5

Less supersymmetry

Let us move towards less supersymmetric theories. In this chapter we will review non-Abelian strings in four-dimensional gauge theories with $\mathcal{N} = 1$. In Chapter 6 we will deal with $\mathcal{N} = 0$.

As was discussed in the Introduction to Part II, the Seiberg–Witten mechanism of confinement [2, 3] relies on a cascade gauge symmetry breaking: the non-Abelian gauge group breaks down to an Abelian subgroup at a higher scale by condensation of the adjoint scalars, and at a lower scale the Abelian subgroup breaks down to a discrete subgroup by condensation of quarks (or monopoles, depending on the type of vacuum considered). This leads to formation of the ANO flux tubes and ensures an Abelian nature of confinement of the monopoles (or quarks, respectively). The gauge group acting in the infrared, where the confinement mechanism becomes operative, is Abelian.

On the other hand, non-supersymmetric QCD-like theories as well as $\mathcal{N} = 1$ SQCD have no adjoint scalars and, as a result, no cascade gauge symmetry breaking occurs. The gauge group acting in the infrared is non-Abelian. Confinement in these theories is non-Abelian. This poses a problem of understanding confinement in theories of this type. Apparently, a straightforward extrapolation of the Seiberg–Witten confinement scenario to these theories does not work.

The discovery of the non-Abelian strings [130, 131, 132, 133] suggests a novel possibility of solving this problem. In the $\mathcal{N} = 2$ gauge theory (4.1.7) the $\mathrm{SU}(N)$ subgroup of the $\mathrm{U}(N)$ gauge group remains unbroken after the squark condensation; the vacuum expectation value $\langle a^a \rangle = 0$, see (4.1.11). This circumstance demonstrates that the formation of the non-Abelian strings does not rely on the presence of adjoint VEVs. This suggests, in turn, that we can give masses to the adjoint fields, make them heavy, and eventually decouple the adjoint fields altogether, without losing qualitative features of the non-Abelian confinement mechanism reviewed above.

This program – moving towards less supersymmetry – was initiated in Ref. [189] which we will discuss in this section. In [189] we considered $\mathcal{N} = 2$ gauge theory (4.1.7), with the gauge group U(N). This theory was deformed by a mass term μ for the adjoint matter fields breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. The breaking terms do not affect classical solutions for the non-Abelian strings. The latter are still 1/2 BPS-saturated. However, at the quantum level the strings "feel" the presence of $\mathcal{N} = 2$ supersymmetry breaking terms. Effects generated by these terms first show up in the sector of the fermion zero modes.

Upon breaking $\mathcal{N} = 2$ supersymmetry in the bulk theory down to $\mathcal{N} = 1$, the fermionic sector of the world sheet theory modifies. The number of the fermion zero modes on the string (and hence the number of the fermion fields in the world sheet theory) does not change. It is determined by the index theorem [104] which we discuss in Section 5.4. However, supersymmetry of the world sheet model changes. The $\mathcal{N} = (2, 2)$ supersymmetry of the undeformed CP($N - 1$) model is broken down to $\mathcal{N} = (0, 2)$ by the bulk mass term μ [190, 191]. Moreover, the superorientational sector of the model gets mixed with the supertranslational one. The world sheet model emerging after deformation is called the heterotic CP($N - 1$) model. We will review in this section how the heterotic world sheet model emerges, as well as the physics of the heterotic $CP(N - 1)$ model which happens to be solvable in the large-N expansion [192]. The solution exhibits supersymmetry breaking at the quantum level.

If the adjoint mass parameter μ is kept finite, the non-Abelian string in the $\mathcal{N} = 1$ model at hand is well-defined and supports confined monopoles. However, at $\mu \to \infty$, as the adjoint superfield becomes very heavy (i.e. we approach the limit of $\mathcal{N} = 1$ SQCD) an infrared problem develops. This is due to the fact that in $\mathcal{N} = 1$ SQCD defined in a standard way the vacuum manifold is no longer an isolated point. Rather, a flat direction develops (a Higgs branch). The presence of the massless states obscures physics of the non-Abelian strings. In particular, the strings become infinitely thick [189]. Thus, one arrives at a dilemma: either one must abandon the attempt to decouple the adjoint superfield, or, if this decoupling is performed, confining non-Abelian strings cease to exist [189].

A way out was suggested in [104]. A relatively insignificant modification of the benchmark $\mathcal{N} = 2$ model cures the infrared problem. All we have to do is to add a neutral meson superfield M coupled to the quark superfields through a superpotential term. Acting together with the mass term of the adjoint superfield, M breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. The limit $\mu \to \infty$ in which the adjoint superfield completely decouples, becomes well-defined. No flat directions emerge. The limiting theory is $\mathcal{N} = 1$ SQCD supplemented by the meson superfield. It supports non-Abelian strings. The junctions of these strings present confined monopoles, or, better to say, what becomes of the monopoles in the theory where there are no adjoint scalar fields. There is a continuous path following which one can trace the monopole evolution in its entirety: from the ’t Hooft–Polyakov monopoles which do not exist without the adjoint scalars to the confined monopoles in the adjoint-free environment.

5.1 Breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$

In Section 5.1 we will outline main results of Ref. [189, 190, 191] where non-Abelian strings were considered in an $\mathcal{N} = 1$ gauge theory obtained as a deformation of the $\mathcal{N} = 2$ theory (4.1.7) by mass terms of the adjoint matter.

5.1.1 Deformed theory and string solutions

Let us add a superpotential mass term to our $\mathcal{N} = 2$ SQCD,

$$\mathcal{W}_{\mathcal{N}=1} = \sqrt{\frac{N}{2}}\,\frac{\mu_1}{2}\,\mathcal{A}^2 + \frac{\mu_2}{2}\,(\mathcal{A}^a)^2\,, \tag{5.1.1}$$

where μ_1 and μ_2 are mass parameters for the chiral superfields belonging to $\mathcal{N} = 2$ gauge supermultiplets, U(1) and SU(N), respectively, while the factor $\sqrt{N/2}$ is included for convenience. Clearly, the mass term (5.1.1) splits these supermultiplets, breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$.

The bosonic part of the SU(N)$\times$U(1) theory has the form (4.1.7) with the potential

$$\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a)_{\mathcal{N}=1} = {} & \frac{g_2^2}{2}\left(\frac{1}{g_2^2}\, f^{abc}\bar{a}^b a^c + \bar{q}_A\, T^a\, q^A - \tilde{q}_A T^a\, \bar{\tilde{q}}^A\right)^2 \\
& + \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi\right)^2 \\
& + \frac{g_2^2}{2}\left|2\tilde{q}_A T^a q^A + \sqrt{2}\mu_2 a^a\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A + \sqrt{N}\mu_1 a\right|^2 \\
& + \frac{1}{2}\sum_{A=1}^{N}\left\{\left|(a + 2T^a a^a)q^A\right|^2\right. \\
& \qquad\left. + \left|(a + 2T^a a^a)\bar{\tilde{q}}_A\right|^2\right\},
\end{aligned} \tag{5.1.2}$$

where the sum over repeated flavor indices A is implied. The potential (5.1.2) differs from the one in (4.1.9) in two ways. First, we use SU(2)$_R$ invariance of the original

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

$\mathcal{N}=2$ theory with the potential (4.1.9) to rotate the FI term. In Eq. (5.1.2) it is the FI D term, while in Chapter 4 we considered the FI F term.

Second, there are $\mathcal{N}=2$ supersymmetry breaking contributions from F terms in Eq. (5.1.2) proportional to the mass parameters μ_1 and μ_2. Note that we set the quark mass differences at zero and redefine a to absorb the average value of the quark mass parameters.

As in Eq. (4.1.9), the FI term triggers spontaneous breaking of the gauge symmetry. The vacuum expectation values of the squark fields can be chosen as

$$\langle q^{kA}\rangle = \sqrt{\xi}\begin{pmatrix} 1 & 0 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & 0 & 1 \end{pmatrix}, \quad \langle \bar{\tilde{q}}^{kA}\rangle = 0,$$
$$k=1,\ldots,N, \quad A=1,\ldots,N, \tag{5.1.3}$$

while the adjoint field VEVs are

$$\langle a^a\rangle = 0, \quad \langle a\rangle = 0, \tag{5.1.4}$$

see (4.1.11).

We see that the quark VEVs have the color-flavor locked form (see (4.1.14)) implying that the $\mathrm{SU}(N)_{C+F}$ global symmetry is unbroken in the vacuum. Much in the same way as in $\mathcal{N}=2$ SQCD, this symmetry leads to the emergence of the orientational zero modes of the Z_N strings.

