

RESEARCH ARTICLE

# On the moments of characteristic polynomials

Bhargavi Jonnadula<sup>1</sup>, Jonathan P. Keating<sup>2</sup> and Francesco Mezzadri<sup>1,\*</sup>

<sup>1</sup>School of Mathematics, University of Bristol, Fry Building, Bristol BS8 1UG, UK

<sup>2</sup>Mathematical Institute, University of Oxford, Andrew Wiles Building, Oxford OX2 6GG, UK

\*Corresponding author. E-mail: [f.mezzadri@bristol.ac.uk](mailto:f.mezzadri@bristol.ac.uk)

**Received:** 26 June 2021; **Revised:** 2 February 2022; **Accepted:** 19 June 2022; **First published online:** 5 August 2022

**Keyword:** random matrix theory, orthogonal polynomials, integrable systems

**2020 Mathematics Subject Classification:** *Primary* - 15B52, 60B20; *Secondary* - 33C50

## Abstract

We calculate the moments of the characteristic polynomials of  $N \times N$  matrices drawn from the Hermitian ensembles of Random Matrix Theory, at a position  $t$  in the bulk of the spectrum, as a series expansion in powers of  $t$ . We focus in particular on the Gaussian Unitary Ensemble. We employ a novel approach to calculate the coefficients in this series expansion of the moments, appropriately scaled. These coefficients are polynomials in  $N$ . They therefore grow as  $N \rightarrow \infty$ , meaning that in this limit the radius of convergence of the series expansion tends to zero. This is related to oscillations as  $t$  varies that are increasingly rapid as  $N$  grows. We show that the  $N \rightarrow \infty$  asymptotics of the moments can be derived from this expansion when  $t = 0$ . When  $t \neq 0$  we observe a surprising cancellation when the expansion coefficients for  $N$  and  $N + 1$  are formally averaged: this procedure removes all of the  $N$ -dependent terms leading to values that coincide with those expected on the basis of previously established asymptotic formulae for the moments. We obtain as well formulae for the expectation values of products of the secular coefficients.

## 1. Introduction

The characteristic polynomials of random matrices have received considerable attention over the past 20 years. One of the principal motivations stems from their connections to the statistical properties of the Riemann zeta-function and other families of  $L$ -functions [37, 36, 33, 15, 13, 14, 35, 25, 27]. In this context, the value distributions of the characteristic polynomials of random unitary, orthogonal and symplectic matrices have been calculated using a variety of approaches. For example, the moments have been computed in all three cases and the results used to develop conjectures for the moments of the Riemann zeta-function  $\zeta(s)$  on its critical line and for the moments of families of  $L$ -functions at the centre of the critical strip. Specifically, if  $A$  is an  $N \times N$  unitary matrix, drawn at random uniformly with respect to Haar measure on the unitary group  $U(N)$ , then for  $\text{Re}\beta > -1/2$

$$\mathbb{E}_{A \in U(N)} [|\det(I - Ae^{-i\theta})|^{2\beta}] = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2\beta)}{\Gamma(j+\beta)^2} \quad (1.1)$$

from which one can deduce that as  $N \rightarrow \infty$

$$\mathbb{E}_{A \in U(N)} [|\det(I - Ae^{-i\theta})|^{2\beta}] \sim \frac{G(1+\beta)^2}{G(1+2\beta)} N^{\beta^2}, \quad (1.2)$$

where  $G(s)$  is the Barnes  $G$ -function, and for  $k \in \mathbb{N}$

$$\mathbb{E}_{A \in U(N)} [|\det(I - Ae^{-i\theta})|^{2k}] \sim \left( \prod_{m=0}^{k-1} \frac{m!}{(m+k)!} \right) N^{k^2}. \quad (1.3)$$

These formulae lead to the conjectures [37] that for  $\text{Re}\beta > -1/2$ , as  $T \rightarrow \infty$

$$\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\beta} dt \sim a(\beta) \frac{G(1 + \beta)^2}{G(1 + 2\beta)} (\log T)^{\beta^2} \tag{1.4}$$

and for  $k \in \mathbb{N}$ , as  $T \rightarrow \infty$

$$\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt \sim a(k) \prod_{m=0}^{k-1} \frac{m!}{(m+k)!} (\log T)^{k^2}, \tag{1.5}$$

where

$$a(s) = \prod_p \left[ \left(1 - \frac{1}{p^s}\right)^{s^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+s)}{m! \Gamma(s)}\right)^2 p^{-m} \right] \tag{1.6}$$

with the product running over primes  $p$ .

Our focus here will primarily be on the Gaussian Unitary Ensemble (GUE) of random complex Hermitian matrices. For an  $N \times N$  matrix  $M$  drawn from the GUE, the joint eigenvalue density function is

$$\frac{1}{\mathcal{Z}_N^{(H)}} \prod_{1 \leq j < k \leq N} |x_j - x_k|^2 \prod_{j=1}^N e^{-\frac{N x_j^2}{2}}, \tag{1.7}$$

where  $\mathcal{Z}_N^{(H)}$  is a normalisation constant. When  $t$  is fixed, Brezin and Hikami [9] calculated the  $N \rightarrow \infty$  asymptotics of the moments of the associated characteristic polynomials to be

$$\mathbb{E}_N^{(H)} [\det(t - M)^{2k}] \sim e^{-Nk} e^{Nk \frac{t^2}{2}} (2\pi N \rho_{sc}(t))^{k^2} \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}, \quad k \in \mathbb{N}, \tag{1.8}$$

where the asymptotic eigenvalue density is given by the Wigner semi-circle law

$$\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}. \tag{1.9}$$

This corresponds precisely to (1.3), where the mean density is constant.

Our purpose is to investigate the series expansion in powers of  $t$  for the moments of the characteristic polynomials of GUE matrices, and of matrices drawn from other unitarily invariant ensembles, when appropriately scaled. We obtain a general formula for the coefficients, which are polynomials in  $N$ . They depend strongly on whether  $N$  is odd or even and diverge when  $N$  grows, meaning that the series expansion has a radius of convergence which shrinks. This is related to increasingly rapid oscillations in  $t$  when  $N$  grows. When  $t = 0$ , as  $N \rightarrow \infty$  we recover the asymptotic formula predicted by (1.8). When  $t \neq 0$ , the series expansion cannot be used straightforwardly to compute the asymptotics of the moments, because of the non-uniform convergence. However, we observe a surprising cancellation when we average formally over consecutive values of  $N$ : this procedure leads to an exact cancellation in the parts of the expansion coefficients that diverge when  $N \rightarrow \infty$ , leaving precisely the values consistent with (1.8). Drawing attention to this observation is our main purpose.

Brezin and Hikami [9] used orthogonal polynomial techniques to arrive at (1.8). Other studies to-date relating to the asymptotics of the moments of characteristic polynomials have relied mainly on the orthogonal polynomial method and saddle point techniques [9, 10, 3], the Riemann–Hilbert method [44], Hankel determinants with Fisher–Hartwig symbols [38, 18, 32] and supersymmetric representations [2, 30, 23, 45]. In the present study, we take a different approach: we express the moments in terms of certain multivariate orthogonal polynomials and take a combinatorial approach to compute the asymptotics of the moments using the properties of these polynomials. By doing so, we discover that even and odd dimensional GUE matrices give different contributions in the large  $N$  limit, and that only a formal average gives formulae consistent with (1.8). In Section 3.2.1, this phenomenon is discussed in detail for the second moment of the characteristic polynomial.

In addition to connections with number theory, characteristic polynomials have found numerous applications in quantum chaos [2], mesoscopic systems [22], quantum chromodynamics [16], and in a variety of combinatorial problems [43, 17]. The asymptotic study of negative moments and ratios of characteristic polynomials is another active area of research, see for example [6, 26, 31, 3, 44, 19, 7, 8, 24, 1]. More recently, the statistics of the maximum of the characteristic polynomial are being extensively studied, motivated by the relations to logarithmically correlated Gaussian processes. For example, see [25, 27, 28, 29] and references therein. We expect that the techniques developed here will have applications to those calculations as well.

This paper is structured as follows. In Section 2, we review the background results we shall need and discuss exact formulae for the moments of characteristic polynomials. In Section 3, we investigate these formulae in the GUE case asymptotically, when the matrix size tends to infinity. In the last section, Section 4, as an application of our results, we compute the correlations of secular coefficients, which are the coefficients of a characteristic polynomial when expanded as a function of the spectral variable.

## 2. Background

A partition  $\mu$  is a sequence of integers  $(\mu_1, \dots, \mu_l)$  such that  $\mu_1 \geq \dots \geq \mu_l > 0$ . Here  $l$  is the length of the partition and we denote  $|\mu| = \mu_1 + \dots + \mu_l$  to be the weight of the partition. We do not distinguish partitions that only differ by a sequence of zeros at the end. A partition can be represented with a *Young diagram* which is a left adjusted table of  $|\mu|$  boxes and  $l(\mu)$  rows such that the first row contains  $\mu_1$  boxes, the second row contains  $\mu_2$  boxes and so on. The conjugate partition  $\mu'$  is defined by transposing the Young diagram of  $\mu$  along the main diagonal.

For a partition  $\mu$ , let  $\Phi_\mu$  be the multivariate symmetric polynomial, with leading coefficient equal to 1, that obey the orthogonality relation

$$\int \Phi_\mu(x_1, \dots, x_n) \Phi_\nu(x_1, \dots, x_n) \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^n w(x_j) dx_j = \delta_{\mu\nu} C_\mu(n) \tag{2.1}$$

for a weight function  $w(x)$ . Here, the lengths of the partitions  $\mu$  and  $\nu$  are less than or equal to the number of variables  $n$ , and  $C_\mu$  is a constant which depends on  $n$ . Polynomial  $\Phi_\mu$  can be expressed as a ratio of determinants, as given in [42],

$$\Phi_\mu(\mathbf{x}) = \frac{1}{\Delta(\mathbf{x})} \det(\varphi_{\mu_j+n-j}(x_k))_{1 \leq j, k \leq n}, \tag{2.2}$$

where  $\varphi_j(x)$  is a monic polynomial of degree  $j$  orthogonal with respect to  $w(x)$ . We focus in particular to the case when  $w(x)$  in (2.1) is one of the following weights

$$w(x) = \begin{cases} e^{-\frac{Nx^2}{2}}, & x \in \mathbb{R}, & \text{Gaussian,} \\ x^\gamma e^{-2Nx}, & x \in \mathbb{R}_+, & \gamma > -1, & \text{Laguerre,} \\ x^{\gamma_1} (1-x)^{\gamma_2}, & x \in [0, 1], & \gamma_1, \gamma_2 > -1, & \text{Jacobi.} \end{cases} \tag{2.3}$$

Note that the Gaussian and Laguerre weights are rescaled with a parameter  $N$  which we later take to infinity.

