g(z)/z^2$, where $g \in X$. Given such a function, define $f_n(z) = \sum_{k \in \mathbb{Z}} f(z + 2kn\pi)$. $(n \in \mathbb{N})$. Since g is bounded on lines $\operatorname{Im} z = \text{constant}$, the series converges. For $|\operatorname{Re} z| \leq n\pi$ we have

$$|f_n(z) - f(z)| \leq 2\|g\| \exp(|\operatorname{Im} z|) \sum_{k=1}^{\infty} (n\pi(2k-1))^{-2} = \alpha n^{-2} \exp(|\operatorname{Im} z|),$$

where $\alpha = \|g\|/4$. Since $f \in X_1$ and f_n has period $2n\pi$, we deduce that

$$|f_n(z)| \leq (1 + \alpha n^{-2}) \exp(|\operatorname{Im} z|)(z \in \mathbb{C}).$$

Therefore $f_n \in X$ and $f_n \to f$ uniformly on compact sets as $n \to \infty$. Since f_n has period $2n\pi$, by [**2**, Theorem 6. 10.1], $f_n \in P_n$. Replacing f_n by $(1 + \alpha n^{-2})^{-1} f_n$, we get the required series.

For the sequence of Lemma 2, $\lim_{n\to\infty} f_n^{(k)}(0) = f^{(k)}(0)$ $(k = 0, 1, 2, \ldots)$, and so $\lim_{n\to\infty} Tf_n = Tf$. Hence $\|T\| = \sup\{|Tf| : f \in P \cap X_1\}$. Let $p_n \in P_n \cap X_1$ be such that

$$Tp_n = \sup\{|Tf| : f \in P_n \cap X_1\}. \tag{4}$$

Since for $p_n^*(z)=\overline{p_n(\bar{z})}$, $Tp_n^*=\overline{Tp_n}=Tp_n$, by replacing p_n by $\frac{1}{2}(p_n+p_n^*)$ we can assume that p_n is real-valued on $\mathbb{R}$. Since X_1 is a normal family, there is a subsequence (p_{n_j}) such that $\lim_{j\to\infty} Tp_{n_j}=\|T\|$ and $\lim_{j\to\infty} p_{n_j}=e\in X_1$, with uniform convergence on compact sets. Therefore $\lim_{j\to\infty} Tp_{n_j}=Te=\|T\|$, and e is an extremal function for T.

If $e'(0)=e''(0)=\ldots=e^{(m)}(0)=0$, then $Te=Te_1$ where e_1 is a constant function, which satisfies (3) for $\mu=0$. Henceforth we assume that one of $e'(0),\ldots,e^{(m)}(0)$ is non-zero. By taking a further subsequence, we can assume that one of $p'_{n_j}(0),\ldots,p_{n_j}^{(m)}(0)$ is non-zero for each j.

LEMMA 3. *If p_n+iq_n (resp. $e+if$) $\in X_1$ for some $q_n\in P_n$ (resp. $f\in X$) with $q_n(\mathbb{R})\subset\mathbb{R}$ (resp. $f(\mathbb{R})\subset\mathbb{R}$), then $Tq_n=0$ (resp. $Tf=0$).*

Proof. By (4), $|Tp_n+iTq_n|=|T(p_n+iq_n)|\leq Tp_n$. Since Tp_n, $Tq_n\in\mathbb{R}$, we get $Tq_n=0$. The second part is similar.

LEMMA 4. *Let a, e be as in Theorem 1. Suppose that there exists $f\in X$, real on $\mathbb{R}$, such that $g=e+if\in X_1$ and $|g(0)|=1$. Then $v(a)=\|a\|$.*

Proof. By Lemma 3, $Tf=0$. Put $g_1=\overline{g(0)}g$, so that $g_1\in X_1$ and $g_1(0)=1$. By Lemma 1 of [3], $Tg_1=(ag_1)(0)\in V(a)$. Hence $v(a)\leq\|a\|=Te=Tg=|Tg_1|\leq v(a)$, where the inequalities follow from the definition of $v(a)$.