Note that VEVs (5.1.3) and (5.1.4) do not depend on supersymmetry breaking parameters μ_1 and μ_2. This is due to the fact that our choice of parameters in (5.1.2) ensures vanishing of the adjoint VEVs, see (5.1.4). In particular, we have the same pattern of symmetry breaking all the way up to very large μ_1 and μ_2, where the adjoint fields decouple. As in $\mathcal{N}=2$ SQCD we assume $\sqrt{\xi}\gg\Lambda_{\mathrm{SU}(2)}$ to ensure weak coupling.

Now, let us discuss the mass spectrum in the $\mathcal{N}=1$ theory at hand. Since both U(1) and $\mathrm{SU}(N)$ gauge groups are broken by squark condensation, all gauge bosons become massive. Their masses are given in Eqs. (4.1.16) and (4.1.17).

To obtain the scalar boson masses we expand the potential (5.1.2) near the vacuum (5.1.3), (5.1.4) and diagonalize the corresponding mass matrix. Then, N^2 components of $2N^2$ (real) component scalar field q^{kA} are eaten by the Higgs mechanism for the U(1) and $\mathrm{SU}(N)$ gauge groups. Other N^2 components are split as follows. One component acquires mass (4.1.17) and becomes the scalar component of a massive $\mathcal{N}=1$ vector U(1) gauge multiplet. Moreover, N^2-1 components acquire masses (4.1.16) and become scalar superpartners of the $\mathrm{SU}(N)$ gauge bosons in $\mathcal{N}=1$ massive gauge supermultiplets.

Other $4N^2$ real scalar components of the fields $\tilde{q}_{Ak}$, a^a and a produce the following states: two states acquire mass

$$m^+_{\mathrm{U}(1)} = g_1\sqrt{\frac{N}{2}\xi\,\lambda^+_1}\,, \tag{5.1.5}$$

while the mass of other two states is given by

$$m^-_{\mathrm{U}(1)} = g_1\sqrt{\frac{N}{2}\xi\,\lambda^-_1}\,, \tag{5.1.6}$$

where $\lambda^\pm_1$ are two roots of the quadratic equation

$$\lambda_i^2 - \lambda_i(2+\omega_i^2) + 1 = 0\,, \tag{5.1.7}$$

for $i = 1$, where we introduced two $\mathcal{N} = 2$ supersymmetry breaking parameters $\omega_{1,2}$ associated with the U(1) and SU(N) gauge groups, respectively,

$$\omega_1 = \frac{g_1\mu_1}{\sqrt{\xi}}\,, \quad \omega_2 = \frac{g_2\mu_2}{\sqrt{\xi}}\,. \tag{5.1.8}$$

Other $2(N^2-1)$ states acquire mass

$$m^+_{\mathrm{SU}(N)} = g_2\sqrt{\xi\lambda^+_2}\,, \tag{5.1.9}$$

while the remaining $2(N^2-1)$ states become massive, with mass

$$m^-_{\mathrm{SU}(N)} = g_2\sqrt{\xi\lambda^-_2}\,, \tag{5.1.10}$$

where $\lambda^\pm_2$ are two roots of the quadratic equation (5.1.7) for $i = 2$. Note that all states come either as singlets or adjoints with respect to the unbroken $\mathrm{SU}(N)_{C+F}$.

When the SUSY breaking parameters ω_i vanish, the masses (5.1.5) and (5.1.6) coincide with the U(1) gauge boson mass (4.1.17). The corresponding states form the bosonic part of a long $\mathcal{N} = 2$ massive U(1) vector supermultiplet [35], see also Section 4.1.2.

If $\omega_1 \neq 0$ this supermultiplet splits into an $\mathcal{N} = 1$ vector multiplet, with mass (4.1.17), and two chiral multiplets, with masses (5.1.5) and (5.1.6). The same happens with the states with masses (5.1.9) and (5.1.10). In the limit of vanishing ω's they combine into bosonic parts of (N^2-1) $\mathcal{N} = 2$ vector supermultiplets with mass (4.1.16). If $\omega_i \neq 0$ these multiplets split into (N^2-1) $\mathcal{N} = 1$ vector multiplets (for the SU(N) group) with mass (4.1.16) and $2(N^2-1)$ chiral multiplets with masses (5.1.9) and (5.1.10). Note that the same splitting pattern was found in [35] in the Abelian case.

Now let us take a closer look at the spectrum obtained above in the limit of large $\mathcal{N}=2$ supersymmetry breaking parameters ω_i, $\omega_i \gg 1$. In this limit the larger masses $m^+_{\mathrm{U}(1)}$ and $m^+_{\mathrm{SU}(N)}$ become

$$m^+_{\mathrm{U}(1)} = m_{\mathrm{U}(1)}\omega_1 = g_1^2\sqrt{\frac{N}{2}}\mu_1\,, \quad m^+_{\mathrm{SU}(N)} = m_{\mathrm{SU}(N)}\omega_2 = g_2^2\mu_2\,. \tag{5.1.11}$$

In the limit $\mu_i \to \infty$ these are the masses of the heavy adjoint scalars a and a^a. At $\omega_i \gg 1$ these fields decouple and can be integrated out.

The low-energy theory in this limit contains massive gauge $\mathcal{N}=1$ multiplets and chiral multiplets with the lower masses m^-. Equation (5.1.7) gives for these masses

$$m^-_{\mathrm{U}(1)} = \frac{m_{\mathrm{U}(1)}}{\omega_1} = \sqrt{\frac{N}{2}}\,\frac{\xi}{\mu_1}\,, \quad m^-_{\mathrm{SU}(2)} = \frac{m_{\mathrm{SU}(2)}}{\omega_2} = \frac{\xi}{\mu_2}\,. \tag{5.1.12}$$

In particular, in the limit of infinite μ_i these masses tend to zero. This reflects the presence of a Higgs branch in $\mathcal{N}=1$ SQCD. To see the Higgs branch and calculate its dimension, please observe that our theory (4.1.7) with the potential (5.1.2) in the limit $\mu_i \to \infty$ flows to $\mathcal{N}=1$ SQCD with the gauge group $\mathrm{SU}(N)\times\mathrm{U}(1)$ and the FI D term. The bosonic part of the action of the latter theory is

$$\begin{aligned} S = \int d^4x \Bigg\{ &\frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 \\ &+ \frac{g_2^2}{2}\left(\bar{q}_A T^a q^A - \tilde{q}_A T^a \bar{\tilde{q}}^A\right)^2 + \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi\right)^2 \Bigg\} \end{aligned} \tag{5.1.13}$$

All F terms disappear in this limit, and we are left with the D terms. We have $4N^2$ real components of the q and $\tilde{q}$ fields while the number of the D term constraints in (5.1.13) is N^2. Moreover, N^2 phases are eaten by the Higgs mechanism. Thus, the dimension of the Higgs branch in Eq. (5.1.13) is

$$4N^2 - N^2 - N^2 = 2N^2\,.$$

The vacuum (5.1.3) corresponds to the base point of the Higgs branch with $\tilde{q}=0$. In other words, flowing from $\mathcal{N}=2$ theory (4.1.7), we do not recover the entire Higgs branch of $\mathcal{N}=1$ SQCD. Instead, we arrive at a single vacuum – a base point of the Higgs branch.

The scale of $\mathcal{N}=1$ SQCD

$$\Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}$$

is expressed in terms of the scale $\Lambda_{\mathrm{SU}(N)}$ of the deformed $\mathcal{N}=2$ theory as follows:

$$\left(\Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}\right)^{2N} = \mu_2^N\, \Lambda^N_{\mathrm{SU}(N)}\,. \tag{5.1.14}$$

To keep the bulk theory at weak coupling in the limit of large μ_i we assume that

$$\sqrt{\xi} \gg \Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}\,. \tag{5.1.15}$$

Now, considering the theory (4.1.7) with the potential (5.1.2), let us return to the case of arbitrary μ_i and discuss non-Abelian string solutions. The BPS saturation is maintained. By the same token, as for the BPS strings in $\mathcal{N}=2$ we use the *ansatz*

$$q^{kA} \equiv \varphi^{kA}\,, \quad \tilde{q}_{Ak} = 0\,. \tag{5.1.16}$$

The adjoint fields are set to zero. Note that Eq. (5.1.16) is an $\mathrm{SU}(2)_R$-rotated version of (4.2.1). The FI F term considered in Chapter 4 is rotated into the FI D term in (5.1.2).