When  $\varphi_n(x)$  in (2.2) is chosen to be one of the monic Hermite, Laguerre and Jacobi polynomials of degree  $n$ , we get their multivariable analogues denoted by  $\mathcal{H}_\mu, \mathcal{L}_\mu^{(\gamma)}$  and  $\mathcal{J}_\mu^{(\gamma_1, \gamma_2)}$ . These multivariate generalisations are the eigenfunctions of differential equations called Calogero–Sutherland Hamiltonians. Several properties such as recursive relations and integration formulas extend to the multivariate case [4, 5].

The Schur polynomials  $S_\lambda$ , indexed by a partition  $\lambda$ , are defined as

$$S_\lambda(x_1, \dots, x_n) = \frac{\det [x_k^{\lambda_j+n-j}]_{1 \leq j, k \leq n}}{\det [x_k^{n-j}]_{1 \leq j, k \leq n}}, \tag{2.4}$$

for  $l(\lambda) \leq n$ , and  $S_\lambda = 0$  for  $l(\lambda) > n$ . The polynomials  $\mathcal{H}_\mu$ ,  $\mathcal{L}_\mu^{(\gamma)}$  and  $\mathcal{J}_\mu^{(\gamma_1, \gamma_2)}$  form a basis for symmetric polynomials of degree  $|\mu|$ . Therefore, Schur polynomials can be expanded as [34]

$$S_\lambda(x_1, \dots, x_n) = \sum_{\nu \subseteq \lambda} \Psi_{\lambda\nu} \Phi_\nu(x_1, \dots, x_n), \tag{2.5}$$

where  $\Phi_\nu(\mathbf{x})$  can be either  $\mathcal{H}_\nu$ ,  $\mathcal{L}_\nu^{(\gamma)}$ , or  $\mathcal{J}_\nu^{(\gamma_1, \gamma_2)}$ . In the following the superscripts  $(H)$ ,  $(L)$  and  $(J)$  indicate Hermite, Laguerre and Jacobi, respectively. The coefficients in (2.5) are

$$\Psi_{\lambda\nu}^{(H)} = \left(\frac{1}{2N}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{C_\lambda(n)}{C_\nu(n)} D_{\lambda\nu}^{(H)}, \tag{2.6}$$

$$\Psi_{\lambda\nu}^{(L)} = \frac{1}{(2N)^{|\lambda| - |\nu|}} \frac{G_\lambda(n, \gamma) G_\lambda(n, 0)}{G_\nu(n, \gamma) G_\nu(n, 0)} D_{\lambda\nu}^{(L)}, \tag{2.7}$$

$$\Psi_{\lambda\nu}^{(J)} = \frac{G_\lambda(n, \gamma_1) G_\lambda(n, 0)}{G_\nu(n, \gamma_1) G_\nu(n, 0)} \left( \prod_{j=1}^n \Gamma(2\nu_j + 2n - 2j + \gamma_1 + \gamma_2 + 2) \right) \mathcal{D}_{\lambda\nu}^{(J)}, \tag{2.8}$$

where

$$C_\lambda(N) = \prod_{j=1}^N \frac{(\lambda_j + N - j)!}{(N - j)!},$$

$$G_\lambda(N, \gamma) = \prod_{j=1}^N \Gamma(\lambda_j + N - j + \gamma + 1), \tag{2.9}$$

and

$$D_{\lambda\nu}^{(H)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \equiv 0 \pmod{2}} \frac{1}{\left(\frac{\lambda_j - \nu_k - j + k}{2}\right)!} \right]_{1 \leq j, k \leq l(\lambda)}, \tag{2.10}$$

$$D_{\lambda\nu}^{(L)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{1}{(\lambda_j - \nu_k - j + k)!} \right]_{1 \leq j, k \leq l(\lambda)}, \tag{2.11}$$

$$\mathcal{D}_{\lambda\nu}^{(J)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{1}{(\lambda_j - \nu_k - j + k)! \Gamma(2n + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 2)} \right]_{1 \leq j, k \leq n}. \tag{2.12}$$

Similarly, when  $\Phi_\lambda$  is one of the  $\mathcal{H}_\lambda$ ,  $\mathcal{L}_\lambda^{(\gamma)}$ ,  $\mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}$ , the Schur expansion is

$$\Phi_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} \Upsilon_{\lambda\mu} \mathcal{S}_\mu(x_1, \dots, x_n), \tag{2.13}$$

with

$$\Upsilon_{\lambda\mu}^{(H)} = \left(\frac{-1}{2N}\right)^{\frac{|\lambda| - |\mu|}{2}} \frac{C_\lambda(n)}{C_\mu(n)} D_{\lambda\mu}^{(H)}, \tag{2.14}$$

$$\Upsilon_{\lambda\mu}^{(L)} = \left(\frac{-1}{2N}\right)^{|\lambda| - |\mu|} \frac{G_\lambda(n, \gamma) G_\lambda(n, 0)}{G_\mu(n, \gamma) G_\mu(n, 0)} D_{\lambda\mu}^{(L)}, \tag{2.15}$$

$$\Upsilon_{\lambda\mu}^{(J)} = (-1)^{|\lambda| + |\mu|} \left( \prod_{j=1}^n \frac{1}{\Gamma(2\lambda_j + 2n - 2j + \gamma_1 + \gamma_2 + 1)} \right) \frac{G_\lambda(n, \gamma_1) G_\lambda(n, 0)}{G_\mu(n, \gamma_1) G_\mu(n, 0)} \tilde{\mathcal{D}}_{\lambda\mu}^{(J)}, \tag{2.16}$$

and

$$\tilde{\mathcal{D}}_{\lambda, \nu}^{(j)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{\Gamma(2n + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 1)}{(\lambda_j - \nu_k - j + k)!} \right]_{1 \leq j, k \leq n}. \tag{2.17}$$

The coefficients  $\Psi_{\lambda, \mu}$  and  $\Upsilon_{\lambda, \mu}$  are nothing but the determinants  $\det(a_{\lambda_j + n - j, \mu_k + n - k})$  and  $\det(b_{\lambda_j + n - j, \mu_k + n - k})$  where  $a_{j, k}$  and  $b_{j, k}$  are the coefficients that appear when the monomial is expanded in the univariate polynomial basis and vice-versa, respectively. Coefficients  $\Psi_{\lambda, \mu}$  and  $\Phi_{\lambda, \mu}$  differ slightly from [34] since we used monic polynomials in the definition (2.2). In this paper, the above results play an important role in studying the correlations of characteristic polynomials and secular coefficients.

### 2.1 Moments of characteristic polynomials

Analogous to the Dual Cauchy identity satisfied by the Schur polynomials, which plays a crucial role in computing the correlations of characteristic polynomials of the unitary group, the multivariate polynomials satisfy the following identity.

**Lemma 2.1** (Generalised dual Cauchy identity). *Let  $k, N \in \mathbb{N}$ . For  $\lambda \subseteq (N^k) \equiv (\underbrace{N, \dots, N}_k)$ , let*

$\tilde{\lambda} = (k - \lambda'_N, \dots, k - \lambda'_1)$ . *Then*

$$\prod_{i=1}^k \prod_{j=1}^N (t_i - x_j) = \sum_{\lambda \subseteq (N^k)} (-1)^{|\tilde{\lambda}|} \Phi_{\lambda}(t_1, \dots, t_k) \Phi_{\tilde{\lambda}}(x_1, \dots, x_N). \tag{2.18}$$

Here  $\lambda = (\lambda_1, \dots, \lambda_k)$  is a sub-partition of  $(N^k)$  indicated by  $\lambda \subseteq (N^k)$  (each  $\lambda_j \leq N$  for  $j = 1, \dots, k$ ) and  $\lambda'$  is the conjugate partition of  $\lambda$ . As the polynomials  $\Phi_{\lambda}$  are orthogonal with respect to the joint eigenvalue density of Hermitian ensembles,

$$p(x_1, \dots, x_N) \propto \Delta(\mathbf{x})^2 \prod_{j=1}^N w(x_j), \tag{2.19}$$

(2.18) gives a compact way to calculate the correlation functions and moments of characteristic polynomials of unitary invariant Hermitian ensembles.

**Proposition 2.1** *Let  $M$  be an  $N \times N$  GUE, LUE or JUE matrix and  $t_1, \dots, t_k \in \mathbb{C}$ . Then, using the generalised dual Cauchy identity [34],*

$$(a) \quad \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^k \det(t_j - M) \right] = \mathcal{H}_{(N^k)}(t_1, \dots, t_k) \tag{2.20}$$

$$(b) \quad \mathbb{E}_N^{(L)} \left[ \prod_{j=1}^k \det(t_j - M) \right] = \mathcal{L}_{(N^k)}^{(N)}(t_1, \dots, t_k) \tag{2.21}$$

$$(c) \quad \mathbb{E}_N^{(J)} \left[ \prod_{j=1}^k \det(t_j - M) \right] = \mathcal{J}_{(N^k)}^{(\gamma_1, \gamma_2)}(t_1, \dots, t_k). \tag{2.22}$$

The moments can be readily computed from the above formulae by taking the limit  $t_j \rightarrow t$  for  $j = 1, \dots, k$ . This leads to a determinantal formula for the moments involving the derivatives of orthogonal polynomials. Alternatively, using the expansions equations (2.14), (2.15), (2.16) and the relations

$$\begin{aligned} S_{\lambda}(t, \dots, t) &= t^{|\lambda|} S_{\lambda}(1, \dots, 1), \\ S_{\lambda}(\underbrace{1, \dots, 1}_k) &= \frac{1}{|\lambda|!} C_{\lambda}(k) \dim V_{\lambda}, \end{aligned} \tag{2.23}$$

where the dimension of the irreducible representation of the symmetric group is

$$\dim V_\lambda = |\lambda|! \frac{\prod_{1 \leq i < j \leq l(\lambda)} \lambda_i - \lambda_j - i + j}{\prod_{j=1}^{l(\lambda)} (\lambda_j + l(\lambda) - j)!}, \tag{2.24}$$

one can compute the moments of the characteristic polynomials.

