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Meromorphic Continuation of Spherical Cuspidal Data Eisenstein Series

Dedicated to the memory of my father Ramazan Alayont.

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Abstract. Meromorphic continuation of the Eisenstein series induced from spherical, cuspidal data on parabolic subgroups is achieved via reworking Bernstein's adaptation of Selberg's third proof of meromorphic continuation.

Introduction

Let G be a reductive group defined over a number field k. The spectral decomposition of the right regular representation on $L^2(G_k \backslash G_{\mathbf{A}})$ is accomplished in [L76], which shows that the regular representation is a direct sum of irreducible cuspidal representations and the direct integrals of Eisenstein series induced from proper parabolic subgroups of G. Meromorphic continuation of Eisenstein series induced from cuspforms on proper parabolic subgroups of G was achieved during the process. In this paper we present a simpler and shorter proof of the meromorphic continuation of the Eisenstein series induced from spherical cuspidal data. The proof employs the continuation principle, based on Selberg's "third proof" of meromorphic continuation of $GL_2(\mathbf{Z})$ Eisenstein series (given in [H83]) extended by Bernstein to a general situation. This principle states that a sufficiently nice holomorphically parametrized system of equations with a unique solution in some non-empty open set has a unique, meromorphic solution almost everywhere. A crucial condition on the system is that, locally, the solutions lie in the image of a finite dimensional space under a weakly holomorphic map. We include a proof of the continuation principle following a rewriting [Ga01a] of Bernstein's proof.

The proof of meromorphic continuation of the Eisenstein series proceeds by induction on the rank of the parabolic subgroup. For a not-self-associate maximal parabolic subgroup, the continuation principle is applied to the following system:

- v_λ is an eigenfunction for certain integral operators;
- all constant terms of v_λ are 0 except for those for P and the conjugate of P;
- P-constant term of v_λ is $\lambda\delta_P^{1/2} f$;
- constant term of v_λ with respect to the conjugate of P is a multiple of $w\lambda\delta_Q^{1/2} f^w$.

For other parabolic subgroups, similar systems are used.

The local finiteness of the system is verified using the compact operator criterion. This criterion assures local finiteness of a system given by a homogeneous equation

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$T_\lambda(\nu) = 0$ where, for every λ, T_λ is invertible modulo a compact operator. In our proof we reduce our system to a system consisting of a single integral operator equation, and we use the fact that the integral operators are compact on the space of rapidly decaying functions.

The first step in proving uniqueness shows that the difference of two solutions in the region of convergence of the Eisenstein series is square-integrable, due to the conditions on the constant terms. Since this difference is also an eigenfunction for the integral operators, by choosing a self-adjoint integral operator we show that the difference must be 0 whenever the eigenvalue is non-real.

The notation and background information used in this paper are mostly from [A78] and [MW95]. The latter is a reworking of [L76] in an adelic setting. The continuation principle and an application to meromorphic continuation of Eisenstein series can also be found in Bernstein's notes.[1]The set-up in these notes is slightly different from ours. The special case of continuation of Eisenstein series induced from cuspidal data on maximal parabolic subgroups in GL_n is proven in [Ga01c], using again the continuation principle. Also [W90] proves meromorphic continuation of Eisenstein series induced from maximal parabolic subgroups in the Fredholm equations setting. The exposition in this paper differs from the usual proofs by being shorter and putting the emphasis on integral equations.

1 Notation and Terminology

Let G be a reductive group defined over a number field k. Fix a minimal parabolic subgroup P_0 of G along with a Levi subgroup M_0 of P_0, both defined over k. Let A_0 be the split component of M_0. From now on, all parabolic (respectively, Levi) subgroups considered will be standard, *i.e.*, will contain P_0 (respectively, M_0). For a parabolic subgroup P, let M_P denote its (standard) Levi component, N_P its unipotent radical and A_P the split component of the center of M_P. We shall omit the subscript P when the parabolic subgroup is understood. Note that $M_G = G$.

Let $X(M_P)$ denote the group of rational characters of M_P, and let

$$\mathfrak{a}_P^* = X(M_P) \otimes_{\mathbf{Z}} \mathbf{R} \qquad \text{and} \qquad \mathfrak{a}_P = \mathrm{Hom}_{\mathbf{Z}}(X(M_P), \mathbf{R}).$$

Then $\mathfrak{a}_P^*$ and $\mathfrak{a}_P$ are naturally dual to each other and $\mathfrak{a}_P^*$ is isomorphic to $X(A_P) \otimes_{\mathbf{Z}} \mathbf{R}$. Restricting a character on G to M_P gives an injection $\mathfrak{a}_G^* \hookrightarrow \mathfrak{a}_P^*$, and a surjection $\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_G$, under duality. Let $\mathfrak{a}_P^G$ be the kernel of this surjection. We obtain two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}_G^* & \longrightarrow & \mathfrak{a}_P^* & \longrightarrow & \mathfrak{a}_P^*/\mathfrak{a}_G^* & \longrightarrow & 0, \\ 0 & \longrightarrow & \mathfrak{a}_P^G & \longrightarrow & \mathfrak{a}_P & \longrightarrow & \mathfrak{a}_G & \longrightarrow & 0. \end{array}$$

The restriction map from A_P to A_G induces a canonical surjection and hence,

$$\mathfrak{a}_P = \mathfrak{a}_P^G \oplus \mathfrak{a}_G \qquad \text{and} \qquad \mathfrak{a}_P^* = (\mathfrak{a}_P^G)^* \oplus \mathfrak{a}_G^*,$$

where $(\mathfrak{a}_P^G)^*$ is the dual of $\mathfrak{a}_P^G$.