With these simplifications the $\mathcal{N}=1$ model (4.1.7) with the potential (5.1.2) reduces to the model (4.2.2) which was exploited in Chapter 4 to obtain non-Abelian string solutions. The reason for this is that the adjoint fields play no role in the string solutions, and we let them vanish, see Eq. (5.1.4). Then $\mathcal{N}=2$ breaking terms vanish, and the potential (5.1.2) reduces to the one in Eq. (4.1.9) (up to an $\mathrm{SU}(2)_R$ rotation).

This allows us to parallel the construction of the non-Abelian strings carried out in Section 4.3. In particular, the elementary string solution is given by Eq. (4.4.4). Moreover, the *bosonic* part of the world sheet theory is nothing but the CP$(N-1)$ sigma model (4.4.9), with the coupling constant β determined by the coupling g_2 of the bulk theory via Eq. (4.4.15) at the scale $\sqrt{\xi}$. The latter scale plays the role of the UV cut off in the world sheet theory.

At small values of the deformation parameter,

$$\mu_2 \ll \sqrt{\xi}\,,$$

the coupling constant g_2 of the four-dimensional bulk theory is determined by the scale $\Lambda_{\mathrm{SU}(N)}$ of the $\mathcal{N}=2$ theory. Then Eq. (4.4.15) implies (see (4.4.35))

$$\Lambda_\sigma = \Lambda_{\mathrm{SU}(N)}\,, \tag{5.1.17}$$

where we take into account that the first coefficient of the β function is N both in the $\mathcal{N}=2$ limit of the four-dimensional bulk theory and in the two-dimensional CP$(N-1)$ model, see (4.4.34).

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

Instead, in the limit of large μ_2,

$$\mu_2 \gg \sqrt{\xi}\,,$$

the coupling constant g_2 of the bulk theory is determined by the scale $\Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}$ of $\mathcal{N} = 1$ SQCD (5.1.13), see (5.1.14). In this limit Eq. (4.4.15) gives

$$\Lambda_\sigma = \frac{\left(\Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}\right)^2}{g_2\sqrt{\xi}}\,, \tag{5.1.18}$$

where we take into account the fact that the first coefficient of the β function in $\mathcal{N} = 1$ SQCD is $2N$.

5.1.2 Heterotic CP$(N-1)$ *model*

In this section we will discuss the fermionic sector of the low-energy effective theory on the world sheet of the non-Abelian string in the deformed bulk theory (4.1.7) with the potential (5.1.2), as well as supersymmetry of the world sheet theory. First, we note that our string is classically 1/2 BPS-saturated. Therefore, in the $\mathcal{N} = 2$ limit (with $\mathcal{N} = 2$ breaking parameters μ_i vanishing) four supercharges out of eight present in the bulk theory are automatically preserved on the string world sheet. They become supercharges in the CP$(N-1)$ model.

What happens when we break $\mathcal{N} = 2$ supersymmetry of the bulk model by switching on parameters μ_i (for simplicity we consider the case $\mu_1 = \mu_2 \equiv \mu$ here)? The 1/2 "BPS-ness" of the string solution requires only two supercharges on the world sheet. However, as we will show in Section 5.4, the number of the fermion zero modes in the string background does not change. This number is fixed by the index theorem. Thus, the number of (classically) massless fermion fields in the world sheet model does not change.

It is well known that the $\mathcal{N} = (2,2)$ supersymmetric sigma model with the CP$(N-1)$ target space does not admit $\mathcal{N} = (0,2)$ supersymmetric deformations [156]. A way out was indicated in [190]: the world-sheet theory is in fact CP$(N-1) \times C$ rather than the CP$(N-1)$ model. The factor C comes from the translational sector. In the $\mathcal{N} = 2$ limit the translational and the orientational sectors of the world-sheet theory are totally decoupled. Breaking $\mathcal{N} = 2$ supersymmetry in the bulk mixes fermions from these two sectors on the world sheet.

The translational sector of the effective theory on the string in the $\mathcal{N} = 2$ limit contains the bosonic field x_{0i}, $i = 1, 2$ (position of string's center in the (1,2) plane), and two fermion fields ζ_L and ζ_R. Two supercharges that survive on the string world sheet at non-zero μ protect x_{0i} and ζ_L. The world-sheet fields $x_{0i}(t,z)$ and $\zeta_L(t,z)$

remain free fields decoupled from all others. This is no longer the case with regards to ζ_R which gets an interaction with fermions of the orientational sector.

As a result, the heterotic $\mathcal{N} = (0, 2)$ model in the gauged formulation (see (4.4.32)) takes the form [190]

$$
\begin{aligned}
S = \int d^2x \Big\{ &\frac{1}{2}\, \bar{\zeta}_R\, i\partial_L\, \zeta_R + \left[2\sqrt{\beta}\, i\, \delta\, \bar{\lambda}_L\, \zeta_R + \text{H.c.}\right] \\
&+ |\nabla_k n^l|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 |n^l|^2 + i\, D(|n^l|^2 - 2\beta) \\
&+ \bar{\xi}_{lR}\, i\nabla_L\, \xi^l_R + \bar{\xi}_{lL}\, i\nabla_R\, \xi^l_L + \frac{1}{e^2}\, \bar{\lambda}_R\, i\partial_L\, \lambda_R + \frac{1}{e^2}\, \bar{\lambda}_L\, i\partial_R\, \lambda_L \\
&+ \left[i\sqrt{2}\, \sigma\, \bar{\xi}_{lR} \xi^l_L + i\sqrt{2}\, \bar{n}_l\, (\lambda_R \xi^l_L - \lambda_L \xi^l_R) + \text{H.c.}\right] + 8\beta\, |\delta|^2\, |\sigma|^2 \Big\}, \qquad (5.1.19)
\end{aligned}
$$

where we omitted the fields $x_{0i}(t, z)$ and $\zeta_L(t, z)$ as irrelevant for the present consideration, while $\partial_{L,R} = \partial_0 \mp i\, \partial_3$. Here $\xi^l_{R,L}$ are fermionic superpartners of the bosonic orientational fields n^l. For convenience we change the normalization of n^l as compared with the one in (4.4.32) absorbing the coupling constant 2β in the kinetic term for n's.

Much in the same way as the $\mathcal{N} = (2, 2)$ CP$(N-1)$ model, this heterotic model should be considered in the strong coupling limit $e^2 \to \infty$. The gauge multiplet consists of the U(1) gauge field A_k, complex scalar σ, fermions $\lambda_{R,L}$ and axillary field D. Integrating over D gives constraint (4.4.3) modified due to the change of normalization of n as follows:

$$|n^l|^2 = 2\beta\,. \qquad (5.1.20)$$

The terms in Eq. (5.1.19) containing the deformation parameter δ break $\mathcal{N} = (2, 2)$ supersymmetry down to $\mathcal{N} = (0, 2)$. The parameter δ is complex and dimensionless. It was calculated in terms of the deformation parameter μ of the bulk theory in [191]. We review this result below.

Integrating over the axillary fields λ we arrive at the constraints

$$\bar{n}_l\, \xi^l_L = 0, \quad \bar{\xi}_{lR}\, n^l = \sqrt{2\beta}\, \delta\, \zeta_R\,, \qquad (5.1.21)$$

replacing those in Eq. (4.4.24) (the latter equation in the gauged formulation for arbitrary N takes the form $\bar{n}_l\, \xi^l = 0$). We see that the constraint (4.4.24) is modified for the right-handed fermions ξ_R implying that the supertranslational sector of the world sheet theory is no longer decoupled from the orientational one. The general structure of the deformation in (5.1.19) is dictated by $\mathcal{N} = (0, 2)$ supersymmetry.

Integrating over A_k and σ produces four-fermion interactions in the model (5.1.19). Once the coefficient in front of $|\sigma|^2$ is modified by the $\mathcal{N} = 2$ breaking deformation the coefficient in front of the four-fermion interaction is modified as well (as compared with Eq. (4.4.31)).

The model (5.1.19) has a U(1) axial symmetry which is broken by the chiral anomaly down to the discrete subgroup Z_{2N} [159]. The σ field is related to the fermion bilinear operator by the following formula:

$$\sigma = -\frac{i}{2\sqrt{2}\beta(1+2|\delta|^2)}\,\bar{\xi}_{lL}\xi_R^l\,. \tag{5.1.22}$$

Moreover, it transforms under the above Z_{2N} symmetry as

$$\sigma \to e^{\frac{2\pi k}{N}\,i}\,\sigma, \quad k=1,\ldots,N-1\,. \tag{5.1.23}$$

We will see below that the Z_{2N} symmetry is spontaneously broken by a condensate of σ, down to Z_2, much in the same way as in the conventional $\mathcal{N}=(2,2)$ model [159]. This is equivalent to saying that the fermion bilinear condensate $\langle\bar{\xi}_{lL}\xi_R^l\rangle$ develops, breaking the discrete Z_{2N} symmetry down to Z_2.