**Proposition 2.2.** *Let  $\lambda = (N^k)$ . The moments of characteristic polynomial are given by [34]*

$$\mathbb{E}_N^{(H)} [\det(t - M)^k] = C_\lambda(k) \sum_{\nu \subseteq \lambda} \left(\frac{-1}{2N}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(H)} t^{|\nu|} \tag{2.25}$$

$$\mathbb{E}_N^{(L)} [\det(t - M)^k] = \left(\frac{-1}{2N}\right)^{Nk} \frac{G_\lambda(k, \gamma) G_\lambda(k, 0)}{G_0(k, 0)} \sum_{\nu \subseteq \lambda} \frac{(-2N)^{|\nu|}}{G_\nu(k, \gamma)} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(L)} t^{|\nu|} \tag{2.26}$$

$$\begin{aligned} \mathbb{E}_N^{(J)} [\det(t - M)^k] &= \left(\prod_{j=N}^{N+k-1} \frac{1}{\Gamma(2j + \gamma_1 + \gamma_2 + 1)}\right) (-1)^{Nk} \frac{G_\lambda(k, \gamma_1) G_\lambda(k, 0)}{G_0(k, 0)} \\ &\times \sum_{\nu \subseteq \lambda} \frac{(-1)^{|\nu|}}{|\nu|! G_\nu(k, \gamma_1)} \dim V_\nu \tilde{D}_{\lambda\nu}^{(J)} t^{|\nu|}. \end{aligned} \tag{2.27}$$

These results give an expansion in powers of the spectral parameter  $t$  for a fixed  $N$ . Equations (2.25), (2.26), (2.27) can also be interpreted as a formal power series in the variable  $t$  for small  $t$ . In the next section, we perform explicit calculations to investigate the  $N$ -dependence of the coefficients and the convergence properties of the sum.

### 3. Asymptotics

In this section, we consider the asymptotics of the moments of characteristic polynomials, appropriately scaled, for the GUE. By exploiting the integral representation of the classical Hermite polynomials, Brezin and Hikami [9] showed that the moments of characteristic polynomials satisfy

$$\mathbb{E}_N^{(H)} [\det(t - M)^{2k}] \sim e^{-Nk} e^{Nk \frac{t^2}{2}} (2\pi N \rho_{sc}(t))^{k^2} \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}, \tag{3.1}$$

as  $N \rightarrow \infty$  with  $t$  fixed, where the asymptotic eigenvalue density is

$$\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}. \tag{3.2}$$

Using (2.25), we show in Section 3.1 that

$$\mathbb{E}_N^{(H)} [\det M^{2k}] \sim e^{-Nk} (2N)^{k^2} \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}, \tag{3.3}$$

which coincides with (3.1) for  $t = 0$ . For  $t \neq 0$ , the radius of convergence of the series expansion of the moments, scaled to compare with (3.1) shrinks when  $N \rightarrow \infty$  and the expansion cannot be used straightforwardly to compute the asymptotics. Moreover, the formula is different for even and odd dimensional GUE matrices. However, we observe that when one averages over consecutive even and odd dimensions, the divergent  $N$ -dependence of the coefficients cancels, leaving terms that do coincide with the expansion of  $\rho_{sc}(x)$ . These cases are discussed in Sections 3.1 and 3.2 in more detail.

3.1 Centre of the semi-circle

Let  $\lambda = (N^{2k})$ . For any finite  $N$ , we have

$$\mathbb{E}_N^{(H)} [\det M^{2k}] = \left(-\frac{1}{2N}\right)^{Nk} C_\lambda(2k) D_{\lambda_0}^{(H)}. \tag{3.4}$$

Proposition 3.1.

$$D_{\lambda_0}^{(H)} = \prod_{j=0}^{k-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m, \quad m \in \mathbb{N},$$

$$D_{\lambda_0}^{(H)} = (-1)^k \frac{m!}{(m+k)!} \prod_{j=0}^{k-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m + 1, \quad m \in \mathbb{N}. \tag{3.5}$$

Proof. The determinant  $D_{\lambda_0}^{(H)}$  can be evaluated as follows. Let  $N = 2m$ , then

$$D_{\lambda_0}^{(H)} = \det \left[ \mathbb{1}_{i-j \equiv 0 \pmod 2} \left( \left( m + \frac{i-j}{2} \right)! \right)^{-1} \right]_{1 \leq i, j \leq k}$$

$$= \prod_{j=0}^{k-1} \frac{1}{(m+j)!^2} \begin{vmatrix} 1 & 0 & m & 0 & \dots & \frac{m!}{(m-k+1)!} & 0 \\ 0 & 1 & 0 & m & \dots & 0 & \frac{m!}{(m-k+1)!} \\ 1 & 0 & m+1 & 0 & \dots & \frac{(m+1)!}{(m-k+2)!} & 0 \\ 0 & 1 & 0 & m+1 & \dots & 0 & \frac{(m+1)!}{(m-k+2)!} \\ & & & & \vdots & & \\ 1 & 0 & m+k-1 & 0 & \dots & \frac{(m+k-1)!}{m!} & 0 \\ 0 & 1 & 0 & m+k-1 & \dots & 0 & \frac{(m+k-1)!}{m!} \end{vmatrix}. \tag{3.6}$$

Perform the row operations  $R_{2j} = R_{2j} - R_{2j-2}, R_{2j-1} = R_{2j-1} - R_{2j-3}$  with  $j$  running from  $k, k-1, \dots, 2$  in that order. Using the Pascal's rule for binomial coefficients, we get

$$D_{\lambda_0}^{(H)} = (k-1)!^2 \prod_{j=0}^{k-1} \frac{1}{(m+j)!^2} \begin{vmatrix} 1 & 0 & m & 0 & \dots & \frac{m!}{(m-k+1)!} & 0 \\ 0 & 1 & 0 & m & \dots & 0 & \frac{m!}{(m-k+1)!} \\ 0 & 0 & 1 & 0 & \dots & \frac{m!}{(m-k+2)!} & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & \frac{m!}{(m-k+2)!} \\ & & & & \vdots & & \\ 0 & 0 & 1 & 0 & \dots & \frac{(m+k-2)!}{m!} & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & \frac{(m+k-2)!}{m!} \end{vmatrix}. \tag{3.7}$$

Next perform  $R_{2j} = R_{2j} - R_{2j-2}$ ,  $R_{2j-1} = R_{2j-1} - R_{2j-3}$  with  $j$  running from  $k, k-1, \dots, 3$  in that order. Repeat this process  $k-2$  more times to reach an upper triangular matrix with determinant given in (3.5). Similarly,  $D_{\lambda^0}^{(H)}$  can be calculated for  $N$  odd.  $\square$

Define

$$D_e(N) = \prod_{j=0}^{k-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m,$$

$$D_o(N) = (-1)^k \frac{m!}{(m+k)!} \prod_{j=0}^{k-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m + 1. \tag{3.8}$$

Using this notation, (3.4) reads

$$\mathbb{E}_N^{(H)} [\det M^{2k}] = \left(-\frac{1}{2N}\right)^{Nk} \times \begin{cases} C_\lambda(2k)D_e(N), & N \text{ even,} \\ C_\lambda(2k)D_o(N), & N \text{ odd.} \end{cases} \tag{3.9}$$

The functions  $C_\lambda(2k)D_e(N)$  and  $C_\lambda(2k)D_o(N)$ ,  $\lambda = (N^{2k})$ , can be expressed in terms of the ratios of factorials,

$$C_{(N^{2k})}(2k)D_e(N) = \prod_{j=0}^{k-1} \frac{(2m+j)!(2m+k+j)!}{(m+j)!^2} \frac{j!}{(k+j)!}, \quad N = 2m,$$

$$C_{(N^{2k})}(2k)D_o(N) = (-1)^k \frac{m!}{(m+k)!} \prod_{j=0}^{k-1} \frac{(2m+1+j)!(2m+1+k+j)!}{(m+j)!^2} \frac{j!}{(k+j)!}, \quad N = 2m + 1. \tag{3.10}$$

Denote

$$\gamma_k = \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}. \tag{3.11}$$

The universal constant  $\gamma_k$  is present in the moments for any finite  $N$ . To compute the large  $N$  limit, we require the asymptotic expansion of (3.10). In Appendix A, we compute the first few terms in this expansion. As  $N \rightarrow \infty$ ,

$$C_{(N^{2k})}(2k)D_e(N) = e^{-Nk} (2N)^{Nk+k^2} \gamma_k \left[ 1 + \frac{k}{6N} (4k^2 + 1) + O(N^{-2}) \right], \quad N \text{ even,}$$

$$C_{(N^{2k})}(2k)D_o(N) = (-1)^k e^{-Nk} (2N)^{Nk+k^2} \gamma_k \left[ 1 + \frac{k}{3N} (2k^2 - 1) + O(N^{-2}) \right], \quad N \text{ odd.} \tag{3.12}$$

Plugging (3.12) in (3.9), the leading order behaviour of the moments for  $N$  even and  $N$  odd is

$$e^{-Nk} (2N)^{k^2} \gamma_k \tag{3.13}$$

which coincides with (3.1) for  $t = 0$ . On the other hand, the sub-leading behaviour depends on the parity of  $N$ .

### 3.2 Away from the centre of the semi-circle

Recall that

$$\mathbb{E}_N^{(H)} [\det (t - M)^{2k}] = C_\lambda(2k) \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{\dim V_\nu}{|\nu|!} D_{\lambda \nu}^{(H)} t^{|\nu|}. \tag{3.14}$$

To compute the asymptotics near the centre of the semi-circle,  $t \neq 0$ , we need to evaluate  $D_{\lambda \nu}^{(H)}$  for a non-empty partition  $\nu$ . In Table. 1, we give the values of  $D_{\lambda \nu}^{(H)}$  when  $\nu$  is a partition of 2 and 4.

**Table 1.** The values of determinant  $D_{\lambda\nu}^{(H)}$  for  $\lambda = (N^{2k})$ . Determinants  $D_e$  and  $D_o$  are given in (3.8).