[1]Lecture notes for the graduate summer school, IAS/ Park City Summer School, 2002.

Let Δ_0 be the set of simple roots for P_0. The roots in Δ_0 are canonically embedded into $\mathfrak{a}_{P_0}^* = \mathfrak{a}_0^*$ and the parabolic subgroups P are in bijection with the subsets Δ_0^P of Δ_0 consisting of the roots which vanish on $\mathfrak{a}_P$. Let Δ_P be the restrictions to $\mathfrak{a}_P$ of elements in $\Delta_0 - \Delta_0^P$. Then Δ_P is a basis of $(\mathfrak{a}_P^G)^*$. For any root α, let $\alpha^\vee$ denote the corresponding co-root. The co-roots are elements of $\mathfrak{a}_0$ and form a basis of $\mathfrak{a}_0^G$. We can obtain a second basis of $(\mathfrak{a}_0^G)^*$ by taking the dual basis $\widehat{\Delta}_0 = \{\varpi_\alpha : \alpha \in \Delta_0\}$ of the basis $\Delta_0^\vee = \{\alpha^\vee : \alpha \in \Delta_0\}$. For a general parabolic P, $\widehat{\Delta}_P = \{\varpi_\alpha : \alpha \in \Delta_P\}$ defined similarly, is a second basis of $(\mathfrak{a}_P^G)^*$.

Let $\mathbf{A}$ be the adele ring of k and v denote an arbitrary place of k. At almost all finite places $K_v = G(\mathfrak{o}_v)$, where $\mathfrak{o}_v$ is the ring of integers of k_v, is a maximal compact subgroup of $G(k_v)$. At the remaining finite places we choose a "special" maximal compact subgroup K_v. At the infinite places we choose a maximal compact subgroup K_v so that $G(k_v) = P_0(k_v)K_v$, and for any standard parabolic subgroup $P = MN$,

$$P(k_v) \cap K_v = (M(k_v) \cap K_v)(N(k_v) \cap K_v).$$

Then $K = \prod K_v$ is a maximal compact subgroup of $G_{\mathbf{A}}$ and for any parabolic subgroup P, $G_{\mathbf{A}} = P_{\mathbf{A}}K$, called the Iwasawa decomposition. We shall refer to the maximal compact group described above as "the maximal compact subgroup of $G_{\mathbf{A}}$".

Given a parabolic subgroup P with Levi component M, define $H_M : M_{\mathbf{A}} \to \mathfrak{a}_P$ by

$$e^{\langle H_M(m),\chi\rangle} = |\chi(m)| = \prod_v |\chi(m_v)|_v$$

for all $\chi \in X(M)$ and $m = \prod_v m_v \in M_{\mathbf{A}}$. The kernel of H_M is denoted by $M_{\mathbf{A}}^1$. We then have the *Langlands decomposition*: any $g \in G_{\mathbf{A}}$ can be written as $g = nmak$ where $n \in N_{\mathbf{A}}$, $m \in M_{\mathbf{A}}^1$, $a \in A_P(\mathbf{R})^0$ and $k \in K$. Here, by abuse of notation, $A_P(\mathbf{R})^0$ denotes the subgroup of $A_{P,\mathbf{A}}$ which is identified with rank of G number of copies of $\mathbf{R}_+^\times$ using the diagonal embedding of $\mathbf{R}$ into $\mathbf{A}_\infty$. In the case of $k = \mathbf{Q}$, this subgroup is indeed $A_P(\mathbf{R})^0$, the identity component of $A_P(\mathbf{R})$. We shall use a_g to denote the a component in the Langlands decomposition for $P = P_0$. Using the Langlands decomposition, H_M defined above can be extended to $H_P \colon G_{\mathbf{A}} \to \mathfrak{a}_P$.

For a parabolic subgroup P, let c_P be the *constant term operator* which is defined by

$$c_P f(g) = \int_{N_{P,k}\backslash N_{P,\mathbf{A}}} f(ng)\, dn$$

for a left $N_{P,k}$-invariant, locally L^1 function f. A function f is *cuspidal* if $c_P f = 0$ almost everywhere, for any proper parabolic subgroup P.

Fix a truncation parameter $0 < T \in \mathbf{R}$. For each parabolic subgroup P, let $\widehat{\tau}_P$ denote the characteristic function of $\{H \in \mathfrak{a}_0 : \beta(H) > T \text{ for all } \beta \in \widehat{\Delta}_P\}$. Then the *truncation* $\wedge^T\varphi$ of a function φ defined on $G_k\backslash G_{\mathbf{A}}$ is

$$\wedge^T\varphi(g) = \sum_P (-1)^{\dim A_P/A_G} \sum_{\gamma\in P_k\backslash G_k} \widehat{\tau}_P(H_{P_0}(\gamma g)) c_P\varphi(\gamma g),$$

where P runs over all parabolic subgroups. For each P, the truncated sum over $P_k\backslash G_k$ is finite. Also the truncation of a continuous function is of rapid decay over Siegel sets [A80]. The Siegel sets considered here will be of the form

$$S_t = \{nmak \in G_{\mathbf{A}} : n \in \omega, m \in \omega_1, k \in K, a \in A_0(\mathbf{R})^0, \alpha(a) > t, \forall \alpha \in \Delta_0\}$$

for ω a compact subset of $N_{0,\mathbf{A}}$ and ω_1 of $M^1_{\mathbf{A}}$ and $t > 0$. We assume ω and ω_1 are large enough and t is small enough so that $G_{\mathbf{A}} = G_k \cdot S_t$.