We can rewrite the indirect interactions between superorientational and supertranslational sectors of the theory coded in the constraint (5.1.21) by shifting the field ξ_R as follows:

$$\bar{\xi}_R \to \bar{\xi}_R - \delta\,\frac{1}{\sqrt{2\beta}}\,\bar{n}\,\zeta_R\,. \tag{5.1.24}$$

Then we return to unmodified constraints

$$\bar{n}_l\,\xi_L^l = 0, \qquad \bar{\xi}_{lR}\,n^l = 0\,. \tag{5.1.25}$$

Performing a rather straightforward algebraic analysis based on the relation between the gauged and O(3) formulations (see Appendix B.4) we get

$$\begin{aligned} S_{1+1} = \beta\int d^2x\,\Big\{&\frac{1}{2}\big(\partial_k S^a\big)^2 + \frac{1}{2}\,\chi_R^a\,i\,\partial_L\,\chi_R^a + \frac{1}{2}\,\chi_L^a\,i\,\partial_R\,\chi_L^a - \frac{c^2}{2}(\chi_R^a\chi_L^a)^2 \\ &+ \frac{c}{2\sqrt{2\beta}}\,\chi_R^a\big(i\,\partial_L\,S^a\,(\alpha\,\zeta_R + \bar{\alpha}\,\bar{\zeta}_R) + i\varepsilon^{abc}S^b i\,\partial_L\,S^c\,(\alpha\,\zeta_R - \bar{\alpha}\,\bar{\zeta}_R)\big) \\ &+ \frac{1}{2\beta}\,\bar{\zeta}_R\,i\,\partial_L\,\zeta_R + |\alpha|^2\,\frac{c^2}{4\beta}\,\bar{\zeta}_R\zeta_R\,i\,\varepsilon^{abc}S^a\,\chi_L^b\chi_L^c\Big\}, \end{aligned} \tag{5.1.26}$$

where

$$c^2 = \frac{1}{1+|\alpha|^2}, \tag{5.1.27}$$

we restrict ourselves to $N=2$ and rename $\chi_{1,2}^a$ of Section 4.4.2, $\chi_{1,2}^a \to \chi_{R,L}^a$.

The relation of the deformation parameter introduced here and the one in the gauged formulation of the theory (see Eqs. (5.1.19)) is as follows:

$$\delta = \frac{\alpha}{\sqrt{1-|\alpha|^2}}\,. \tag{5.1.28}$$

As a result of the shift (5.1.24), crucial bifermionic terms of the type $\chi_R \partial_L S \zeta_R$ appear in the second line of Eq. (5.1.26).

Now the problem is to actually derive the heterotic world sheet CP($N-1$) model from the bulk theory. The general structure of the theory (5.1.19) is fixed by $\mathcal{N} = (0,2)$ supersymmetry. In order to derive this theory as an effective low-energy theory on the string we have to calculate the deformation parameter δ in terms of the bulk parameters. The kinetic cross-terms $\chi_R \partial S \zeta_R$ in the formulation (5.1.26), bilinear in the fermion fields, allow us to do so. In Ref. [191] the μ-deformation of the fermion zero modes was considered. Both supertranslational and superorientational fermion zero modes on the string were found in the limits of small and large μ by solving the Dirac equations. The overlap of the translational and orientational fermion zero modes gives the kinetic cross-term $\chi_1 \partial S \zeta_R$ in (5.1.19).

This derivation provides us with a relation between the bulk and world sheet deformation parameters, namely [191],

$$\delta = \begin{cases} \text{const}\ \dfrac{g_2^2 \mu}{M_{\text{SU}(N)}}\,, & \text{small}\ \ \mu\,, \\[2ex] \text{const}\ \dfrac{\mu}{|\mu|}\sqrt{\ln \dfrac{g_2^2|\mu|}{M_{\text{SU}(N)}}}\,, & \text{large}\ \ \mu\,, \end{cases} \tag{5.1.29}$$

where the mass of the gauge boson $M_{\text{SU}(N)}$ is given in Eq. (4.1.16). The constants here are determined by the profile functions of the string solution [191].

The physical reason for the logarithmic behavior of the world sheet deformation parameter at large μ is as follows. In the large-μ limit certain states in the bulk theory become light [189, 191], see Section 5.1.1. This reflects the presence of the Higgs branch in $\mathcal{N} = 1$ SQCD to which our bulk theory flows in the $\mu \to \infty$ limit. The argument of the logarithm in (5.1.29) is the ratio of $M_{\text{SU}(N)}$ and the small mass of the light states associated with this would-be Higgs branch [191].

To conclude this section, let us mention that more general deformations of $\mathcal{N} = 2$ theory (4.1.7) preserving $\mathcal{N} = 1$ supersymmetry were also considered in [190, 191]. In particular, deformations of (4.1.7) with unequal quark masses with a polynomial superpotential

$$\mathcal{W} = \text{Tr} \sum_{k=1}^{N} \frac{c_k}{k+1}\, \Phi^{k+1} \tag{5.1.30}$$

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

do not spoil the BPS nature of string solutions if the critical points of the superpotential coincide with the quark mass parameters. Associated $\mathcal{N} = (0, 2)$ heterotic deformations of the CP($N-1$) world-sheet theory were derived in [191]. For polynomial deformations (5.1.30) a non-polynomial (logarithmic) response was found in the world-sheet model. The heterotic CP($N-1$) model in the geometric formulation is presented in Appendix B.5.

5.1.3 Large-N solution

The $\mathcal{N} = (2, 2)$ model as well as the non-supersymmetric $CP(N-1)$ model were solved by Witten in the large-N limit [159]. The same method was used in [192] to study the $\mathcal{N} = (0, 2)$ heterotic $CP(N-1)$ model. In this section we will briefly review this analysis.

Since the action (5.1.19) is quadratic in the fields n^l and ξ^l we can integrate out these fields and then minimize the resulting effective action with respect to the fields from the gauge multiplet. The large-N limit ensures the corrections to the saddle point approximation to be small. In fact, this procedure boils down to calculating a small set of one-loop graphs with the n^l and ξ^l fields propagating in loops. After integrating n^l and ξ^l out, we must check self-consistency.

Integration over n^l and ξ^l in (5.1.19) yields the following determinants:

$$\left[\det\left(-\partial_k^2 + iD + 2|\sigma|^2\right)\right]^{-N}\left[\det\left(-\partial_k^2 + 2|\sigma|^2\right)\right]^{N}, \tag{5.1.31}$$

where we dropped the gauge field A_k. The first determinant here comes from the boson loops while the second from fermion loops. Note, that the n^l mass is given by $iD + 2|\sigma|^2$ while that of the fermions ξ^l is $2|\sigma|^2$. If supersymmetry is unbroken (i.e. $D = 0$) these masses are equal, and the product of the determinants reduces to unity, as it should be.

Calculation of the determinants in Eq. (5.1.31) is straightforward. We easily get the following contribution to the effective action:

$$\frac{N}{4\pi}\left\{\left(iD + 2|\sigma|^2\right)\left[\ln\frac{M_{\rm uv}^2}{iD + 2|\sigma|^2} + 1\right] - 2|\sigma|^2\left[\ln\frac{M_{\rm uv}^2}{2|\sigma|^2} + 1\right]\right\}, \tag{5.1.32}$$

where quadratically divergent contributions from bosons and fermions do not depend on D and σ and cancel each other. Here $M_{\rm uv}$ is an ultraviolet cut off. Remembering that the action in (5.1.19) presents an effective low-energy theory on the string world sheet one can readily identify the UV cut off in terms of the bulk parameters,

$$M_{\rm uv} = M_{{\rm SU}(N)}\,. \tag{5.1.33}$$

Invoking Eq. (4.4.15) we conclude that the bare coupling constant β in (5.1.19) can be parameterized as

$$\beta = \frac{N}{8\pi} \ln \frac{M_{\rm uv}^2}{\Lambda_\sigma^2}\,. \tag{5.1.34}$$

Substituting this expression in (5.1.19) and adding the one-loop correction (5.1.32) we see that the term proportional to $iD \ln M_{\rm uv}^2$ is canceled out, and the effective action is expressed in terms of the renormalized coupling constant,

$$\beta_{\rm ren} = \frac{N}{8\pi} \ln \frac{iD + 2|\sigma|^2}{\Lambda_\sigma^2}\,. \tag{5.1.35}$$