$D_{\lambda\nu}^{(H)}$	$N = 2m$	$N = 2m + 1$
$D_{\lambda 0}^{(H)}$	$D_e$	$D_o$
$D_{\lambda(2)}^{(H)}$	$mkD_e$	$mkD_o$
$D_{\lambda(1^2)}^{(H)}$	$-mkD_e$	$-(m + 1)kD_o$
$D_{\lambda(4)}^{(H)}$	$\frac{1}{2}m(m - 1)k(k + 1)D_e$	$\frac{1}{2}m(m - 1)k(k + 1)D_o$
$D_{\lambda(3,1)}^{(H)}$	$-\frac{1}{2}m(m - 1)k(k + 1)D_e$	$-\frac{1}{2}m(m + 1)k(k + 1)D_o$
$D_{\lambda(2^2)}^{(H)}$	$m^2k^2D_e$	$m(m + 1)k^2D_o$
$D_{\lambda(2,1^2)}^{(H)}$	$-\frac{1}{2}m(m + 1)k(k - 1)D_e$	$-\frac{1}{2}m(m + 1)k(k - 1)D_o$
$D_{\lambda(1^4)}^{(H)}$	$\frac{1}{2}m(m + 1)k(k - 1)D_e$	$\frac{1}{2}(m + 2)(m + 1)k(k - 1)D_o$

Therefore,

$$\mathbb{E}_N^{(H)} [\det(t - M)^{2k}] = \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{\dim V_\nu}{|\nu|!} t^{|\nu|} \text{poly}_{\frac{|\nu|}{2}}(N, k) \times \begin{cases} C_\lambda(2k)D_e, & N \text{ even,} \\ C_\lambda(2k)D_o, & N \text{ odd,} \end{cases} \tag{3.15}$$

where  $\text{poly}_j(N, k)$  denotes a polynomial of degree  $j$  in the variables  $N, k$ , and the explicit expressions are given in Table 1 for  $j \leq 4$ . By referring to (3.10), it is interesting to see that the universal constant  $\gamma_k$  is a factor of the moments for any finite  $N$ . The first few terms in the moments of characteristic polynomials are

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= \left(-\frac{1}{2N}\right)^{Nk} C_\lambda(2k)D_e \\ &\times \left[ 1 + \left(\frac{2^2N^2}{4!}\right) Nk t^4 + \left(\frac{2^3N^3}{6!}\right) 2Nk(2k - N)t^6 \right. \\ &\quad \left. + \left(\frac{2^4N^4}{8!}\right) Nk(4N^2 - 17Nk + 16k^2 + 2)t^8 \right. \\ &\quad \left. + O(t^{10}) \right], \tag{3.16} \end{aligned} \quad N \text{ even,}$$

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= \left(-\frac{1}{2N}\right)^{Nk} C_\lambda(2k)D_o \left[ 1 + \left(\frac{2N}{2!}\right) kt^2 + \left(\frac{2^2N^2}{4!}\right) (k^2 - Nk) t^4 \right. \\ &\quad \left. + \left(\frac{2^3N^3}{6!}\right) k(2N^2 - 3Nk + k^2)t^6 \right. \\ &\quad \left. + \left(\frac{2^4N^4}{8!}\right) k(-4N^3 + 15N^2k - 6Nk^2 - 2N + k^3 - 4k)t^8 \right. \\ &\quad \left. + O(t^{10}) \right], \tag{3.16} \end{aligned} \quad N \text{ odd.}$$

Up to a factor of  $(-1)^k$ , both  $C_\lambda D_e$  and  $C_\lambda D_o$  have the same leading term,

$$e^{-Nk} (2N)^{Nk+k^2} \gamma_p, \tag{3.17}$$

but they differ at sub-leading order as shown in (3.12). In Appendix A, we give the expansion of  $C_\lambda(N)D_e$  and  $C_\lambda(N)D_o$  up to  $O(N^{-6})$ . Note that the fact that the coefficients are polynomial functions of  $N$  means

that the radius of convergence of the expansion shrinks as  $N \rightarrow \infty$ . This means that for  $t \neq 0$  we cannot use this expansion straightforwardly to determine the large- $N$  asymptotics.

3.2.1 Second moment

The correlations of characteristic polynomials are connected to the correlation functions of random matrices [40, 20, 41]. In particular,

$$R_1^{(N)}(t) = \frac{N!}{(N-1)!} \frac{\mathcal{Z}_{N-1}^{(H)}}{\mathcal{Z}_N^{(H)}} \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_{N-1}^{(H)} [\det(t - M)^2], \tag{3.18}$$

where  $R_1^{(N)}$  is the one-point density of eigenvalues of matrix size  $N$ . As the second moment of the characteristic polynomial is related to the density of states, it is natural to expect the semi-circle law in the limit  $N \rightarrow \infty$  as given in (3.1) for a fixed  $t$  inside the support of the spectrum. Re-writing (3.1) for  $k = 1$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{2N} e^{N\left(1-\frac{t^2}{2}\right)} \mathbb{E}_N^{(H)} [\det(t - M)^2] = \pi \rho_{sc}(t), \tag{3.19}$$

which as an expansion in  $t$  reads

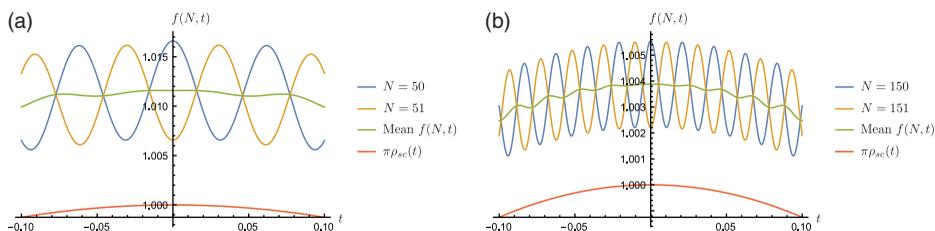
$$\lim_{N \rightarrow \infty} \frac{1}{2N} e^{N\left(1-\frac{t^2}{2}\right)} \mathbb{E}_N^{(H)} [\det(t - M)^2] = 1 - \frac{1}{8}t^2 - \frac{1}{128}t^4 - \frac{1}{1024}t^6 + O(t^8). \tag{3.20}$$

We now make an observation that we consider surprising: for  $k = 1$ , starting with (3.14) there is a formal procedure that leads to (3.20). Inserting the asymptotics of  $C_\lambda D_e$  and  $C_\lambda D_o$  in (3.16), one obtains

$$\begin{aligned} & e^{-\frac{Nt^2}{2}} \mathbb{E}_N^{(H)} [\det(t - M)^2] \\ &= 2Ne^{-N} \left[ 1 + \left(-\frac{1}{2}N - \frac{5}{12} + O(N^{-1})\right) t^2 + \left(\frac{1}{6}N^3 + \frac{19}{72}N^2 + \frac{17}{216}N - \frac{811}{77,760} + O(N^{-1})\right) t^4 \right. \\ &\quad + \left(-\frac{1}{45}N^5 - \frac{31}{540}N^4 - \frac{323}{6480}N^3 - \frac{3667}{291,600}N^2 + \frac{799}{1,749,600}N - \frac{640,879}{587,865,600} + O(N^{-1})\right) t^6 \\ &\quad \left. + O(N^7)O(t^8) \right], \quad N \text{ even,} \\ & e^{-\frac{Nt^2}{2}} \mathbb{E}_N^{(H)} [\det(t - M)^2] \\ &= 2Ne^{-N} \left[ 1 + \left(\frac{1}{2}N + \frac{1}{6} + O(N^{-1})\right) t^2 + \left(-\frac{1}{6}N^3 - \frac{19}{72}N^2 - \frac{17}{216}N - \frac{101}{19,440} + O(N^{-1})\right) t^4 \right. \\ &\quad + \left(\frac{1}{45}N^5 + \frac{31}{540}N^4 + \frac{323}{6480}N^3 + \frac{3667}{291,600}N^2 - \frac{799}{1,749,600}N - \frac{15,853}{18,370,800} + O(N^{-1})\right) t^6 \\ &\quad \left. + O(N^7)O(t^8) \right], \quad N \text{ odd.} \end{aligned} \tag{3.21}$$

Treating the above expansions as a formal series in  $N$  and taking their average gives (3.20). In Appendix B, it is shown that the average over even and odd  $N$  coincides with the semi-circle law up to  $O(t^{10})$ . Also, a general expression for the coefficient of  $t^{2j}$  in (3.14) is given for  $k = 1$ .

**Remark 1.** For a fixed  $j$ , notice that the leading order terms in the coefficients of  $t^{2j}$ , which depend on  $N$ , differ only by a sign for  $N$  even and odd. Therefore, after a formal average of both the series in (3.21), the coefficient of  $t^{2j}$  is equal to  $c_{2j} + O(N^{-1})$ . The constant  $c_{2j}$  turns out to be the appropriate coefficient of  $t^{2j}$  in the semi-circle law.



**Figure 1.** We plot in (a)  $f(N, t)$ , defined in (3.22), as a function of  $t$  when  $t$  is close to the origin for  $N = 50, 51$ ; (b) denotes the same for  $N = 150, 151$ .

**Remark 2.** The behaviour exhibited by (3.21) can be understood by considering Figure 1. For a fixed  $N$ , the plot shows the behaviour of

$$f(N, t) = \frac{1}{2N} e^{N\left(1 - \frac{t^2}{2}\right)} \mathbb{E}_N^{(H)} [\det(t - M)^2] \tag{3.22}$$

as a function of  $t$ . The function  $f(N, t)$  is oscillatory. The magnitude of the oscillations decreases as  $N$  increases, but their length-scale decreases. Moreover, the peaks and troughs of  $f(N, t)$  for  $N$  even coincides with the troughs and peaks of  $N$  odd, respectively. For a fixed  $j$ , this explains the divergent polynomial part of  $N$  in the coefficients of  $t^j$  and the sign change of leading order terms in the  $N$  even and odd cases. Formally averaging the two series of  $f(N, t)$  when  $N$  is even and odd cancels out these oscillations and leaves a result that converges to the semi-circle law.