Let f be an automorphic form on $M_k\backslash M_{\mathbf{A}}$ where M is the Levi component of P and $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$ (for a definition of an automorphic form, see [MW95, I.2.17]). For $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$, the continuous character sending $m \in M_{\mathbf{A}}$ to $e^{\langle H_M(m),\lambda\rangle}$ will be denoted by $\lambda(m)$ or m^λ. We define φ_λ on $G_{\mathbf{A}}$, attached to f, λ, by

$$\varphi_\lambda(g) = \varphi_\lambda(nmk) = \lambda(m) f(m) \delta_P^{1/2}(m),$$

where δ_P is the modulus function of P. Then the *Eisenstein series* induced from the parabolic P with data f is

$$E_\lambda(g) = \sum_{\gamma \in P_k\backslash G_k} \varphi_\lambda(\gamma g)$$

whenever the series converges. The Eisenstein series converges for all λ with real part in a positive open cone in $(\mathfrak{a}_P^G)^*$ depending on f, and the resulting $E_\lambda(g)$ is an automorphic form on $G_k\backslash G_{\mathbf{A}}$ [Go67]. In this paper we consider Eisenstein series induced from spherical cuspforms, which are cuspidal automorphic forms right-invariant under K.

Proposition 1.1 *Let $P = MN$ be a standard parabolic subgroup, f a spherical cuspform on $M_{\mathbf{A}}$ and $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$. The following properties hold:*

(i) *Let $R = M'N'$ be a standard parabolic subgroup. Then*

$$c_R E_\lambda = \sum_w E^{M' \cap w^{-1} P w}(M(w)\varphi_\lambda),$$

where the sum is over all $w \in W$ such that $w\alpha > 0$ for $\alpha \in \Delta_0^R$ and $M \subset wM'w^{-1}$. The operator $M(w)$ is the intertwining operator *defined as*

$$M(w) f(g) = \int_{w^{-1} P_k w \cap N'_k \backslash N'_{\mathbf{A}}} f(wng)\, dn.$$

(ii) *For any $\eta \in C_c^\infty(G_{\mathbf{A}})^{inv}$, the space of compactly supported smooth K-conjugation invariant functions on $G_{\mathbf{A}}$, E_λ is an eigenfunction for the integral operator attached to η. More precisely, for any η there exists μ_λ, holomorphic as a function of λ such that*

$$\eta E_\lambda = \int_{G_{\mathbf{A}}} \eta(g) g \cdot E_\lambda \, dg = \mu_\lambda E_\lambda.$$

2 Selberg–Bernstein Continuation Principle

In this section we give a simple proof of the continuation principle following the rewriting of Bernstein's proof in [Ga01a].

Let V and W_i, $i \in I$ be topological vector spaces, and S a complex manifold. An *S-parametrized system of linear equations* $X = \{X_s\}$ on V consists of equations

$$X_s : T_{i,s}(v) = w_{i,s} \text{ for all } i \in I,$$

where $T_{i,s}\colon V \to W_i$ are continuous linear maps. Such a system is *holomorphic in S* if the maps $s \mapsto T_{i,s}$ and $s \mapsto w_{i,s}$ are weakly holomorphic, meaning their compositions with the functionals on the target space are holomorphic $\mathbf{C}$-valued functions. We take the weak operator topology on $\operatorname{Hom}(V, W)$, so the functionals on $\operatorname{Hom}(V, W)$ are of the form $T \mapsto \lambda(T(v))$ where $\lambda \in W^*$ and $v \in V$. An S-parametrized system of equations is said to have a *locally finite envelope* if locally the solutions lie in the image of a finite dimensional space under a family of weakly holomorphic maps, *i.e.,* $\operatorname{Sol} X_s \subset f_s(V_0)$ with $f_s\colon V_0 \to V$ weakly holomorphic. If there is a system X' defined on V' and a family of weakly holomorphic maps $h_s\colon V' \to V$ such that $\operatorname{Sol} X_s \subset h_s(\operatorname{Sol} X'_s)$, then X' is said to *dominate* X. If there is a locally finite system dominating X, then X is also locally finite.

It can be shown, by adapting the proof of [R91, Theorem 3.31], that weak holomorphy of a function $f\colon S \to V$ implies holomorphy of f for a locally convex, quasi-complete space V. A *quasi-complete* space is a space where every bounded closed subset is complete. In particular, all Fréchet and LF-spaces are quasi-complete. Also $\operatorname{Hom}(V, W)$ with the weak operator topology is quasi-complete if V is an LF-space and W is a quasi-complete space.

Theorem 2.1 (Continuation Principle) *Let V be a locally convex, quasi-complete topological vector space, W_i, $i \in I$, locally convex topological vector spaces and S a connected complex manifold. Let X be an S-parametrized, holomorphic system of linear equations on V with a locally finite envelope. Suppose that on a non-empty open subset Ω of S, X_s has a unique solution v_s. Then v_s is meromorphic on Ω and has a meromorphic continuation to S which is the unique solution of X_s outside a proper analytic subset of S.*

Proof Since the statement is local, we can assume X has a finite envelope, *i.e.,* there exists V_0 with $\dim V_0 = n$ and a weakly holomorphic family of operators $f_s\colon V_0 \to V$ so that $\operatorname{Sol} X_s \subset f_s(V_0)$ for all $s \in S$.