Assembling all contributions together we get the effective potential as a function of the D and σ fields in the form

$$V_{\rm eff} = \int d^2x\, \frac{N}{4\pi}\left\{-\left(iD + 2|\sigma|^2\right) \ln \frac{iD + 2|\sigma|^2}{\Lambda_\sigma^2} + iD \right.$$
$$\left. + 2|\sigma|^2 \ln \frac{2|\sigma|^2}{\Lambda_\sigma^2} + 2|\sigma|^2 u \right\}, \tag{5.1.36}$$

where instead of the deformation parameter δ we introduced a more convenient (dimensionless) parameter u which does not scale with N,

$$u = \frac{16\pi}{N}\beta\, |\delta|^2. \tag{5.1.37}$$

Minimizing this potential with respect to D and σ we arrive at the following relations:

$$\beta_{\rm ren} = \frac{N}{8\pi} \ln \frac{iD + 2|\sigma|^2}{\Lambda_\sigma^2} = 0\,,$$
$$\ln \frac{iD + 2|\sigma|^2}{2|\sigma|^2} = u\,. \tag{5.1.38}$$

Equations (5.1.38) represent the *master set* which determines the vacua of the theory. Solutions can be readily found,

$$2|\sigma|^2 = \Lambda_\sigma^2\, e^{-u}\,, \qquad \sigma = \frac{1}{\sqrt{2}}\Lambda_\sigma \exp\left(-\frac{u}{2} + \frac{2\pi\, i\, k}{N}\right), \quad k = 0, \ldots, N-1,$$
$$iD = \Lambda_\sigma^2\,(1 - e^{-u})\,. \tag{5.1.39}$$

The phase factor of σ does not follow from (5.1.38), but we know of its existence from the fact of the spontaneous breaking of the discrete chiral Z_{2N} down to Z_2, see the discussion in Section 5.1.2. Substituting this solution in Eq. (5.1.36) we get the expression for the vacuum energy density [192],

$$\mathcal{E}_{\rm vac} = \frac{N}{4\pi}\, i\,D = \frac{N}{4\pi}\,\Lambda_\sigma^2\,(1-e^{-u})\,. \tag{5.1.40}$$

We see that the vacuum energy does not vanish! The $\mathcal{N}=(0,2)$ supersymmetry is spontaneously broken. The breaking of $\mathcal{N}=(0,2)$ supersymmetry was first conjectured in Ref. [193]. Of course, the large-N solution presented above is the most clear-cut demonstration of its spontaneous breaking. However, even in the absence of this solution one can present general arguments in favor of this scenario. Indeed, addition of the extra right-handed field ζ_R in the CP($N-1$) model changes Witten's index from N to zero [191].

Another argument comes from the bulk theory in the limit of large deformation parameter μ. The breaking of supersymmetry for the $\mathcal{N}=(0,2)$ world-sheet theory could be argued from consistency with the absence of localized BPS solutions of the required type in $\mathcal{N}=1$ gauge theories. The argument could go along the following lines: If the $\mathcal{N}=(0,2)$ world-sheet theory is to have supersymmetric vacua, it would need to have N degenerate vacua due to the breaking of the discrete Z_N symmetry. This is likely to imply the existence of BPS kinks preserving one supercharge. From the bulk point of view, these configurations would be 1/4 BPS-saturated confined monopoles. However, such solutions are not supported by the allowed central charges in the $\mathcal{N}=1$ superalgebra and, therefore, are not expected. Thus, the breaking of world sheet supersymmetry is consistent with the absence of such bulk BPS configurations.

Needless to say, the linear N dependence of the vacuum energy we see in Eq. (5.1.40) was expected.

It is instructive to discuss the first condition in (5.1.38). That $\beta_{\rm ren}=0$ was a result of Witten's analysis [159] too. This fact, $\beta_{\rm ren}=0$, implies that in quantum theory (unlike the classical one)

$$\langle\, |n^l|^2\,\rangle = 0\,, \tag{5.1.41}$$

i.e. the global SU(N) symmetry is not spontaneously broken in the vacuum and, hence, there are no massless Goldstone bosons. All bosons get a mass.

If the deformation parameter u vanishes, the vacuum energy vanishes too and supersymmetry is not broken, in full accord with Witten's analysis [159] and with the fact that the Witten index is N in this case [123]. The σ field develops a vacuum expectation value (5.1.39) breaking Z_{2N} symmetry.[1] As we switch on the

[1] The vacuum structure (5.1.39) of the $\mathcal{N}=(2,2)$ model at $u=0$ was also obtained by Witten for arbitrary N in [157] using a superpotential of the Veneziano–Yankielowicz type [194].

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

deformation parameter u, the D component develops a VEV; hence, $\mathcal{N} = (0, 2)$ supersymmetry is spontaneously broken. The vacuum energy density no longer vanishes.

In the limit $\mu \to \infty$, the deformation parameter u behaves logarithmically with μ,

$$u = \text{const} \left(\ln \frac{M_{\text{SU}(N)}}{\Lambda^{\mathcal{N}=1}_{\text{SU}(N)}} \right) \left(\ln \frac{g_2^2 |\mu|}{M_{\text{SU}(N)}} \right), \tag{5.1.42}$$

where the constant above does not depend on N. At any finite u the σ-field condensate does not vanish, labeling N distinct vacua as indicated in Eq. (5.1.39). In each vacuum Z_{2N} symmetry is spontaneously broken down to Z_2. As we explain in Section 5.1.4 we cannot trust the world-sheet theory at very large values of μ due to the presence of the Higgs branch in $\mathcal{N} = 1$ SQCD. Therefore, the vacuum structure outlined above persists in the whole window of the allowed values of the deformation parameter u.

The mass spectrum of the heterotic CP($N-1$) model was determined in Ref. [192] in the large-N limit. In the regime of large u the masses of the n^l bosons and ξ^l fermions become drastically different. They are

$$m_n = \sqrt{iD + 2|\sigma|^2} = \Lambda_\sigma, \quad m_\xi = \sqrt{2}|\sigma| = \Lambda_\sigma \exp\left(-\frac{u}{2}\right), \tag{5.1.43}$$

where we used Eqs. (5.1.39). The fermions are much lighter than their bosonic counterparts.

Much in the same way as in the $\mathcal{N} = (2, 2)$ CP($N-1$) model [159], the fields belonging to the U(1) gauge multiplet, introduced as axillary fields in (5.1.19), acquire kinetic terms at one loop and become dynamical. Moreover, the photon acquires mass proportional to the VEV of the σ field due to the chiral anomaly. The σ field also becomes massive, with mass determined by $\langle\sigma\rangle$ in (5.1.39).

Due to the spontaneous supersymmetry breaking the theory always has a massless Goldstino. At small u its role is played by ζ_R with a small admixture of other fermions, while in the large u limit the gaugino λ_R becomes massless. In the large u limit when $\langle\sigma\rangle$ is small the low-energy effective theory contains the light (but massive!) photon, two light σ states and only one fermion: the massless Goldstino λ_R.

As was shown above, in the $\mathcal{N} = (0, 2)$ theory supersymmetry is spontaneously broken. The vacuum energy density does not vanish, see (5.1.40). This means that strings under consideration are no longer BPS and their tensions get a shift (5.1.40) with respect to the classical value $T_{\text{cl}} = 2\pi\xi$. However, this shift is the same for all N elementary strings. Their tensions are strictly degenerate; Z_{2N} symmetry is

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

spontaneously broken down to Z_2. The order parameter (the σ field VEV) remains nonvanishing at any finite value of the bulk parameter μ.

The kinks that interpolate between different vacua of the world sheet theory are described by the n^l fields. Their masses are given in Eq. (5.1.43). In the $\mathcal{N}=(0,2)$ theory the masses of the boson and fermion superpartners are split. The bosonic kinks have masses $\sim \Lambda_\sigma$ in the large-μ limit, while the fermionic kinks become light. Still their masses remain finite and nonvanishing at any finite μ.

We already know that, from the standpoint of the bulk theory, these kinks are confined monopoles [189, 104, 192]. The fact that tensions of all elementary strings are the same ensures that these monopoles are free to move along the string, since with their separation increasing, the energy of the configuration does not change. This means they are in the deconfinement phase on the string.[2] The kinks are deconfined both in $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,2)$ $CP(N-1)$ theories. In other words, individual kinks are present in the physical spectrum of (1+1) dimensional theory. The monopoles, although attached to strings, are free to move on the strings. We will see in Chapter 6 that this is not the case for monopoles in non-supersymmetric theories. Kinks in non-supersymmetric CP($N-1$) models are in the confinement phase on the string, therefore a monopole and an antimonopole attached to the string come close to each other forming a meson-like configuration.