**Remark 3.** As discussed, the size of oscillations in  $f(N, t)$  decreases as  $N$  increases. Therefore,  $f(N, t)$  converges to the semi-circle law as  $N \rightarrow \infty$  independent of whether  $N$  is even or odd. But, at finite but large  $N$ , the average of  $f(N, t)$  between two consecutive integers is a significantly smoother approximation to the semi-circle law.

### 3.2.2 Higher moments

For higher moments, the correlations of characteristic polynomials are related to the correlation functions of eigenvalues as

$$R_k^{(N)}(t_1, \dots, t_k) = \frac{N!}{(N - k)!} \frac{\mathcal{Z}_{N-k}^{(H)}}{\mathcal{Z}_N^{(H)}} \exp\left(-\frac{N}{2} \sum_{j=1}^k t_j^2\right) \Delta^2(t_1, \dots, t_k) \mathbb{E}_{N-k}^{(H)} \left[ \prod_{j=1}^k \det(t_j - M)^2 \right], \tag{3.23}$$

where  $R_k^{(N)}(t_1, \dots, t_k)$  denotes a  $k$ -point correlation function of a GUE matrix of size  $N$ . The correlations of characteristic polynomials of matrices of size  $N - k$  are related to the correlation functions of eigenvalues of matrices of size  $N$ . The Dyson sine-kernel for the  $k$ -point correlation function and (3.1) for the moments of characteristic polynomials are recovered in the Dyson limit:  $t_i - t_j \rightarrow 0, N \rightarrow \infty$  and  $N(t_i - t_j)$  is kept finite when  $|t_j| < 2, j = 1, \dots, k$ .

In terms of the Schur polynomials,  $\lambda = (N^{2k})$ ,

$$\mathbb{E}_N^{(H)} \left[ \prod_{j=1}^{2k} \det(t_j - M) \right] = C_\lambda(2k) \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{1}{C_\nu(2k)} D_{\lambda\nu}^{(H)} S_\nu(t_1, \dots, t_{2k}). \tag{3.24}$$

Computing the asymptotics of moments of characteristic polynomials in the Dyson limit using (3.24) is highly non-trivial. Instead, we fix  $t_j = t, j = 1, \dots, 2k$ , and give an expansion of the moments as a function of  $t$  in the large  $N$  limit.

As  $N \rightarrow \infty$ , up to  $O(t^2)$ ,

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= (2N)^{k^2} e^{-Nk} \gamma_k [1 + O(N^3)O(t^4)], & N \text{ even} \\ \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= (2N)^{k^2} e^{-Nk} \gamma_k \\ &\quad \times \left[ 1 + t^2 \left( Nk + \frac{k^2}{3}(2k^2 - 1) + O(N^{-1}) \right) + O(N^3)O(t^4) \right], & N \text{ odd.} \end{aligned} \tag{3.25}$$

Note that the coefficient of  $t^2$  is identically zero for even  $N$ , where as for odd  $N$  it is a polynomial in  $N$  and  $k$ . Treating the above expansions as a formal series in  $N$  and  $t$  and taking their average gives

$$(2N)^{k^2} e^{-Nk} \gamma_k \left( 1 + \frac{Nkt^2}{2} + O(N^2)O(t^4) \right) \left( 1 - \frac{k^2 t^2}{8} + O(t^4) \right) \left( 1 + \frac{k}{12N}(8k^2 - 1) + O(N^{-2}) \right). \tag{3.26}$$

By comparing with (3.1), the terms in the first and second parenthesis of (3.26) are the expansions of  $e^{\frac{Nkt^2}{2}}$  and  $\pi\rho_{sc}(t)$  up to  $O(t^2)$ , respectively.

Similarly, as  $N \rightarrow \infty$ , up to  $O(t^4)$ ,

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= (2N)^{k^2} e^{-Nk} \gamma_k \left[ 1 + t^4 \frac{N^3 k}{6} \left( 1 + \frac{k}{6N}(4k^2 + 1) \right) \right. \\ &\quad \left. + \frac{k^2}{72N^2}(16k^4 - 16k^2 - 11) + \frac{k}{6480N^3}(320k^8 - 1200k^6 \right. \\ &\quad \left. + 708k^4 + 1265k^2 - 756) + O(N^{-4}) \right] + O(N^5)O(t^6), & N \text{ even,} \end{aligned}$$

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^{2k}] &= (2N)^{k^2} e^{-Nk} \gamma_k \left[ 1 + t^2 \left( Nk + \frac{k^2}{3}(2k^2 - 1) + O(N^{-1}) \right) \right. \\ &\quad \left. + t^4 \frac{N^3 k}{6} \left( -1 - \frac{2k}{3N}(k^2 - 2) - \frac{k^2}{18N^2}(4k^4 - 22k^2 + 13) \right) \right. \\ &\quad \left. - \frac{k}{405N^3}(20k^8 - 210k^6 + 483k^4 - 385k^2 + 54) + O(N^{-4}) \right] \\ &\quad \left. + O(N^5)O(t^6) \right], & N \text{ odd.} \end{aligned} \tag{3.27}$$

Taking average of the above series and factorising gives

$$\begin{aligned} &(2N)^{k^2} e^{-Nk} \gamma_k \left( 1 + \frac{Nkt^2}{2} + \frac{N^2 k^2 t^4}{8} + O(N^3)O(t^6) \right) \left( 1 - \frac{k^2 t^2}{8} + \frac{t^4}{128} k^2 (k^2 - 2) + O(t^6) \right) \\ &\times \left[ 1 + \frac{1}{N} \left( \frac{k}{12}(8k^2 - 1) + \frac{kt^2}{96}(13k^2 - 1) + O(t^4) \right) + \frac{1}{N^2} \left( \frac{k^2}{144}(32k^4 - 56k^2 + 17) + O(t^2) \right) \right. \\ &\quad \left. + O(N^{-3}) \right], \end{aligned} \tag{3.28}$$

where the first two brackets correspond to the expansion of  $e^{\frac{Nkt^2}{2}}$  and  $\pi\rho_{sc}(t)$ , respectively, up to  $O(t^4)$ , and the last factor is sub-leading.

### 4. Secular coefficients

Consider a matrix  $M$  of size  $N$ . Its characteristic polynomial can be expanded as

$$\det(t - M) = \prod_{j=1}^N (t - x_j) = \sum_{j=0}^N (-1)^j \text{Sc}_j(M) t^{N-j}, \tag{4.1}$$

where  $\text{Sc}_j$  is the  $j$ th secular coefficient of the characteristic polynomial. We have

$$\text{Sc}_1(M) = \text{Tr}M, \quad \text{Sc}_N(M) = \det(M). \tag{4.2}$$

These secular coefficients are nothing but the elementary symmetric polynomials  $e_j$  defined as

$$e_j(x_1, \dots, x_N) = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq N} x_{k_1} x_{k_2} \dots x_{k_j} \tag{4.3}$$

for  $j \leq N$  and  $e_j = 0$  for  $j > N$ .

The correlations of secular coefficients and their connections to combinatorics have been studied previously [21, 17]. For example, the joint moments of secular coefficients of the unitary group are connected to the enumeration of magic squares: matrices with positive entries with prescribed row and column sum. In a similar way, the joint moments of secular coefficients of Hermitian ensembles, such as the GUE, are connected to matching polynomials of closed graphs. In this section, we compute these correlations and indicate their combinatorial properties.

*Gaussian ensemble:* Elementary symmetric polynomials can be expanded in terms of multivariate Hermite polynomials as

$$e_r = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \Psi_{(1^r)(1^{r-2j})}^{(H)} \mathcal{H}_{(1^{r-2j})}, \tag{4.4}$$

where

$$\Psi_{(1^r)(1^{r-2j})}^{(H)} = (-1)^j \frac{(N-r+2j)!}{(2N)j!(N-r)!}. \tag{4.5}$$

Equivalently, we have

$$e_{2r} = \sum_{j=0}^r \Psi_{(1^{2r})(1^{2j})}^{(H)} \mathcal{H}_{(1^{2j})}, \quad e_{2r+1} = \sum_{j=0}^r \Psi_{(1^{2r+1})(1^{2j+1})}^{(H)} \mathcal{H}_{(1^{2j+1})}, \tag{4.6}$$

with

$$\begin{aligned} \Psi_{(1^{2r})(1^{2j})}^{(H)} &= (-1)^{r-j} \frac{1}{(2N)^{r-j}(r-j)!} \frac{(N-2j)!}{(N-2r)!}, \\ \Psi_{(1^{2r+1})(1^{2j+1})}^{(H)} &= (-1)^{r-j} \frac{1}{(2N)^{r-j}(r-j)!} \frac{(N-2j-1)!}{(N-2r-1)!}. \end{aligned} \tag{4.7}$$

Because of the orthogonality of the  $\mathcal{H}_\mu$ ,

$$\mathbb{E}_N^{(H)}[\text{Sc}_r] = \mathbb{E}_N^{(H)}[e_r] = \begin{cases} (-1)^{\frac{r}{2}} \frac{1}{(2N)^{\frac{r}{2}}} \frac{N!}{(N-r)!}, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases} \tag{4.8}$$

These expectations are nothing but the coefficients of Hermite polynomial of degree  $N$ . Thus,

$$\mathbb{E}_N^{(H)}[\det(t - M)] = \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \mathbb{E}_N^{(H)}[\text{Sc}_{2j}(M)] t^{N-2j} = h_N(t), \tag{4.9}$$

which coincides with (2.20) for  $p = 1$ . The expectation  $|\mathcal{N}^j \mathbb{E}_N^{(H)}[\text{Sc}_{2j}(M)]|$  is equal to the number of  $2j$  matchings in the complete graph [17, 21].