Let $K_s^+ = f_s^{-1}(\operatorname{Sol} X_s)$. Then X_s has a unique solution if and only if $\dim \ker(f_s) = \dim K_s^+$. Subspaces $\ker(f_s)$ and K_s^+ can be described by systems of linear equations as follows. Let A and B_i be separating families of linear functionals on V and W_i, respectively, and fix a basis $\{e_1, \dots, e_n\}$ of V_0. A system of linear equations for $\ker(f_s)$ is of the form

$$\ker(f_s) = \{x_1 e_1 + \cdots + x_n e_n \in V_0 : \sum_j a_{\alpha j} x_j = 0 \text{ for all } \alpha \in A\},$$

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where (suppressing the dependence on s) $a_{\alpha j} = \alpha(f_s(e_j))$ depends on s holomorphically, by definition of the weak holomorphy of f_s. Similarly,

$$K_s^+ = \{x_1e_1 + \cdots + x_ne_n \in V_0 : \sum_j b_{\beta j}x_j = c_\beta, \text{ for all } \beta \in B = \bigcup_i B_i\},$$

where $c_\beta = \beta(w_{i,s})$ and $b_{\beta j} = \beta(T_{i,s}f_s)(e_j)$ depend on s holomorphically. This follows from weak holomorphy of $w_{i,s}$ and of $T_{i,s}f_s$. Let $Y = \{Y_s\}$ denote this system.

Define the holomorphic matrices M_s, N_s and Q_s by

$$M_s(\alpha, j) = a_{\alpha j} \quad \alpha \in A, j = 1, \dots, n,$$
$$N_s(\beta, j) = b_{\beta j} \quad \beta \in B, j = 1, \dots, n,$$
$$Q_s(\beta, j) = \begin{cases} b_{\beta j} & \text{if } 1 \le j \le n, \beta \in B, \\ c_\beta & \text{if } j = n+1, \beta \in B. \end{cases}$$

Then $\ker(f_s)$ is the nullspace of the matrix M_s, hence $\dim\ker(f_s) = n - \operatorname{rank} M_s$ and K_s^+ is the inverse image of $(c_\beta)_{\beta\in B}$ under N_s, so $\dim K_s^+ = n - \operatorname{rank} N_s$ if there is a solution. A solution exists if and only if $\operatorname{rank} N_s = \operatorname{rank} Q_s$. Therefore, the condition $\dim\ker(f_s) = \dim K_s^+$ can be rewritten as $\operatorname{rank} M_s = \operatorname{rank} N_s = \operatorname{rank} Q_s$. Each matrix assumes maximal rank in an open dense subset which is the complement of a proper analytic subset. Let S_0 be the intersection of these sets. Since $\Omega \cap S_0$ is not empty and the ranks are equal on Ω, the maximal ranks of the three matrices are equal to some number r. Hence, on S_0, X_s has a solution and the solution is unique.

To finish the proof, we show that the solution is meromorphic. It is enough to show that the system Y has a meromorphic solution, since the solution of the system X will then be a weakly holomorphic image of this meromorphic solution, and hence will be weakly meromorphic. Since V is quasi-complete, this solution is then meromorphic.

Choose an $r \times r$-minor D_{s_0} of N_{s_0} of full rank for some $s_0 \in S_0$. Suppose that its entries $b_{\beta j}$ are indexed by $\beta \in \{\beta_1, \dots, \beta_r\}$ and $j \in \{j_1, \dots, j_r\}$. Let D_s be the corresponding $r \times r$-minor of N_s and S_1 the set of $s \in S_0$ where D_s assumes full rank. Consider the system Z defined on the span of $e_{j_1}, \dots e_{j_r}$ consisting of the equations

$$\sum_{j\in\{j_1,\dots,j_r\}} b_{\beta j}x_j = c_\beta \text{ for all } \beta \in \{\beta_1, \dots, \beta_r\}.$$

By Cramer's rule, the system Z has a unique solution. Further this solution is meromorphic since the coefficients of the matrix are holomorphic. Extend this solution to V_0 by letting $x_j = 0$ for all $j \notin \{j_1, \dots, j_r\}$. The extension is a solution of Y because the r equations defining Z span the set of equations defining Y for all $s \in S_0$. ■

The compact operator criterion assuring the local finiteness of a homogeneous system is as follows.

Proposition 2.2 *Let V, W be Banach spaces. Suppose X is the system given by $T_s(v) = 0$ with $T_s : V \to W$ continuous linear, and the map $s \mapsto T_s$ holomorphic for the operator norm topology on* $\mathrm{Hom}(V, W)$. *Suppose that for some $s_0 \in S$, T_{s_0} has a left inverse modulo compact operators,* i.e., *there exists an $A\colon W \to V$ and a compact $K\colon V \to V$ such that $A \circ T_{s_0} = K + id_V$. Then X is of finite type in some neighborhood of s_0.*

Proof Since K is compact, $V_0 = \ker(K + \mathrm{id}_V)$ is of finite dimension and $V_1 = \mathrm{im}(K + \mathrm{id}_V)$ is closed and of finite co-dimension. Let pr_{V_0} and pr_{V_1} be projections from V to V_0 and V_1, respectively. Consider a new system X' on V defined by the equation $T'_s(v) = 0$, where

$$T'_s = pr_{V_1} \circ A \circ T_s\colon V \to V_1.$$

Since $\mathrm{Sol}\, X_s \subset \mathrm{Sol}\, X'_s$, X' dominates X, and hence it is enough to show that X' has a finite envelope in a neighborhood of s_0.

Consider the family of operators $\varphi_s = pr_{V_0} \oplus T'_s\colon V \to V_0 \oplus V_1$. The holomorphy of $s \mapsto T_s$ implies the holomorphy, and hence the continuity, of $s \mapsto \varphi_s$. By definition, φ_{s_0} is a bijection and hence, by the open mapping theorem, an isomorphism. The subset of invertible maps is open in the operator norm topology of $\mathrm{Hom}(V, V_0 \oplus V_1)$, which then implies that φ_s is an isomorphism in a neighborhood of s_0.