Proof of Theorem 1. If $f\in P_n$ is real on $\mathbb{R}$, so that for some ν, $f(z)=\sum_{k=-\nu}^{\nu} a_k\exp(ikz/n)$ with $a_\nu\neq 0$, we prove later that we may write

$$f(z)=\lambda\prod_{k=1}^{2\nu}\sin((z-z_k)/2n)\ (z\in\mathbb{C}),$$

where $\lambda\in\mathbb{R}$ and we may assume that $-n\pi<\operatorname{Re} z_k\leq n\pi$. For $p=p_{n_j}$ as in (4), we factorize p' and $1\pm p$: the same 'ν' appears for each function. Since p'^2 and $1-p^2$ are non-negative on $\mathbb{R}$ we get for some $\lambda>0$, $-n\pi<\operatorname{Re}\alpha_k$, $\operatorname{Re}\beta_k\leq n\pi$, where we write n for n_j,

$$p'^2(z)(1-p^2(z))^{-1}=\lambda^2\prod_{k=1}^{2\nu}\sin^2((z-\beta_k)/2n)\Big/\prod_{k=1}^{4\nu}\sin((z-\alpha_k)/2n). \tag{5}$$

If $\alpha_k\in\mathbb{R}$, then $p(\alpha_k)=\pm1$, which gives $p'(\alpha_k)=0$, and we find that p'^2 has a zero of order at least that of $(1-p^2)$ at α_k. Hence by cancellation we can assume in (5) that $\alpha_k\notin\mathbb{R}$, and by similar reasoning that $\alpha_k\neq\beta_j$, for all j, k. The α_k are in complex conjugate pairs. Hence for some μ, for all $z\in\mathbb{C}$,

$$p'^2(z)(1-p^2(z))^{-1}=\lambda^2\prod_{k=1}^{\mu}\sin^2((z-\beta_k)/2n)[\sin((z-\alpha_k)/2n)\sin((z-\bar{\alpha}_k)/2n)]^{-1}. \tag{6}$$

Write $q(z)=\lambda^{-1}p'(z)/\prod_{k=1}^{\mu}\sin((z-\beta_k)/2n)$. Thus

$$q(z)=(\text{constant})\prod_{k=\mu+1}^{2\nu}\sin((z-\beta_k)/2n),$$

from (5). By (6),

$$p^2(z)+q^2(z)\prod_{k=1}^{\mu}\sin((z-\alpha_k)/2n)\sin((z-\bar{\alpha}_k)/2n)=1 \quad (z\in\mathbb{C}). \tag{7}$$

The function $x\to\prod_{k=1}^{\mu}|\sin((x-\alpha_k)/2n)|^2$ is continuous, periodic and non-zero on $\mathbb{R}$, and hence is bounded below by some $\delta^2>0$. Hence by (7), $p^2(x)+\delta^2q^2(x)\leq 1$ $(x\in\mathbb{R})$, and so $P_n\cap X_1$ contains the function

$$x\to p(x)+i\delta q(x)\sin^j(x/2n)\cos^{\mu-j}(x/2n), \quad (j=0,1,2,\ldots,\mu).$$

Since one of $p'(0),\ldots,p^{(m)}(0)$ is non-zero, from the definition of q, (at least) one of $q(0),q'(0),\ldots,q^{(m-1)}(0)$ is non-zero.

Suppose if possible that $\mu\geq m$. Choose $j\leq m$ such that $q^{(m-j)}(0)\neq 0$ and $q^{(k)}(0)=0$ $(k<m-j)$. By Lemma 3 and (2) we have

$$0=T(q(z)\sin^j(z/2n)\cos^{\mu-j}(z/2n))=m!q^{(m-j)}(0)/((m-j)!(2n)^j):$$

the function to which T is applied has a leading term in z^m. Hence $\mu\leq m-1$.

By Hadamard's factorization theorem, any $f\in X$ can be written

$$f(z)=\beta\exp(\alpha z)z^{\lambda}\prod_{k=1}^{\infty}(1-a_k^{-1}z)\exp(z/a_k),$$

where a_k are the zeros. By (6) each function $p_{n_j}'^2$, $1-p_{n_j}^2$ has, counting multiplicity, at most $2(m-1)$ zeros in $-n_j\pi<\operatorname{Re}z\leq n_j\pi$ which the other does not have. Each zero of e'^2, $(1-e^2)$ is a limit of zeros of $p_{n_j}'^2$, $(1-p_{n_j}^2)$. Therefore the factorizations of e'^2 and $1+e$, $1-e$ give, where $\alpha,\beta\in\mathbb{R}$, $\beta>0$ (since e'^2 and $1-e^2$ are non-negative on $\mathbb{R}$) and $\mu_1,\mu_2\leq m-1$, $a_k,b_k\in\mathbb{C}$,