By using (4.4), the second moment of the secular coefficient can also be computed. Similar to the univariate case, multivariate Hermite polynomials  $\mathcal{H}_\lambda$  corresponding to even and odd  $|\lambda|$  do not mix. Hence, we obtain

$$\mathbb{E}_N^{(H)}[\text{Sc}_{2j}(M)\text{Sc}_{2k+1}(M)] = 0, \tag{4.10}$$

and

$$\begin{aligned}
 \mathbb{E}_N^{(H)}[\text{Sc}_{2r}(M)\text{Sc}_{2s}(M)] &= \sum_{j=0}^r \sum_{k=0}^s \Psi_{(1^{2r})(1^{2j})}^{(H)} \Psi_{(1^{2s})(1^{2k})}^{(H)} \mathbb{E}_N^{(H)}[\mathcal{H}_{(1^{2j})} \mathcal{H}_{(1^{2k})}] \\
 &= \sum_{j=0}^{\min(r,s)} \frac{1}{N^{2j}} \Psi_{(1^{2r})(1^{2j})}^{(H)} \Psi_{(1^{2s})(1^{2j})}^{(H)} C_{(1^{2j})}(N) \\
 &= \left(-\frac{1}{2N}\right)^{r+s} \sum_{j=0}^{\min(r,s)} \frac{2^{2j}}{(r-j)!(s-j)!} \frac{N!(N-2j)!}{(N-2r)!(N-2s)!}. \tag{4.11}
 \end{aligned}$$

Similarly, we write

$$\mathbb{E}_N^{(H)}[\text{Sc}_{2r+1}(M)\text{Sc}_{2s+1}(M)] = \left(-\frac{1}{2N}\right)^{r+s} \sum_{j=0}^{\min(r,s)} \frac{2^{2j}}{(r-j)!(s-j)!} \frac{(N-1)!(N-2j-1)!}{(N-2r-1)!(N-2s-1)!}. \tag{4.12}$$

Computing higher order correlations requires evaluating integrals involving a sequence of multivariate Hermite polynomials. Busbridge [11, 12] calculated these integrals for the univariate case, but the results are still unknown for the multivariate generalisation. Instead, we take a different approach by first expressing the product  $\prod_j \text{Sc}_j(M)^{b_j}$  in terms of the  $\mathcal{H}_\mu$  and then using orthogonality for the  $\mathcal{H}_\mu$ .

**Proposition 4.1.** Consider a partition  $\lambda = (\lambda_1, \dots, \lambda_l)$ . We have

$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^l \text{Sc}_{\lambda_j}(M)\right] = \begin{cases} \sum_{\mu} \frac{1}{(2N)^{\frac{|\mu|}{2}} \frac{|\mu|!}{2}} K_{\lambda'\mu} \chi_{(2|\mu|/2)}^\mu C_\mu(N), & \text{if } |\lambda| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

Here  $K_{\lambda\mu}$  are Kostka numbers<sup>1</sup> and  $\chi_\nu^\mu$  is the character of the symmetric group.

*Proof.* For a partition  $\lambda$ , denote

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots \tag{4.14}$$

Elementary symmetric polynomials  $e_\lambda$  can be expanded in Schur basis as follows:

$$e_\lambda = \sum_{\mu} K_{\lambda'\mu} S_\mu, \tag{4.15}$$

where  $K_{\lambda\mu}$  are the Kostka numbers [39] and  $\mu$  is a partition of  $|\lambda|$ . Using (2.5),

$$e_\lambda = \sum_{\mu \vdash |\lambda|} \sum_{\nu \subseteq \mu} K_{\lambda'\mu} \Psi_{\mu\nu}^{(H)} \mathcal{H}_\nu. \tag{4.16}$$

When  $|\lambda|$  is odd,  $\mathbb{E}_N^{(H)}[e_\lambda] = 0$  due to the orthogonality of multivariate Hermite polynomials. When  $|\lambda|$  is even,

$$\begin{aligned}
 \mathbb{E}_N^{(H)}[e_\lambda] &= \mathbb{E}_N^{(H)}\left[\prod_{j=1}^l \text{Sc}_{\lambda_j}(M)\right] \\
 &= \mathbb{E}_N^{(H)}\left[\sum_{\mu} \sum_{\nu} K_{\lambda'\mu} \Psi_{\mu\nu}^{(H)} \mathcal{H}_\nu\right] \\
 &= K_{\lambda'\mu} \Psi_{\mu 0}^{(H)}. \tag{4.17}
 \end{aligned}$$

<sup>1</sup>The Kostka numbers are non-negative integers that count the number of semi-standard Young tableau of shape  $\lambda$  and weight  $\mu$ .

It can be shown that [34]

$$\Psi_{\mu 0}^{(H)} = \frac{1}{(2N)^{\frac{\mu}{2}} \frac{|\mu|!}{2}} \chi_{(2|\mu|/2)}^{\mu} C_{\mu}(N). \tag{4.18}$$

Putting everything together completes the proof. □

*Laguerre ensemble:* All the calculations discussed for the Gaussian ensemble can be extended to the Laguerre and the Jacobi ensembles.

The polynomials  $e_r$  can be expanded as

$$e_r = \sum_{j=0}^r \Psi_{(1^r)(1^j)}^{(L)} \mathcal{L}_{(1^j)}^{(\gamma)}, \tag{4.19}$$

where

$$\Psi_{(1^r)(1^j)}^{(L)} = \frac{1}{(2N)^{r-j}} \frac{1}{(r-j)!} \frac{(N-j)!}{(N-r)!} \frac{\Gamma(N-j+\gamma+1)}{\Gamma(N-r+\gamma+1)}. \tag{4.20}$$

By using the orthogonality of the multivariate Laguerre polynomials we arrive at

$$\mathbb{E}_N^{(L)}[\text{Sc}_r] = \mathbb{E}_N^{(L)}[e_r] = \frac{1}{(2N)^r} \frac{1}{r!} \frac{N!}{(N-r)!} \frac{\Gamma(N+\gamma+1)}{\Gamma(N-r+\gamma+1)}, \tag{4.21}$$

which are the absolute values of the coefficients of the Laguerre polynomial of degree  $N$ . For the characteristic polynomial, we have

$$\mathbb{E}_N^{(L)}[\det(t - M)] = \sum_{j=0}^N (-1)^j \mathbb{E}_N^{(L)}[\text{Sc}_j(M)] t^{N-j} = I_N^{(\gamma)}(t). \tag{4.22}$$

The correlations of secular coefficients can be computed similar to the Gaussian case.

**Proposition 4.2.** *Let  $\lambda = (\lambda_1, \dots, \lambda_l)$ , we have*

$$\mathbb{E}_N^{(L)}\left[\prod_{j=1}^l \text{Sc}_{\lambda_j}(M)\right] = \sum_{\mu \vdash |\lambda|} \frac{1}{(2N)^{|\lambda|}} \frac{G_{\mu}(N, \gamma) G_{\mu}(N, 0)}{G_0(N, \gamma) G_0(N, 0)} \frac{\chi_{(1^{|\mu|})}^{\mu}}{|\lambda|!} K_{\lambda' \mu}. \tag{4.23}$$

*Proof.* The proof is similar to the Gaussian case. By writing

$$e_{\lambda} = \sum_{\mu} \sum_{\nu \subseteq |\lambda|} K_{\lambda' \mu} \Psi_{\mu \nu}^{(L)} \mathcal{L}_{\nu}^{(\gamma)}, \tag{4.24}$$

and using the orthogonality of the multivariate Laguerre polynomials along with the result [34]

$$\Psi_{\mu 0}^{(L)} = \frac{1}{(2N)^{|\mu|}} \frac{G_{\mu}(N, \gamma) G_{\mu}(N, 0)}{G_0(N, \gamma) G_0(N, 0)} \frac{\chi_{(1^{|\mu|})}^{\mu}}{|\mu|!} \tag{4.25}$$

proves the proposition. □

*Jacobi ensemble.* The  $e_r$  can be expanded as

$$e_r = \sum_{j=0}^r \Psi_{(1^r)(1^j)}^{(J)} \mathcal{L}_{(1^j)}^{(\gamma_1, \gamma_2)}, \tag{4.26}$$

where  $\Psi_{\lambda \nu}^{(J)}$  is given in (2.8). The expected values of the  $e_r$  are related to the coefficients of the Jacobi polynomial of degree  $N$ .

$$\mathbb{E}_N^{(J)}[\det(t - M)] = \sum_{j=0}^N (-1)^j \mathbb{E}_N^{(J)}[\text{Sc}_j(M)] t^{N-j} = J_N^{(\gamma_1, \gamma_2)}(t) \tag{4.27}$$

**Acknowledgements.** FM is grateful for support from the University Research Fellowship of the University of Bristol. JPK is pleased to acknowledge support from a Royal Society Wolfson Research Merit Award and ERC Advanced Grant 740900 (LogCorRM).

**Appendix**

**A. Asymptotics of ratio of factorials**

The asymptotics of the ratio of factorials can be computed as follows. First we look at  $C_\lambda(2k)D_e$  with  $\lambda = (2m, \dots, 2m)$ . Consider

$$\frac{(2m + j)!(2m + k + j)!}{(m + j)!^2} = (2m)^k \frac{(2m + j)!^2}{(m + j)!^2} \prod_{a=1}^k \left(1 + \frac{j + a}{2m}\right). \tag{A.1}$$

Now, one can see that

$$\frac{(2m + j)!}{(m + j)!} = 2^{j+1} \frac{\Gamma(2m)}{\Gamma(m)} \prod_{a=0}^j \frac{1 + \frac{a}{2m}}{1 + \frac{a}{m}}. \tag{A.2}$$

Using the duplication formula for the Gamma functions

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z) \tag{A.3}$$

and Stirling’s series

$$\Gamma(z + h) \sim \sqrt{2\pi} e^{-z} z^{z+h-\frac{1}{2}} \prod_{j=2}^{\infty} \exp\left(\frac{(-1)^j B_j(h)}{j(j-1)z^{j-1}}\right), \quad z \rightarrow \infty, \tag{A.4}$$

the asymptotic expansion for the ratio of Gamma functions can be found. Here  $B_j$  is the Bernoulli polynomial of degree  $j$ . Combining all the formulae, up to first order correction,

$$C_{((2m)2k)}(2k)D_e = e^{-2mk} 2^{4mk+2k^2} m^{2mk+k^2} \left(\prod_{j=0}^{k-1} \frac{j!}{(k+j)!}\right) \left[1 + \frac{k}{12m}(4k^2 + 1) + O(m^{-2})\right]. \tag{A.5}$$

Similarly for the case  $C_\lambda(2k)D_o$ , we obtain

$$\frac{(2m + 1 + k + j)!(2m + 1 + j)!}{(m + j)!^2} = (2m + 1)^k \frac{(2m + 1 + j)!^2}{(m + j)!^2} \prod_{a=1}^k \left(1 + \frac{j + a}{2m + 1}\right). \tag{A.6}$$