Observe that $\mathrm{Sol}\, X'_s = \varphi_s^{-1}(V_0 \oplus \{0\})$. The proof of the finite envelope property will be finished if we show φ_s^{-1} is weakly holomorphic. This follows because the inversion map (restricted to the invertible elements) from $\mathrm{Hom}(V, V_0 \oplus V_1)$ to $\mathrm{Hom}(V_0 \oplus V_1, V)$ is differentiable in the operator norm topology. ∎

3 Meromorphic Continuation of Eisenstein Series

Theorem 3.1 *Let $P = MN$ be a parabolic subgroup and f a cuspform on $M_k \backslash M_{\mathbf{A}}$ which is right-invariant under the maximal compact subgroup of $M_{\mathbf{A}}$. The Eisenstein series E_λ induced from data f on P has a meromorphic continuation to $(\mathfrak{a}_P^G)^*_{\mathbf{C}}$ with singularities only on a proper analytic subset.*

Proof The proof proceeds by induction on $\dim(A_P/A_G) = r$.

First assume P is a maximal proper parabolic subgroup which is not self-associate, *i.e.*, the only Weyl group element which conjugates M to itself is the identity element. Let Q be the unique parabolic subgroup such that M_Q is conjugated to M by a non-trivial Weyl group element and w the shortest such element: $wM_Qw^{-1} = M_P$.

We shall prove meromorphic continuation of the Eisenstein series to open sets $S \subset (\mathfrak{a}_P^G)^*_{\mathbf{C}}$ intersecting the region of convergence of the Eisenstein series with $\mathrm{Re}\, S \subset (\mathfrak{a}_P^G)^*$ bounded. Choose $\Lambda \in (\mathfrak{a}_P^G)^*$ in the fundamental Weyl chamber so that $\Lambda/2$ dominates all Weyl group translates of $\mathrm{Re}\, S$.

Without loss of generality, we can assume that f has a central character which is trivial on $A_G(\mathbf{A})$. For a Siegel set S_t, define the space $V_{\Lambda,t}$ by

$$V_{\Lambda,t} = \Big\{ h \in L^1_{loc}(G_{\mathbf{A}}) : h(\gamma g) = h(g)\ \forall \gamma \in G_k,\ h(zg) = h(g)\ \forall z \in A_{G,\mathbf{A}}, \\ h \text{ is right } K\text{-invariant and } \int_{A_G(\mathbf{R})^\circ \backslash S_t} |h(g)|^2 a_g^{-\Lambda}\, dg < \infty \Big\}$$

and a norm on $V_{\Lambda,t}$ by

$$\|h\|_t = \Big(\int_{A_G(\mathbf{R})^\circ \backslash S_t} |h(g)|^2 a_g^{-\Lambda}\, dg \Big)^{1/2}.$$

During the proof we fix ω and ω_1 in the definition of S_t so that S_t covers $G_k \backslash G_{\mathbf{A}}$ for t small enough.

Claim 3.2 *For t and u small enough, $V_{\Lambda,t} = V_{\Lambda,u}$ and the norms $\|\cdot\|_t$ and $\|\cdot\|_u$ are equivalent.*

Proof Without loss of generality, assume $t > u$, so that $S_t \subset S_u$. Then $V_{\Lambda,u} \subset V_{\Lambda,t}$ and for $h \in V_{\Lambda,u}$ we have $\|h\|_t \leq \|h\|_u$.

Suppose now $h \in V_{\Lambda,t}$. From reduction theory, there exist a compact subset C of S_t and finitely many $\gamma \in G_k$ such that $S_u - S_t$ can be covered by the union of γC. For each such $\gamma \in G_k$ there exists $c \in \mathbf{R}$ such that for all $g \in C$, we have $a_{\gamma g}^{-\Lambda} \leq c a_g^{-\Lambda}$, and so

$$\int_{A_G(\mathbf{R})^\circ \backslash \gamma C} |h(g)|^2 a_g^{-\Lambda}\, dg \leq c \int_{A_G(\mathbf{R})^\circ \backslash C} |h(g)|^2 a_g^{-\Lambda}\, dg = c\|h\|_t^2.$$

Hence $\|h\|_u$ is bounded by a multiple of $\|h\|_t$. This proves the reverse inclusion $V_{\Lambda,t} \subset V_{\Lambda,u}$, and the equivalence of the norms. ∎

Let V be equal to $V_{\Lambda,t}$ with t small as above and define a system X_λ on V by

$$\begin{aligned}
(\eta - \mu_\lambda)v_\lambda &= 0 \quad \text{for all } \eta \in C_c^\infty(G_{\mathbf{A}})^{\mathrm{inv}},\\
c_R(v_\lambda) &= 0 \quad \text{for any maximal } R \text{ not equal to } P \text{ or } Q,\\
c_P(v_\lambda) &= \varphi_\lambda,\\
(z - w\lambda\delta_Q^{1/2}(z))c_Q(v_\lambda) &= 0 \quad \text{where } z \in A_{Q,\mathbf{A}},\\
c_R(v_\lambda) &= 0 \quad \text{for any } R \text{ properly contained in } Q,\\
(\eta' - \nu_\lambda)(w\lambda\delta_Q^{1/2})^{-1}c_Q(v_\lambda) &= 0 \quad \text{for all } \eta' \in C_c^\infty(M_{Q,\mathbf{A}})^{\mathrm{inv}},
\end{aligned}$$

where μ_λ is the eigenvalue of η acting on φ_λ, and ν_λ is that of η' acting on the cusp-form $f^w = (w\lambda\delta_Q^{1/2})^{-1}M(w)\varphi_\lambda$. Here $w\lambda$ is the character $w\lambda(x) = \lambda(wxw^{-1})$. The last three equations say that $c_Q(v_\lambda)$ lies in a finite dimensional space consisting of the cuspforms of the same type as f^w.