$$\varphi(z)=e'^2(z)(1-e^2(z))^{-1}=\beta\exp(\alpha z)\prod_{k=1}^{\mu_1}(z-b_k)^2\prod_{k=1}^{\mu_2}(z-a_k)^{-1}(z-\bar{a}_k)^{-1}, \tag{8}$$

where we are defining φ. By the same argument as for p, we can assume that $a_k\notin\mathbb{R}$ and $a_k,\bar{a}_k\neq b_j$ for all j,k.

Since e is not constant, there is a disc $\Delta=\{|z-\xi|<\eta\}$, $\xi\in\mathbb{R}$, on which $|e|<1$. On Δ, $\sigma(z)=\sin^{-1}(e(z))$ is analytic. For $z\in\Delta\cap\mathbb{R}$, $\sigma'(z)=e'(z)(1-e^2(z))^{-1/2}=\psi(z)$, where by (8),

$$\psi(z)=\pm\beta^{1/2}\exp(\tfrac{1}{2}\alpha z)\prod_{k=1}^{\mu_1}(z-b_k)\Big/\prod_{k=1}^{\mu_2}[(z-a_k)(z-\bar{a}_k)]^{1/2}.$$

Since $(z-a_k)(z-\bar{a}_k)>0$, we have the usual square root here. Hence on $\Delta\cap\mathbb{R}$, for some $\theta\in\mathbb{R}$ we have

$$e(z)=\sin\left(\theta+\int_{\xi}^{z}\psi(t)\,dt\right). \tag{9}$$

The right-hand side of (9) defines a function analytic in a neighbourhood of $\mathbb{R}$, which must therefore equal e.

Suppose if possible that $|\psi|>1$ for all $t>$ (or $<$) some $t_0\in\mathbb{R}$. Then by (9) for certain $x_n\in\mathbb{R}$ with $|x_n|\to\infty$ as $n\to\infty$, we have $e(x_n)=0$ and, by (8), $e'(x_n)^2=\varphi(x_n)=\psi^2(x_n)>1$. This contradicts Bernstein's inequality $|e'(x)|\leqslant 1$ $(x\in\mathbb{R})$. Therefore in (8), $\alpha=0$ and $\mu_2\geqslant\mu_1$. Hence by (8), $\varphi(x)\leqslant K$ $(x\in\mathbb{R})$ for some $K>0$, and $e^2(x)+K^{-1}e'^2(x)\leqslant 1$ $(x\in\mathbb{R})$.

Equation (9) holds for any z if we integrate along a curve not through any a_k or $\bar{a}_k$, and take the square root continuously in ψ. If $|\psi|\leqslant\delta<1$ for all large $|z|$, then (9) shows that $e'(z)/\exp(\delta|z|)$ is bounded on $\mathbb{C}$. Hence if $\gamma>0$, $j\in\mathbb{N}$ and $\delta+j\gamma\leqslant 1$, then $e'(z)\sin^j\gamma z\in X$, and $e(z)+iK^{-1/2}e'(z)\sin^j\gamma z\in X_1$. Choose γ, j such that $e^{(m-j+1)}(0)\neq 0$ and $e^{(k)}(0)=0$ $(1\leqslant k\leqslant m-j)$. Then by Lemma 3 and (2), $0=T(e'(z)\sin^j\gamma z)\neq 0$. Thus $|\psi|\leqslant\delta<1$ for all large $|z|$ is not possible, which gives $\mu_2=\mu_1$ and $\beta\geqslant 1$. Since $\beta>1$ would give $|\psi|>1$ for all large real x, we have $\beta=1$. Equation (8) now gives us (3).

Define $\sigma_k(z)=|a_k|-|a_k|^{-1}$ $(\operatorname{Re}a_k)z$ $(z\in\mathbb{C}, k=1,2,\ldots,\mu)$. For $t\in\mathbb{R}$, we have $\sigma_k^2(t)\leqslant(t-a_k)(t-\bar{a}_k)$, with equality when $t=0$. Then (3) gives

$$e^2(t)+e'^2(t)\prod_{k=1}^{\mu}\frac{\sigma_k^2(t)}{(t-b_k)^2}\leqslant 1\quad(t\in\mathbb{R}),\tag{10}$$

with equality when $t=0$.