Let  $z = m + \frac{1}{2}$ , then

$$\frac{(2m + 1 + j)!}{(m + j)!} = \frac{\Gamma(2z + j + 1)}{\Gamma(z + \frac{1}{2} + j)} = 2^{j+1} z \frac{\Gamma(2z)}{\Gamma(z + \frac{1}{2})} \prod_{a=1}^j \frac{1 + \frac{a}{2z}}{1 + \frac{2a-1}{2z}}, \tag{A.7}$$

and

$$\frac{m!}{(m + k)!} = \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z + k + \frac{1}{2})} = \frac{1}{z^k} \prod_{a=1}^k \frac{1}{1 + \frac{2a-1}{2z}} \tag{A.8}$$

Combining the above formulae and using (A.3) and (A.4),

$$C_{(2m+1)2k}D_o \equiv C_{((2z)2k)}D_o = (-1)^k e^{-2zk} z^{k^2+2kz} 2^{2k^2+4kz} \left(\prod_{j=0}^{k-1} \frac{j!}{(k+j)!}\right) \times \left[1 + \frac{k}{6z}(2k^2 - 1) + O(z^{-2})\right]. \tag{A.9}$$

Higher order corrections can also be calculated with some effort or using any commercial software like Mathematica. Writing in terms of the matrix size  $N$ , as  $N \rightarrow \infty$ , we have

$$\begin{aligned}
 C_{(N^{2k})}D_e &= e^{-Nk}(2N)^{Nk+k^2} \left( \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \right) \left[ 1 + \frac{k}{6N}(4k^2 + 1) + \frac{k^2}{72N^2}(16k^4 - 16k^2 - 11) \right. \\
 &+ \frac{k}{6480N^3}(320k^8 - 1200k^6 + 708k^4 + 1265k^2 - 756) \\
 &+ \frac{k^2}{155,520N^4}(1280k^{10} - 10,240k^8 + 25,248k^6 - 6400k^4 - 56,371k^2 + 51,408) \\
 &+ \frac{k}{6,531,840N^5} \left( 7168k^{14} - 98,560k^{12} + 499,072k^{10} - 982,688k^8 - 399,844k^6 \right. \\
 &\quad \left. + 4,606,735k^4 - 5,598,936k^2 + 1,607,040 \right) \\
 &+ \frac{k^2}{1,175,731,200N^6} \left( 143,360k^{16} - 3,010,560k^{14} + 25,294,080k^{12} - 103,093,760k^{10} \right. \\
 &\quad \left. + 158,864,016k^8 + 298,943,760k^6 - 1,697,420,809k^4 + 2,663,679,600k^2 \right. \\
 &\quad \left. - 1,390,123,296 \right) + O\left(\frac{1}{N^7}\right) \Big], \tag{A.10}
 \end{aligned}$$

$N$  even.

$$\begin{aligned}
 C_{(N^{2k})}D_o &= (-1)^k e^{-Nk}(2N)^{Nk+k^2} \left( \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \right) \left[ 1 + \frac{k}{3N}(2k^2 - 1) + \frac{k^2}{18N^2}(4k^4 - 10k^2 + 7) \right. \\
 &+ \frac{k}{810N^3}(40k^8 - 240k^6 + 516k^4 - 455k^2 + 108) \\
 &+ \frac{k^2}{9720N^4}(80k^{10} - 880k^8 + 3828k^6 - 8356k^4 + 9509k^2 - 4320) \\
 &+ \frac{k}{204,120N^5} \left( 224k^{14} - 3920k^{12} + 28,616k^{10} - 113,428k^8 + 266,818k^6 \right. \\
 &\quad \left. - 372,127k^4 + 255,528k^2 - 51,840 \right) \\
 &+ \frac{k^2}{18,370,800N^6} \left( 2240k^{16} - 57,120k^{14} + 628,320k^{12} - 3,919,160k^{10} \right. \\
 &\quad \left. + 15,363,624k^8 - 39,481,170k^6 + 65,605,589k^4 - 62,864,640k^2 + 25,046,496 \right) \\
 &+ O\left(\frac{1}{N^7}\right) \Big], \tag{A.11}
 \end{aligned}$$

$N$  odd.

**B. More on the second moment**

Fix  $\lambda = (N, N)$ . The second moment of the characteristic polynomial is given by

$$\mathbb{E}_N^{(H)} [\det(t - M)^2] = \left(\frac{-1}{2N}\right)^N C_\lambda(2) \sum_{\nu \subseteq \lambda} \frac{1}{|\nu|!} (-2N)^{\frac{|\nu|}{2}} D_{\lambda\nu}^{(H)} \dim V_\nu t^{|\nu|}. \tag{B.1}$$

Let  $\nu = (\nu_1, \nu_2) \subseteq \lambda$ . Since  $|\nu|$  is even, either both  $\nu_1, \nu_2$  are even or both of them are odd. For  $N = 2m$ ,  $m \in \mathbb{N}$ ,

$$D_{\lambda\nu}^{(H)} = \begin{cases} \frac{1}{\left(m - \frac{\nu_1}{2}\right)! \left(m - \frac{\nu_2}{2}\right)!}, & \nu_1, \nu_2 \text{ are even,} \\ -\frac{1}{\left(m - \frac{\nu_1+1}{2}\right)! \left(m - \frac{\nu_2-1}{2}\right)!}, & \nu_1, \nu_2 \text{ are odd.} \end{cases} \tag{B.2}$$

Therefore,

$$C_\lambda(2)D_{\lambda\nu}^{(H)} = (2m)!(2m + 1)! \begin{cases} \frac{1}{(m - \frac{\nu_1}{2})! (m - \frac{\nu_2}{2})!}, & \nu_1, \nu_2 \text{ are even,} \\ -\frac{1}{(m - \frac{\nu_1+1}{2})! (m - \frac{\nu_2-1}{2})!}, & \nu_1, \nu_2 \text{ are odd.} \end{cases} \tag{B.3}$$

Similarly, for  $N = 2m + 1, m \in \mathbb{N}$ , we have

$$D_{\lambda\nu}^{(H)} = \begin{cases} -\frac{1}{(m - \frac{\nu_1}{2})! (m - \frac{\nu_2-2}{2})!}, & \nu_1, \nu_2 \text{ are even,} \\ \frac{1}{(m - \frac{\nu_1-1}{2})! (m - \frac{\nu_2-1}{2})!}, & \nu_1, \nu_2 \text{ are odd,} \end{cases} \tag{B.4}$$

and

$$C_\lambda(2)D_{\lambda\nu}^{(H)} = (2m + 1)!(2m + 2)! \begin{cases} -\frac{1}{(m - \frac{\nu_1}{2})! (m - \frac{\nu_2-2}{2})!}, & \nu_1, \nu_2 \text{ are even,} \\ \frac{1}{(m - \frac{\nu_1-1}{2})! (m - \frac{\nu_2-1}{2})!}, & \nu_1, \nu_2 \text{ are odd.} \end{cases} \tag{B.5}$$

For a partition of length 2,  $\nu = (\nu_1, \nu_2)$ ,

$$\frac{1}{|\nu|!} \dim V_\nu = \frac{\nu_1 - \nu_2 + 1}{(\nu_1 + 1)! \nu_2!} \tag{B.6}$$

Inserting (B.3), (B.5), (B.6) in (B.1), and observing that  $\nu$  runs over all partitions such that  $0 \leq |\nu| \leq 2N$  gives

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^2] &= \left(-\frac{1}{2N}\right)^N C_\lambda(2)D_{\lambda 0}^{(H)} \sum_{k=0}^N (-2N)^k t^{2k} \\ &\times \left[ \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \frac{2k + 1 - 4j}{(2k + 1 - 2j)!(2j)!} - \frac{2k - 1 - 4j}{(2k - 2j)!(2j + 1)!} \right) \frac{\left(\frac{N}{2}\right)!^2}{\left(\frac{N}{2} - k + j\right)! \left(\frac{N}{2} - j\right)!} \right. \\ &\quad \left. + \frac{1}{k!(k + 1)!} \frac{\left(\frac{N}{2}\right)!^2}{\left(\frac{N}{2} - \frac{k}{2}\right)!^2} \mathbb{1}_{k=0 \pmod{2}} \right]. \end{aligned} \tag{B.7}$$

for  $N$  even. Similarly, for  $N$  odd, one gets

$$\begin{aligned} \mathbb{E}_N^{(H)} [\det(t - M)^2] &= \left(-\frac{1}{2N}\right)^N C_\lambda(2)D_{\lambda 0}^{(H)} \sum_{k=0}^N (-2N)^k t^{2k} \\ &\times \left[ \sum_{j=0}^{\lfloor \frac{k-2}{2} \rfloor} \left( -\frac{2k - 1 - 4j}{(2k - 2j)!(2j + 1)!} + \frac{2k - 3 - 4j}{(2k - 2j - 1)!(2j + 2)!} \right) \frac{\left(\frac{N-1}{2}\right)! \left(\frac{N+1}{2}\right)!}{\left(\left(\frac{N+1}{2}\right) - k + j\right)! \left(\left(\frac{N-1}{2}\right) - j\right)!} \right. \\ &\quad \left. + \frac{1}{(2k)!} \frac{\left(\frac{N-1}{2}\right)!}{\left(\left(\frac{N-1}{2}\right) - k\right)!} - \frac{1}{k!(k + 1)!} \frac{\left(\frac{N-1}{2}\right)! \left(\frac{N+1}{2}\right)!}{\left(\left(\frac{N-1}{2}\right) - \frac{k-1}{2}\right)!^2} \mathbb{1}_{k=1 \pmod{2}} \right]. \end{aligned} \tag{B.8}$$

We have

$$C_\lambda(2)D_{\lambda 0}^{(H)} = \begin{cases} \frac{N!(N + 1)!}{\left(\frac{N}{2}\right)!^2}, & N \text{ even,} \\ -\frac{N!(N + 1)!}{\left(\frac{N-1}{2}\right)! \left(\frac{N+1}{2}\right)!}, & N \text{ odd.} \end{cases} \tag{B.9}$$