The image spaces for the maps in the system are chosen as follows. Integral operators η map V to itself and the constant term operators c_R map to

$$V_R = \Big\{ h \in L^1_{loc}(M_{R,\mathbf{A}}) : h \text{ is left } M_{R,k}, \text{ right } K \cap M_{R,\mathbf{A}} \text{ and } A_{G,\mathbf{A}}\text{-invariant and} \\ \int_{A_G(\mathbf{R})^\circ \backslash (S_t \cap M_{R,\mathbf{A}})} |h(m)|^2 a_m^{-\Lambda-2\rho}\, dm < \infty \text{ for all } t > 0 \Big\},$$

where the factor $a_m^{-2\rho}$ is equal to δ_R^{-1}.

Being a Hilbert space, the space V is quasi-complete and locally convex. The spaces V_R are locally convex because their topologies are given by seminorms.

Claim 3.3 *Integral operator η is continuous from V to itself.*

Proof Let t be small enough that $V = V_{\Lambda,t}$. It is enough to show that η is continuous from $V_{\Lambda,u}$ to $V_{\Lambda,t}$ with u such that the Siegel set S_u covers $S_t \cdot \operatorname{supp} \eta$ and $V = V_{\Lambda,u}$.

$$\|\eta f\|_t^2 = \int_{A_G(\mathbf{R})^\circ \backslash S_t} |\eta f(x)|^2 a_x^{-\Lambda}\, dx \leq \int_{A_G(\mathbf{R})^\circ \backslash S_t} \int_G |\eta(g) f(xg)|^2 a_x^{-\Lambda}\, dg dx.$$

Interchanging the order of integration along with a change of variable $x \mapsto xg$ gives

$$\|\eta f\|_t^2 \leq \int_G |\eta(g)|^2 \int_{A_G(\mathbf{R})^\circ \backslash S_t g} |f(x)|^2 a_{xg^{-1}}^{-\Lambda}\, dx dg.$$

There exists a constant c such that $a_{xg^{-1}}^{-\Lambda} \leq c a_x^{-\Lambda}$ for all $g \in \operatorname{supp} \eta$. Hence,

$$\|\eta f\|_t^2 \leq c \int_G |\eta(g)|^2 \int_{A_G(\mathbf{R})^\circ \backslash S_t g} |f(x)|^2 a_x^{-\Lambda}\, dx dg.$$

Since for all $g \in \operatorname{supp} \eta$, $S_t g \subset S_u$, the inner integral is less than $\|f\|_u^2$, consequently

$$\|\eta f\|_t^2 \leq c \operatorname{vol}(\operatorname{supp} \eta) |\sup \eta(g)|^2 \|f\|_u^2.$$

We have shown before that the norms $\|\cdot\|_t$ and $\|\cdot\|_u$ are comparable if both t and u are small enough, hence the continuity of η from $V_{\Lambda,u} = V_\Lambda$ to $V_{\Lambda,t} = V_\Lambda$ follows. ■

Continuity of the constant term operators follows from a Fubini argument. Let R be a parabolic subgroup. Constant term maps are continuous if they are bounded

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with respect to each seminorm on V_R and with respect to the norm $\|\cdot\|_t$ on V. This condition holds because for any $u > 0$,

$$\begin{aligned}\int_{A_G(\mathbf{R})^\circ\backslash(S_u\cap M_{R,\mathbf{A}})} &|c_R h(m)|^2 a_m^{-\Lambda-2\rho}\, dm \\ &\leq \int_{A_G(\mathbf{R})^\circ\backslash(S_u\cap M_{R,\mathbf{A}})} \int_{N_{R,k}\backslash N_{R,\mathbf{A}}} |h(nm)|^2 a_m^{-\Lambda-2\rho}\, dndm \\ &\leq c\cdot \int_{A_G(\mathbf{R})^\circ\backslash S_u} |h(g)|^2 a_g^{-\Lambda}\, dg = c\,\|h\|_u^2,\end{aligned}$$

and the norms with respect to u and t on V are comparable for u, t small enough.

The equations in X involve only λ, φ_λ and μ_λ, all of which are holomorphic, so the system X is a holomorphic S-parametrized system.

Claim 3.4 *X has a locally finite envelope.*

Proof To prove the claim, we use the compact operator criterion. The compact operator will be one of the integral operators restricted to the space of rapidly decreasing functions.