5.1.4 Limits of applicability

As was discussed above, both the string solution and the bosonic part of the world sheet theory for the non-Abelian strings in $\mathcal{N}=1$ with the potential (5.1.2) are identical to those in $\mathcal{N}=2$. However, the occurrence of the Higgs branch in the limit $\mu\to\infty$ manifests itself at the quantum level [189]. At the classical level light fields appearing in the bulk theory in the large-μ limit do not enter the string solution. The string is "built" of heavy fields. However, at the quantum level couplings to the light fields lead to a dramatic effect: an effective string thickness becomes large due to long-range tails of the string profile functions associated with the light fields. As a matter of fact, we demonstrated [189, 191] that in the fermion sector this effect is seen already at the classical level. Some of the fermion zero modes on the string solution acquire long-range tails and become non-normalizable in the limit $\mu\to\infty$.

Below we will estimate the range of validity of the description of non-Abelian string dynamics by the CP($N-1$) model (5.1.19). To this end let us note that higher derivative corrections to (5.1.19) run in powers of

$$\Delta\,\partial_k\,, \tag{5.1.44}$$

[2] We stress that these monopoles are confined in the bulk theory being attached to strings.

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

where Δ is the string transverse size (thickness). At small μ it is quite clear that $\Delta \sim 1/g\sqrt{\xi}$. A typical energy scale on the string world sheet is given by the scale Λ_σ of the CP($N-1$) model which, in turn, is given by (5.1.17) at small μ. Thus, $\partial \to \Lambda_\sigma$, and higher-derivative corrections run in powers of $\Lambda_\sigma/g\sqrt{\xi}$. At small μ the higher-derivative corrections are suppressed by powers of $\Lambda_\sigma/g\sqrt{\xi} \ll 1$ and can be ignored. However, with μ increasing, the fermion zero modes acquire long-range tails [189, 191]. This means that an effective thickness of the string grows. The thickness is determined by masses of the lightest states (5.1.12) of the bulk theory,

$$\Delta \sim \frac{1}{m^-} = \frac{\mu}{\xi}\,. \tag{5.1.45}$$

The higher-derivative terms are small if $(\Delta\,\Lambda_\sigma) \ll 1$. Substituting here the scale of the CP($N-1$) model given at large μ by (5.1.18) and the scale of $\mathcal{N}=1$ SQCD (5.1.14) we arrive at the constraint

$$\mu \ll \mu^*\,, \tag{5.1.46}$$

where the critical value of μ is

$$g_2^2\mu^* = \frac{g_2^2\xi}{\Lambda_\sigma} = \frac{M^3_{\mathrm{SU}(N)}}{\left(\Lambda^{\mathcal{N}=1}_{\mathrm{SU}(N)}\right)^2}\,. \tag{5.1.47}$$

If the condition (5.1.46) is met, the $\mathcal{N}=2$ CP($N-1$) model gives a good description of world-sheet physics. A hierarchy of relevant scales in our theory is displayed in Fig. 5.1.

If we increase μ above the critical value (5.1.47) the non-Abelian strings become thick and their world-sheet dynamics is no longer described by $\mathcal{N}=2$ CP($N-1$) sigma model. The higher-derivative corrections on the world-sheet explode. Note that the physical reason for the growth of the string thickness Δ is the presence of the Higgs branch in $\mathcal{N}=1$ SQCD. Although the classical string solution (4.4.20) retains a finite transverse size, the Higgs branch manifests itself at the quantum level. In particular, the fermion zero modes feel the Higgs branch and acquire long-range logarithmic tails.

Now, let us abstract ourselves from the fact that the theory (5.1.19) is a low-energy effective model on the world sheet of the non-Abelian string. Let us consider

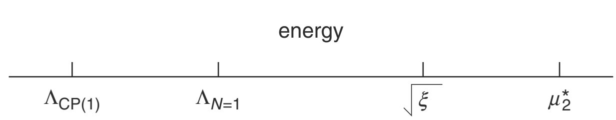
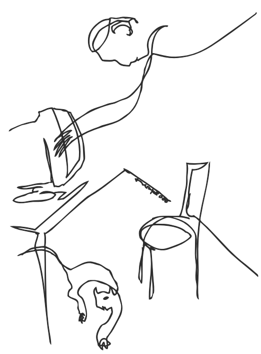


Figure 5.1. Relevant scale hierarchy in the limit $\mu \gg \sqrt{\xi}$.

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

this model *per se*, with no reference to the underlying four-dimensional theory. Then, of course, the parameter u can be viewed as arbitrary. One can address a subtle question: what happens in the limit $u \to \infty$? In this limit the σ field VEV tends to zero (see Eq. (5.1.39)) and N degenerate vacua coalesce. Moreover, the U(1) gauge field, σ and the fermionic kinks ξ become massless (in addition to the λ_R field which, being the Goldstino in this limit, is necessarily massless). The model seemingly becomes conformal. It is plausible to interpret this conformal fixed point as a phase transition point from the kink deconfinement phase to the Coulomb/confining phase.

A similar phenomenon occurs in two-dimensional conformal $\mathcal{N} = (4, 4)$ supersymmetric gauge theory [195]. In this theory the same tube metric $|d\sigma|^2/|\sigma|^2$ appears (as in (5.1.19), see [192]) and the point $\sigma = 0$ is interpreted as a transition point between two distinct phases.



5.2 The *M* model

In Section 5.1 we learned that the occurrence of the Higgs branch in $\mathcal{N} = 1$ SQCD obscures physics of the non-Abelian strings. Thus, it is highly desirable to get rid of the Higgs branch, keeping $\mathcal{N} = 1$. This was done in [104]. Below we will review key results pertinent to the issue.

To eliminate light states we will introduce a particular $\mathcal{N} = 2$ breaking deformation in the U(N) theory with the potential (5.1.2). Namely, we uplift the quark mass matrix m_A^B (see Eq. (4.1.9) where this matrix is assumed to be diagonal) to the superfield status,

$$m_A^B \to M_A^B ,$$

and introduce the superpotential

$$\mathcal{W}_M = Q M \tilde{Q} . \tag{5.2.1}$$

The matrix M represents N^2 chiral superfields of the mesonic type (they are color singlets). Needless to say, we have to add a kinetic term for M_A^B,

$$S_{M\text{kin}} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \frac{2}{h}\, \mathrm{Tr}\, \bar{M} M\,, \tag{5.2.2}$$

where h is a new coupling constant (it supplements the set of the gauge couplings). In particular, the kinetic term for the scalar components of M takes the form

$$\int d^4x \left\{ \frac{1}{h}\left|\partial_\mu M^0\right|^2 + \frac{1}{h}\left|\partial_\mu M^a\right|^2 \right\}, \tag{5.2.3}$$

where we use the decomposition

$$M_B^A = \frac{1}{2}\,\delta_B^A\, M^0 + (T^a)_B^A\, M^a\,. \tag{5.2.4}$$

At $h = 0$ the matrix field M becomes sterile, it is frozen and in essence returns to the status of a constant numerical matrix. The theory acquires flat directions (a moduli space). With nonvanishing h these flat directions are lifted, and M is determined by the minimum of the scalar potential, see below.

The uplift of the quark mass matrix to superfield is a crucial step which allows us to lift the Higgs branch which would develop in this theory in the large-μ limit if M were a constant matrix. We will refer to this theory as the M model.

The potential $V(q^A, \tilde{q}_A, a^a, a, M^0, M^a)$ of the M model is

$$\begin{aligned} V(q^A, \tilde{q}_A, a^a, a, M^0, M^a) &= \frac{g_2^2}{2}\left(\frac{1}{g_2^2}\, f^{abc}\, \bar{a}^b a^c + \mathrm{Tr}\, \bar{q}\, T^a q - \mathrm{Tr}\, \tilde{q}\, T^a\, \bar{\tilde{q}}\right)^2 \\ &+ \frac{g_1^2}{8}\left(\mathrm{Tr}\, \bar{q} q - \mathrm{Tr}\, \tilde{q}\bar{\tilde{q}} - N\xi\right)^2 + \frac{g_2^2}{2}\left|2\mathrm{Tr}\, \tilde{q} T^a q + \sqrt{2}\mu_2 a^a\right|^2 \\ &+ \frac{g_1^2}{2}\left|\mathrm{Tr}\, \tilde{q} q + \sqrt{N}\mu_1 a\right|^2 + \frac{1}{2}\,\mathrm{Tr}\left\{\left|(a + 2\,T^a\, a^a)q + \frac{1}{\sqrt{2}} q(M^0 + 2T^a M^a)\right|^2 \right. \\ &+ \left.\left|(a + 2\,T^a\, a^a)\bar{\tilde{q}} + \frac{1}{\sqrt{2}}\bar{\tilde{q}}(M^0 + 2T^a M^a)\right|^2\right\} + \frac{h}{4}|\mathrm{Tr}\, \tilde{q} q|^2 + h|\mathrm{Tr}\, q T^a \tilde{q}|^2\,. \end{aligned} \tag{5.2.5}$$

The last two terms here are F terms of the M field. In Eq. (5.2.5) we also introduced the FI D-term for the U(1) field, with the FI parameter ξ.