Since in (3) a_j, $\bar{a}_j\neq b_k$, it follows that $e'(z)/\prod_{j=1}^{\mu}(z-b_j)$ is entire. Hence $f(z)=e'(z)\prod_{j=1}^{\mu}\sigma_j(z)(z-b_j)^{-1}\in X$, since $e'\in X$ and $\prod_{j=1}^{\mu}\sigma_j(z)(z-b_j)^{-1}$ tends to a limit as $|z|\to\infty$. By (10) and since $f(\mathbb{R})\subseteq\mathbb{R}$, $|(e+if)(x)|\leqslant 1$ $(x\in\mathbb{R})$, and so $e+if\in X_1$. The case $t=0$ in (10) gives $|(e+if)(0)|=1$. By Lemma 4, $v(a)=\|a\|$.

To derive the factorization of a function in P_n given, use the Hadamard factorization, grouping the zeros in subsets of period $2n\pi$. If the function is real on $\mathbb{R}$, there is no factor $\exp(\beta z)$ "left over". This completes the proof of Theorem 1.

Remarks. For each m there exist elements a such that in (3) we have $\mu=m-1$. For example, let $\xi_i\in\mathbb{C}$ be such that the element a of (1) has minimum norm. We can show that in fact $\xi_i\in\mathbb{R}$. There exists $\Phi\in A'$ such that $\|\Phi\|=1$, $\Phi(a)=\|a\|$, and $\Phi(h^j)=0$ $(0\leqslant j<m)$. The function $e(z)=\Phi(\exp(izh))$ is an extremal for a, with $e^{(j)}(0)=0$ $(0\leqslant j<m)$. If we replace e by $\frac{1}{2}(e+e^*)$, (3) becomes

$$e'^2(z)\prod_{k=1}^{m-1}(z-a_k)(z-\bar{a}_k)=z^{2m-2}(1-e^2(z)).$$

In the case $m=3$, this a is $-i(h^3-\xi_1 h)$, and (9) becomes, for a certain α,

$$e(z)=\sin\int_0^z\psi(t)\,dt,\quad\text{where}\quad\psi(t)=t^2/[(t-\alpha)(t+\alpha)(t-\bar{\alpha})(t+\bar{\alpha})]^{1/2}.$$

We require $\int_0^\alpha\psi=\pm\pi/2$, and calculation with elliptic functions gives $\xi_1\simeq 0{\cdot}73$, $\alpha\simeq 0{\cdot}97+i2.10$, $\|h^3-\xi_1 h\|\simeq 0{\cdot}37$.

We can prove (omitted) the following necessary and sufficient condition that a given function $e \in X_1$, real on $\mathbb{R}$, is the extremal for T as in (2): for a, T given, e is in fact unique, and not constant.

(i) e satisfies an equation of the form of (3), and

(ii) there exist sequences c_k, $d_k \in \mathbb{R}$ such that for any $f \in X$, the contour integral

$$\int_\Gamma T_w\left(\frac{e'(w)}{w-z}\right)\frac{f(z)}{e'(z)}\,dz, \tag{11}$$

where T_w denotes T with the differentiations carried out with respect to w, and contours $\Gamma \to \infty$, gives as the sum of the residues of the integrand (which equals 0) a fixed multiple of $Tf - \sum_k c_k f(d_k)$. Further, $c_k e(d_k) \geq 0$ for all k.

For $\mu = 1$ in (3), e is a translate of $\cos\sqrt{z^2+\theta^2}$, $0 \leq \theta$.

The functions found here are also the extremals for operators of the form $Tf = \sum_{j=1}^{n} \beta_j f(\alpha_j)$, where α_j, $\beta_j \in \mathbb{R}$, and we require the maximum over $f \in X_1$: this T has an extremal e with $\mu \leq m-2$ in (3). Boas [**2**, Theorem 11.4.1] gives this result for $Tf = f(\delta) - f(-\delta)$.

From (3), if $e'(z) = 0$ then, with at most $(m-1)$ exceptions, $e(z) = \pm 1$. Thus on $\mathbb{R}$, outside the interval spanned by the b_j's which happen to be real, e oscillates between ± 1.

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