From the properties of E_λ, only two of the constant terms are non-zero: $c_P E_\lambda = \varphi_\lambda$ and $c_Q E_\lambda = M(w)\varphi_\lambda$. Hence the truncation is

$$\wedge^T E_\lambda = E_\lambda - E^P(\widehat{\tau}_P\varphi_\lambda) - E^Q(\widehat{\tau}_Q M(w)\varphi_\lambda).$$

Let V_1 be the space of cuspforms of M_Q of the same type as f^w and let $V' = \mathbf{C}\oplus V_1 \oplus V_0$ where $V_0 = \wedge^T V$. Define T_λ on V' by

$$T_\lambda(a, f_1, h) = aE^P(\widehat{\tau}_P\varphi_\lambda) + E^Q(\widehat{\tau}_Q w\lambda\delta_Q^{1/2} f_1) + h,$$

and let X' be the system on V' given by the homogeneous equation

$$T'_\lambda(v_\lambda) = (\eta - \mu_\lambda)T_\lambda(v_\lambda) = 0.$$

The system X' is holomorphic since the sums involved in the T_λ are locally finite. Note also that X' dominates X. To see this, given $v_\lambda \in \operatorname{Sol} X_\lambda$, let $h = \wedge^T v_\lambda$, $f_1 = (w\lambda)^{-1}\delta_Q^{-1/2} c_Q(v_\lambda)$. Then by definition of X, $f_1 \in V_1$, $T_\lambda(1, f_1, h) = v_\lambda$ and $(1, f_1, h) \in \operatorname{Sol} X'_\lambda$.

Therefore it is enough to show finiteness of X'. The map $\lambda \mapsto T_\lambda$ is holomorphic with respect to the operator norm topology because of the sums being locally finite sums of holomorphic functions. If we show that every T'_λ has an inverse modulo a compact operator, locally finiteness follows from the compact operator criterion. Define $A\colon V \to V'$ by $A(v) = (0, 0, -\frac{1}{\mu_\lambda}\wedge^T v)$. On the subspace $V_0 \subset V'$, $A\circ T'_\lambda$ is given by

$$A\circ T'_\lambda(0, 0, h) = (0, 0, -\tfrac{1}{\mu_\lambda}\wedge^T (\eta - \mu_\lambda)h),$$

so it differs from the identity by $-1/\mu_\lambda \wedge^T \eta$. As V_0 lies inside the space of rapidly decreasing functions and the operator η is compact on the space of rapidly decreasing functions, the operator $-1/\mu_\lambda \wedge^T \eta$ is compact on V_0. The complement of V_0 in V' is finite-dimensional, hence $A \circ T'_\lambda$ differs from the identity by a compact operator on the whole V'. ∎

Claim 3.5 *X has a unique solution in a non-empty open subset of the region of convergence of the Eisenstein series.*

Proof The Eisenstein series in the region of convergence is a solution of the system X. Suppose that v' is another solution. Then the difference $v = v' - E_\lambda$ is an eigenfunction for the integral operators, and all constant terms of v are 0 except $c_Q(v)$. Hence $v - c_Q(v)$ is a rapidly decreasing function ([Ga01b] provides an elementary proof for a special case; a more general statement is I.2.12, [MW95]), and in particular is bounded. The term $c_Q(v)$ is of the form

$$c_Q(v)(g) = c_Q(v)(nmk) = w\lambda(m)\delta_Q^{1/2}(m)h(m),$$

where $g = nmk$ is the Iwasawa decomposition with respect to Q and h is a cuspform on $M_{Q,\mathbf{A}}$. In the direction of any $\alpha \in \Delta_0^Q$, the cuspform is of rapid decay and the characters have moderate growth, so the product is of rapid decay. In the direction of α_Q, the only root in Δ_Q, $w\lambda$ has a sufficiently large negative order in the region of convergence, $\delta_Q^{1/2}$ has a small positive order, and h is bounded. Combining all, $c_Q(v)$ is bounded in that direction. Therefore v is bounded on the Siegel set and hence is square-integrable on $A_G(\mathbf{R})^\circ G_k\backslash G_{\mathbf{A}}$.

Pick an integral operator $\eta \in C_c^\infty(G_{\mathbf{A}})^{\mathrm{inv}}$ such that the eigenvalue μ_λ is non-constant, and consider the open subset of the region of convergence of the Eisenstein series where μ_λ is non-real. Assume without loss of generality that the integral operator η on $L^2(A_G(\mathbf{R})^\circ G_k\backslash G_{\mathbf{A}})$ is self-adjoint. We then have

$$\mu_\lambda \cdot \langle v_\lambda, v_\lambda\rangle = \langle \eta v_\lambda, v_\lambda\rangle = \langle v_\lambda, \eta v_\lambda\rangle = \overline{\mu_\lambda} \cdot \langle v_\lambda, v_\lambda\rangle.$$

Since μ_λ is not real, this implies $v_\lambda = 0$, proving uniqueness for λ in this open set. ∎

Now we can apply the continuation principle to conclude the meromorphic continuation of the Eisenstein series to $(\mathfrak{a}_P^G)^*_{\mathbf{C}}$ with singularities only on a proper analytic set. This finishes the proof in the first case.

Consider the second case: P a self-associate maximal parabolic subgroup. The above proof applies to this case with minor changes. We note the differences. Let w be the shortest non-trivial Weyl group element which conjugates M to itself. The only non-zero constant term of E_λ is $c_P E_\lambda = \varphi_\lambda + M(w)\varphi_\lambda$ and the truncation of E_λ is $\wedge^T E_\lambda = E_\lambda - E^P(\widehat{\tau}_P(\varphi_\lambda + M(w)\varphi_\lambda))$. The system of equations X_λ consists of the equations

$$\begin{aligned}
(\eta-\mu_\lambda)\nu_\lambda &= 0 && \text{for all } \eta\in C_c^\infty(G_{\mathbf{A}})^{\mathrm{inv}},\\
c_R(\nu_\lambda) &= 0 && \text{for any maximal } R \text{ not equal to } P,\\
(z-w\lambda\delta_P^{1/2}(z))(c_P(\nu_\lambda)-\varphi_\lambda) &= 0 && \text{where } z\in A_{P,\mathbf{A}},\\
c_R(c_P(\nu_\lambda)-\varphi_\lambda) &= 0 && \text{for any } R \text{ properly contained in } P,\\
(\eta'-\nu_\lambda)(w\lambda\delta_P^{1/2})^{-1}(c_P(\nu_\lambda)-\varphi_\lambda) &= 0 && \text{for all } \eta'\in C_c^\infty(M_{P,\mathbf{A}})^{\mathrm{inv}},
\end{aligned}$$