The FI term triggers the spontaneous breaking of the gauge symmetry. The VEV's of the squark fields and adjoint fields are given by (5.1.3) and (5.1.4), respectively, while the VEV's of M field vanish,

$$\langle M^a \rangle = 0, \quad \langle M^0 \rangle = 0\,. \tag{5.2.6}$$

The color-flavor locked form of the quark VEV's in Eq. (5.1.3) and the absence of VEVs of the adjoint field a^a and the meson fields M^a in Eqs. (5.1.4) and (5.2.6) result in the fact that, while the theory is fully Higgsed, a diagonal $\mathrm{SU}(N)_{C+F}$ symmetry survives as a global symmetry, much in the same way as in μ-deformations of $\mathcal{N}=2$ SQCD. Namely, the global rotation

$$q \to UqU^{-1}, \quad a^a T^a \to U a^a T^a U^{-1}, \quad M \to U^{-1} M U, \tag{5.2.7}$$

is not broken by the VEVs (5.1.3), (5.1.4) and (5.2.6). Here U is a matrix from SU(N). As usual, this symmetry leads to the emergence of orientational zero modes of the Z_N strings in the theory with the potential (5.2.5).

At large μ one can readily integrate out the adjoint fields $\mathcal{A}^a$ and $\mathcal{A}$. The bosonic part of the action of the M model takes the form

$$\begin{aligned} S = \int d^4x \Bigg\{ & \frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 + \mathrm{Tr}\,|\nabla_\mu q|^2 + \mathrm{Tr}\,|\nabla_\mu \bar{\tilde{q}}|^2 \\ & + \frac{1}{h}|\partial_\mu M^0|^2 + \frac{1}{h}|\partial_\mu M^a|^2 + \frac{g_2^2}{2}\left(\mathrm{Tr}\,\bar{q}\,T^a q - \mathrm{Tr}\,\tilde{q} T^a\,\bar{\tilde{q}}\right)^2 \\ & + \frac{g_1^2}{8}\left(\mathrm{Tr}\,\bar{q}q - \mathrm{Tr}\,\tilde{q}\bar{\tilde{q}} - N\xi\right)^2 + \mathrm{Tr}|qM|^2 + \mathrm{Tr}|\bar{\tilde{q}}M|^2 \\ & + \frac{h}{4}\left|\mathrm{Tr}\,\tilde{q}q\right|^2 + h\left|\mathrm{Tr}\,qT^a\tilde{q}\right|^2 \Bigg\}. \end{aligned} \tag{5.2.8}$$

The vacuum of this theory is given by Eqs. (5.1.3) and (5.2.6). The mass spectrum of elementary excitations over this vacuum consists of the $\mathcal{N}=1$ gauge multiplets for the U(1) and SU(N) sectors, with masses given in Eqs. (4.1.16) and (4.1.17). In addition, we have chiral multiplets $\tilde{q}$ and M, with masses

$$m_{\mathrm{U}(1)} = \sqrt{\frac{hN\xi}{4}} \tag{5.2.9}$$

for the U(1) sector, and

$$m_{\mathrm{SU}(N)} = \sqrt{\frac{h\xi}{2}} \tag{5.2.10}$$

for the SU(N) sector.

It is worth emphasizing that there are no massless states in the bulk theory. At $h=0$ the theory with the potential (5.2.5) develops a Higgs branch in the large-μ limit (see Section 5.1). If $h \neq 0$, M becomes a fully dynamical field. The Higgs branch is lifted, as follows from Eqs. (5.2.9) and (5.2.10).

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

The $\mathcal{N} = 1$ SQCD with the M field, the M model, belongs to the class of theories introduced by Seiberg [196] to provide a dual description of conventional $\mathcal{N} = 1$ SQCD with the SU(N_c) gauge group and N_f flavors of fundamental matter, where

$$N_c = N_f - N$$

(for reviews see Refs. [197, 198]). There are significant distinctions, however.

Let us outline the main differences of the M model (5.2.8) from those introduced [196] by Seiberg:

(i) The theory (5.2.8) has the U(N) gauge group rather than SU(N);

(ii) It has the FI D term instead of a linear in M superpotential in Seiberg's models;

(iii) Following [104] we consider the case $N_f = N$ which would correspond to Seiberg's $N_c = 0$ in the original SQCD. The theory (5.2.8) is asymptotically free, while Seiberg's dual theories are most meaningful (i.e. have the maximal predictive power with regards to the original strongly coupled $\mathcal{N} = 1$ SQCD) below the left edge of the conformal window, in the range $N_f < (3/2)\, N_c$, which would correspond to $N_f > 3N$ rather than $N_f = N$. Note that at $N_f > 3N$ the theory (5.2.8) is not asymptotically free and is thus uninteresting from our standpoint.

In addition, it is worth noting that at $N_f > N$ the vacuum (5.1.3), (5.2.6) becomes metastable: supersymmetry is broken [199]. The $N_c = N_f - N$ supersymmetry-preserving vacua have vanishing VEV's of the quark fields and a non-vanishing VEV of the M field.[3] The latter vacua are associated with the gluino condensation in pure SU(N) theory, $\langle\lambda\lambda\rangle \neq 0$, arising upon decoupling N_f flavors [197]. In the case $N_f = N$ to which we limit ourselves the vacuum (5.1.3), (5.2.6) preserves supersymmetry. Thus, despite a conceptual similarity between Seiberg's models and ours, dynamical details are radically different.

Now, it is time to pass to solutions for non-Abelian BPS strings in the M model [104]. Much in the same way as in Section 5.1 we use the *ansatz* (5.1.16). Moreover, we set the adjoint fields and the M fields to zero. With these simplifications the $\mathcal{N} = 1$ model with the potential (5.2.5) reduces to the model (4.2.2) which we used previously in the original construction of the non-Abelian strings.

In particular, the solution for the elementary string is given by (4.4.4). Moreover, the bosonic part of the effective world-sheet theory is again described by the CP($N-1$) sigma model (4.4.9) with the coupling constant β determined by (4.4.15). The scale of this CP($N-1$) model is given by Eq. (5.1.18) in the limit of large μ.

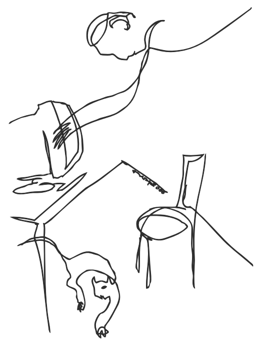
The full construction of the world-sheet theory in the M model has not been yet carried out. One can conjecture as to what the fermion part of this theory is. There are good reasons to expect that we will get the heterotic $\mathcal{N} = (0, 2)$ CP($N-1$)

[3] This is correct for the version of the theory with the ξ-parameter introduced via superpotential.

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

theory much in the same way as in Section 5.1.2 (see also Appendix B.5). The relation between the bulk and world sheet deformation parameters are likely to change, but all consequences (such as spontaneous SUSY breaking at the quantum level) presumably will stay intact.

To conclude this section let us note a somewhat related development: *non*-BPS non-Abelian strings were considered in metastable vacua of a dual description of $\mathcal{N} = 1$ SQCD at $N_f > N$ in Ref. [200].



5.3 Confined non-Abelian monopoles

As was mentioned, the effective low-energy Lagrangian describing world-sheet physics of the non-Abelian string in the M model, must be supersymmetric, presumably, $\mathcal{N} = (0, 2)$. The heterotic sigma model dynamics is known (see Section 5.1.3); in particular, we will have N degenerate vacua and kinks that interpolate between them, similar to the kinks that emerge in $\mathcal{N} = 2$ SQCD. These kinks are interpreted as (confined) non-Abelian monopoles [165, 132, 133], the descendants of the 't Hooft–Polyakov monopole.

Let us discuss what happens with these monopoles as we deform our theory and eventually end up with the M model. It is convenient to split this deformation into several distinct stages. We will describe what happens with the monopoles as one passes from one stage to another. Some of these steps involving deformations of $\mathcal{N} = 2$ SQCD were already discussed in Section 4.5. Here we focus on deformations of $\mathcal{N} = 2$ SQCD leading to the M model.

A qualitative evolution of the monopoles under consideration as a function of the relevant parameters is presented in Fig. 4.3.