The asymptotics of the ratio of the factorials are already discussed in Appendix A. For the special case in (B.9),

$$\begin{aligned} \frac{N!(N+1)!}{\left(\frac{N}{2}\right)!^2} &\sim e^{-N}(2N)^{N+1} \prod_{j=1}^{\infty} \exp\left(\frac{(-1)^j(2^{j+1}-2)B_{j+1}}{j(j+1)N^j}\right), \\ \frac{N!(N+1)!}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!} &\sim e^{-N}(2N)^{N+1} \prod_{j=1}^{\infty} \exp\left(\frac{(-1)^{j+1}2^{j+1}B_{j+1}}{j(j+1)N^j}\right), \end{aligned} \tag{B.10}$$

where  $B_n$  are Bernoulli numbers. The explicit asymptotic expansion is

$$\begin{aligned} C_{\lambda}(2)D_{\lambda_0}^{(H)} &= e^{-N}(2N)^{N+1} \left[ 1 + \frac{5}{6N} - \frac{11}{72N^2} + \frac{337}{6480N^3} + \frac{985}{31,104N^4} - \frac{360,013}{6,531,840N^5} \right. \\ &\quad - \frac{46,723,609}{1,175,731,200N^6} + \frac{224,766,221}{1,410,877,440N^7} + \frac{41,757,020,981}{338,610,585,600N^8} \\ &\quad \left. - \frac{889,926,952,101,377}{1,005,673,439,232,000N^9} + O(N^{-10}) \right], \quad N \text{ even,} \\ C_{\lambda}(2)D_{\lambda_0}^{(H)} &= -e^{-N}(2N)^{N+1} \left[ 1 + \frac{1}{3N} + \frac{1}{18N^2} - \frac{31}{810N^3} - \frac{139}{9720N^4} + \frac{9871}{204,120N^5} \right. \\ &\quad + \frac{324,179}{18,370,800N^6} - \frac{8,225,671}{55,112,400N^7} - \frac{69,685,339}{1,322,697,600N^8} \\ &\quad \left. + \frac{1,674,981,058,019}{1,964,205,936,000N^9} + O(N^{-10}) \right], \quad N \text{ odd.} \end{aligned} \tag{B.11}$$

Substituting the above asymptotic series in

$$(2N)^{-1} e^{N-\frac{Nt^2}{2}} \mathbb{E}_N^{(H)} [\det(t-M)^2] \tag{B.12}$$

and taking the average over  $N$  even and odd gives

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{2N} e^N \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_N^{(H)} [\det(t-M)^2] \\ &= 1 - \frac{1}{8}t^2 - \frac{1}{128}t^4 - \frac{1}{1024}t^6 - \frac{5}{32768}t^8 - \frac{7}{262144}t^{10} + O(t^{12}). \end{aligned} \tag{B.13}$$

The R.H.S. in (B.13) coincides with  $\pi\rho_{sc}(t)$  up to  $O(t^{10})$ .

### References

- [1] G. Akemann, E. Strahov and T. R. Würfel, Averages of products and ratios of characteristic polynomials in polynomial ensembles, *Annales Henri Poincaré* **21**(12) (2020), 3973–4002.
- [2] A. Andreev and B. Simons, Correlators of spectral determinants in quantum chaos, *Phys. Rev. Lett.* **75**(12) (1995), 2304.
- [3] J. Baik, P. Deift and E. Strahov, Products and ratios of characteristic polynomials of random Hermitian matrices, *J. Math. Phys.* **44**(8) (2003), 3657–3670.
- [4] T. H. Baker and P. J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, *Commun. Math. Phys.* **188**(1) (1997), 175–216.
- [5] T. H. Baker and P. J. Forrester, The Calogero-Sutherland model and polynomials with prescribed symmetry, *Nuclear Phys. B* **492**(3) (1997), 682–716.
- [6] M. Berry and J. P. Keating, Clusters of near-degenerate levels dominate negative moments of spectral determinants, *J. Phys. A Math. General* **35**(1) (2001), L1.
- [7] A. Borodin and E. Strahov, Averages of characteristic polynomials in random matrix theory, *Commun. Pure Appl. Math.* **59**(2) (2006), 161–253.
- [8] J. Breuer and E. Strahov, A universality theorem for ratios of random characteristic polynomials, *J. Approximation Theory* **164**(6) (2012), 803–814.

- [9] E. Brézin and S. Hikami, Characteristic polynomials of random matrices, *Commun. Math. Phys.* **214**(1) (2000), 111–135.
- [10] E. Brézin and S. Hikami, Characteristic polynomials of random matrices at edge singularities, *Phys. Rev. E* **62**(3) (2000), 3558.
- [11] I. W. Busbridge, The evaluation of certain integrals involving products of Hermite polynomials, *J. London Math. Soc.* **1**(2) (1939), 93–97.
- [12] I. W. Busbridge, Some integrals involving Hermite polynomials. *J. London Math. Soc.* **1**(2) (1948), 135–141.
- [13] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, Autocorrelation of random matrix polynomials, *Commun. Math. Phys.* **237**(3) (2003), 365–395.
- [14] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, Integral moments of L-functions, *Proc. London Math. Soc.* **91**(1) (2005), 33–104.
- [15] J. B. Conrey and S. M. Gonek, High moments of the Riemann zeta-function, *Duke Math. J.* **107**(3) (2001), 577–604.
- [16] P. H. Damgaard and S. M. Nishigaki, Universal spectral correlators and massive Dirac operators, *Nuclear Phys. B* **518**(1–2) (1998), 495–512.
- [17] P. Diaconis and A. Gamburd, Random matrices, magic squares and matching polynomials, *Electron. J. Comb.* **11** (2004), R2.
- [18] P. Forrester and N. Frankel, Applications and generalizations of Fisher–Hartwig asymptotics, *J. Math. Phys.* **45**(5) (2004), 2003–2028.
- [19] P. Forrester and J. Keating, Singularity dominated strong fluctuations for some random matrix averages, *Commun. Math. Phys.* **250**(1) (2004), 119–131.
- [20] P. J. Forrester, *Log-gases and random matrices (LMS-34)* (Princeton University Press, Princeton, 2010).
- [21] P. J. Forrester and A. Gamburd, Counting formulas associated with some random matrix averages, *J. Comb. Theory Ser. A* **113**(6) (2006), 934–951.
- [22] Y. Fyodorov, *Mesoscopic Quantum Physics* ed E Akkermans et al (1995).
- [23] Y. V. Fyodorov, Negative moments of characteristic polynomials of random matrices: Ingham–Siegel integral as an alternative to Hubbard–Stratonovich transformation, *Nuclear Phys. B* **621**(3) (2002), 643–674.
- [24] Y. V. Fyodorov, J. Grella and E. Strahov, On characteristic polynomials for a generalized chiral random matrix ensemble with a source, *J. Phys. A Math. Theoret.* **51**(13) (2018), 134003.
- [25] Y. V. Fyodorov, G. A. Hiary and J. P. Keating, Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function, *Phys. Rev. Lett.* **108**(17) (2012), 170601.
- [26] Y. V. Fyodorov and J. P. Keating, Negative moments of characteristic polynomials of random GOE matrices and singularity-dominated strong fluctuations, *J. Phys. A Math. General* **36**(14) (2003), 4035.
- [27] Y. V. Fyodorov and J. P. Keating, Freezing transitions and extreme values: random matrix theory, and disordered landscapes, *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.* **372** (2007) (2014), 20120503.
- [28] Y. V. Fyodorov and P. Le Doussal, Moments of the position of the maximum for GUE characteristic polynomials and for log-correlated Gaussian processes, *J. Stat. Phys.* **164**(1) (2016), 190–240.
- [29] Y. V. Fyodorov and N. J. Simm, On the distribution of the maximum value of the characteristic polynomial of GUE random matrices, *Nonlinearity* **29**(9) (2016), 2837.
- [30] Y. V. Fyodorov and E. Strahov, Characteristic polynomials of random Hermitian matrices and Duistermaat–Heckman localisation on non-compact Kähler manifolds, *Nuclear Phys. B* **630**(3) (2002), 453–491.
- [31] Y. V. Fyodorov and E. Strahov, An exact formula for general spectral correlation function of random Hermitian matrices, *J. Phys. A Math. General* **36**(12) (2003), 3203.
- [32] T. Geroni, On the asymptotics of some large Hankel determinants generated by Fisher–Hartwig symbols defined on the real line, *J. Math. Phys.* **46**(4) (2005), 043516.
- [33] C. P. Hughes, J. P. Keating and N. O’Connell, Random matrix theory and the derivative of the Riemann zeta function, *Proc. R. Soc. London Ser. A Math. Phys. Eng. Sci.* **456**(2003) (2000), 2611–2627.
- [34] B. Jonnadula, J. P. Keating and F. Mezzadri, Symmetric function theory and unitary invariant ensembles. arXiv preprint arXiv:2003.02620 (2020), R24.
- [35] J. Keating and N. Snaith, Random matrix theory and number theory, in *The Handbook on Random Matrix Theory* (2011), 491–509.
- [36] J. P. Keating and N. C. Snaith, Random matrix theory and L-functions at  $s = 1/2$ , *Commun. Math. Phys.* **214**(1) (2000), 91–100.
- [37] J. P. Keating and N. C. Snaith, Random matrix theory and  $\zeta(1/2 + it)$ , *Commun. Math. Phys.* **214**(1) (2000), 57–89.
- [38] I. Krasovsky, Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant, *Duke Math. J.* **139**(3) (2007), 581–619.
- [39] I. G. Macdonald, *Symmetric functions and Hall polynomials* (Oxford University Press, Oxford, 1998).
- [40] M. L. Mehta, *Random matrices* (Elsevier, Amsterdam, 2004).
- [41] F. Mezzadri, N. C. Snaith and N. Hitchin, *Recent perspectives in random matrix theory and number theory*, vol. **322** (Cambridge University Press, Cambridge, 2005).
- [42] A. N. Sergeev and A. P. Veselov, Jacobi–Trudy formula for generalised Schur polynomials, *Mosc. Math. J.* **14**(1) (2014), 161–168.

- [43] E. Strahov, Moments of characteristic polynomials enumerate two-rowed lexicographic arrays, *Electron. J. Comb.* **10** (2002).
- [44] E. Strahov and Y. V. Fyodorov, Universal results for correlations of characteristic polynomials: Riemann-Hilbert approach, *Commun. Math. Phys.* **241**(2–3) (2003), 343–382.
- [45] R. J. Szabo, Microscopic spectrum of the QCD Dirac operator in three dimensions, *Nuclear Phys. B* **598**(1–2) (2001), 309–347.