where μ_λ and ν_λ are as before. Let V' be defined as in the first case, and define T_λ on V' by $T_\lambda(a,f_1,h)=E^P(\widehat{\tau}_P(a\varphi_\lambda+w\lambda\delta_P^{1/2}f_1))+h$. If X' is the system consisting of the single homogeneous equation $(\eta-\mu_\lambda)T_\lambda(\nu_\lambda)=0$, then X' dominates X and the compact operator criterion, applied in the same way as in the first case, shows that X', and hence X, has a locally finite envelope.

The proof of uniqueness proceeds as before. The only non-zero constant term of the difference of two solutions of the system is the P-constant term, which is of the form $w\lambda(m)\delta_P^{1/2}(m)h(m)$, and it is bounded in the region of convergence. Hence the proof of continuation in the second case follows.

The continuation of the Eisenstein series induced from all maximal parabolic subgroups implies in particular that the constant terms of these Eisenstein series have continuations. These constant terms are either $M(w)\varphi_\lambda$ or $\varphi_\lambda+M(w)\varphi_\lambda$, hence each $M(w)\varphi_\lambda$ has a continuation when w is a reflection s_α corresponding to a simple root α. For a general Weyl group element w, there exists a sequence of simple roots $\alpha_1,\dots,\alpha_j$ such that $w=s_{\alpha_j}\cdots s_{\alpha_1}$ and $M(w)\varphi_\lambda=M(w_{\alpha_j})\cdots M(w_{\alpha_1})\varphi_\lambda$, where each intertwining operator $M(w_{\alpha_i})$ comes from a maximal parabolic subgroup case (IV.4.1, [MW95]). Therefore the continuation of $M(w)\varphi_\lambda$ for an arbitrary w is obtained using the maximal parabolic case iteratively.

Assume now that the continuation is achieved for all Eisenstein series made from spherical cuspidal data on parabolic subgroups of rank at most $r-1$, and we shall prove meromorphic continuation of the Eisenstein series induced from a parabolic subgroup P of rank r.

Let the system of equations X be given by

$$\begin{aligned}
(\eta-\mu_\lambda)\nu_\lambda &= 0 && \text{for all } \eta\in C_c^\infty(G_{\mathbf{A}})^{\mathrm{inv}},\\
c_Q(\nu_\lambda) &= \sum_w E^{M_Q\cap w^{-1}Pw}(M(w)\varphi_\lambda) && \text{for any maximal } Q.
\end{aligned}$$

All the Eisenstein series appearing in the constant term equations are made from spherical cuspidal data $M(w)\varphi_\lambda$ induced from smaller rank parabolic subgroups. We showed above that each $M(w)\varphi_\lambda$ has a continuation, and by induction on the rank, the Eisenstein series made from $M(w)\varphi$ has a continuation. Therefore the system X is holomorphically parametrized.

Let $V_0=\wedge^T V$ and define T_λ on $V'=\mathbf{C}\oplus V_0$ by

$$T_\lambda(a,h)=h-a\sum_{Q\neq G}(-1)^{\dim A_Q/A_G}E^Q\Big(\sum_w\widehat{\tau}_QE^{M_Q\cap w^{-1}Pw}(M(w)\varphi_\lambda)\Big).$$

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Consider the system X' consisting of the equation $T'_\lambda(\nu_\lambda) = (\eta - \mu_\lambda)T_\lambda(\nu_\lambda) = 0$ on V'. A similar argument as in the maximal proper parabolic subgroup case proves the finiteness of X' and hence that of X.

The Eisenstein series in the region of convergence is a solution of this system. Moreover, since the constant terms are specified explicitly in the system, the difference between any solution and the Eisenstein series is of rapid decay. Therefore, proceeding as before, one shows uniqueness in an open subset in the region of convergence. Hence the Eisenstein series meromorphically continues to $(\mathfrak{a}_P^G)^*_{\mathbf{C}}$, using once again the continuation principle.

We have meromorphically continued the Eisenstein series as a function in a weighted L^2-space. However, we would like the continuation to be an automorphic form as well, *i.e.*, we want representatives E_λ in the L^2-equivalence class of the continuation such that E_λ is smooth, left G_k-invariant, right K-finite, $\mathfrak{z}$-finite and of moderate growth. The smoothness follows from the fact that the solution is an eigenfunction for a smoothing operator, any of the integral operators. In particular, for any solution ν_λ of the system X_λ, the smooth representative can be chosen as $\frac{1}{\mu_\lambda}\eta \cdot \nu_\lambda$ for an integral operator η. The left G_k-invariance is proved by realizing that for any $\gamma \in G_k$, $\nu_\lambda(\gamma\,\cdot\,)$ is also a meromorphic continuation of the Eisenstein series, hence it must be equal to ν_λ everywhere. To prove K-finiteness, we observe that the Eisenstein series is right K-invariant, so for any $k \in K$, we consider $\nu_\lambda(\,\cdot\, k)$ and realize again that this is a meromorphic continuation of the Eisenstein series. The $\mathfrak{z}$-finiteness also follows similarly, since the Eisenstein series are eigenfunctions for the actions of the elements of $\mathfrak{z}$. The moderate growth property is checked using L^2-ness along with the smoothness property. ■

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