(i) We start from $\mathcal{N} = 2$ SQCD turning off the $\mathcal{N} = 2$ breaking parameters h and μ's as well as the FI parameter in the potential (5.2.5), i.e. we start from the Coulomb branch of the theory,

$$\mu_1 = \mu_2 = 0, \quad h = 0, \quad \xi = 0, \quad M \neq 0\,. \tag{5.3.1}$$

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

As was explained in Section 5.2, the field M is frozen in this limit and can take arbitrary values (the notorious flat direction). The matrix M^A_B plays the role of a fixed mass matrix for the quark fields. As a first step let us consider the diagonal matrix M, with distinct diagonal entries,

$$M^A_B = \text{diag}\{M_1, \ldots, M_N\}. \tag{5.3.2}$$

Shifting the field a one can always make $\sum_A M_A = 0$ in the limit $\mu_1 = 0$. Therefore $M^0 = 0$. If all M_A's are different the gauge group SU(N) is broken down to U(1)$^{(N-1)}$ by a VEV of the SU(N) adjoint scalar (see (4.1.11)),

$$\langle a^k_l \rangle = -\frac{1}{\sqrt{2}}\, \delta^k_l M_l\,. \tag{5.3.3}$$

Thus, as was already discussed in Section 4.5, there are 't Hooft–Polyakov monopoles embedded in the broken gauge SU(N). Classically, on the Coulomb branch the masses of $(N-1)$ elementary monopoles are proportional to

$$|(M_A - M_{A+1})|/g_2^2\,.$$

In the limit $(M_A - M_{A+1}) \to 0$ the monopoles tend to become massless, formally, in the classical approximation. Simultaneously their size becomes infinite [112]. The mass and size are stabilized by highly quantum confinement effects. The monopole confinement occurs in the Higgs phase, at $\xi \neq 0$.

(ii) Now let us make the FI parameter ξ non-vanishing. This triggers the squark condensation. The theory is in the Higgs phase. We still keep $\mathcal{N} = 2$ breaking parameters h and μ's vanishing,

$$\mu_1 = \mu_2 = 0, \quad h = 0, \quad \xi \neq 0, \quad M \neq 0. \tag{5.3.4}$$

The squark condensation leads to formation of the Z_N strings. Monopoles become confined by these strings. As we discussed in Section 4.5, $(N-1)$ elementary monopoles become junctions of pairs of elementary strings.

Now, if we reduce $|\Delta M_A|$,

$$\Lambda_{\text{CP}(N-1)} \ll |\Delta M_A| \ll \sqrt{\xi}\,, \tag{5.3.5}$$

the size of the monopole along the string $\sim |(M_A - M_{A+1})|^{-1}$ becomes larger than the transverse size of the attached strings. The monopole becomes a *bona fide* confined monopole (the lower left corner of Fig. 4.3). At nonvanishing ΔM_A the effective theory on the string world sheet is the CP($N-1$) model with twisted mass terms [165, 132, 133], see Section 4.4.4. Two Z_N strings attached to an elementary monopole correspond to two "neighboring" vacua of the CP($N-1$) model. The monopole (a.k.a. the string junction of two Z_N strings) manifests itself as a kink interpolating between these two vacua.

(iii) Next, we switch off the mass differences ΔM_A still keeping the $\mathcal{N}=2$ breaking parameters vanishing,

$$\mu_1 = \mu_2 = 0, \quad h = 0, \quad \xi \neq 0, \quad M = 0\,. \tag{5.3.6}$$

The values of the twisted masses in CP($N-1$) model coincide with ΔM_A while the size of the twisted-mass sigma-model kink/confined monopole is of the order of $\sim |(M_A - M_{A+1})|^{-1}$.

As we decrease ΔM_A approaching $\Lambda_{\mathrm{CP}(N-1)}$ and then getting below the scale $\Lambda_{\mathrm{CP}(N-1)}$, the monopole size grows, and, classically, it would explode. This is where quantum effects in the world sheet theory take over. It is natural to refer to this domain of parameters as the "regime of highly quantum dynamics." While the thickness of the string (in the transverse direction) is $\sim \xi^{-1/2}$, the z-direction size of the kink representing the confined monopole in the highly quantum regime is much larger, $\sim \Lambda^{-1}_{\mathrm{CP}(N-1)}$, see the lower right corner in Fig. 4.3.

In this regime the confined monopoles become non-Abelian. They no longer carry average magnetic flux since

$$\langle n^l \rangle = 0\,, \tag{5.3.7}$$

in the strong coupling limit of the CP($N-1$) model [159]. The kink/monopole belongs to the fundamental representation of the SU$(N)_{C+F}$ group [159, 120].

(iv) Thus, with vanishing ΔM_A we still have confined "monopoles" (interpreted as kinks) stabilized by quantum effects in the world sheet CP($N-1$) model. Now we can finally switch on the $\mathcal{N}=2$ breaking parameters μ_i and h,

$$\mu_i \neq 0, \quad h \neq 0, \quad \xi \neq 0, \quad M = 0\,. \tag{5.3.8}$$

Note that the last equality here is automatically satisfied in the vacuum, see Eq. (5.2.6).

As was discussed in Section 5.2 the effective world sheet description of the non-Abelian string is given by a heterotic deformation of the supersymmetric CP($N-1$) model. This two-dimensional theory has N vacua which should be interpreted as N elementary non-Abelian strings in the quantum regime, with kinks interpolating between these vacua. These kinks should be interpreted as non-Abelian confined monopoles/string junctions.

Note that although the adjoint fields are still present in the theory (5.2.5) their VEV's vanish (see (5.1.4)) and the monopoles cannot be seen in the semiclassical

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

approximation. They are seen solely as world sheet kinks. Their mass and inverse size are determined by Λ_σ which in the limit of large μ_i is given by Eq. (5.1.18).

(v) At the last stage, we take the limit of large masses of the adjoint fields in order to eliminate them from the physical spectrum altogether,

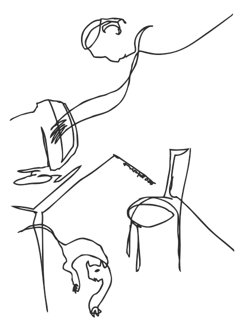
$$\mu_i \to \infty, \qquad h \neq 0, \qquad \xi \neq 0, \qquad M = 0\,. \tag{5.3.9}$$

The theory flows to $\mathcal{N} = 1$ SQCD extended by the M field.

In this limit we get a remarkable result: although the adjoint fields are eliminated from our theory and the monopoles cannot be seen in any semiclassical description, our analysis shows that confined non-Abelian monopoles still exist in the theory (5.2.8). They are seen as kinks in the effective world sheet theory on the non-Abelian string.

(vi) The confined monopoles are in the highly quantum regime, so they carry no average magnetic flux (see Eq. (5.3.7)). Thus, they are non-Abelian. Moreover, they acquire global flavor quantum numbers. In fact, they belong to the fundamental representation of the global $\mathrm{SU}(N)_{C+F}$ group (see Refs. [159, 120] where this phenomenon is discussed in the context of the $\mathrm{CP}(N-1)$-model kinks).

It is quite plausible that the emergence of these non-Abelian monopoles can shed light on mysterious objects introduced by Seiberg: "dual magnetic" quarks which play an important role in the description of $\mathcal{N} = 1$ SQCD at strong coupling [196, 197].



5.4 Index theorem

In this section we will discuss an index theorem establishing the number of the fermion zero modes on the string. For definiteness we will consider the M model [104]. Similar theorems can be easily proved for ordinary $\mathcal{N} = 1$ SQCD (5.1.13) as well as for theories with potentials (5.1.2) or (5.2.5) at intermediate values of μ. They generalize index theorems obtained long ago for simple non-supersymmetric models [201].

https://doi.org/10.1017/9781009402200.006 Published online by Cambridge University Press

The fermionic part of the action of the model (5.2.8) is

$$
\begin{aligned}
S_{\text{ferm}} = \int d^4x \Bigg\{ & \frac{i}{g_2^2}\bar{\lambda}^a \bar{\not{D}}\lambda^a + \frac{i}{g_1^2}\bar{\lambda}\bar{\not{\partial}}\lambda + \mathrm{Tr}\big[\bar{\psi} i \bar{\not{\nabla}}\psi\big] + \mathrm{Tr}\big[\tilde{\psi} i \not{\nabla}\bar{\tilde{\psi}}\big] \\
& + \frac{2i}{h}\mathrm{Tr}\big[\bar{\zeta}\bar{\not{\partial}}\zeta\big] + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\big[\bar{q}(\lambda\psi) - (\tilde{\psi}\lambda)\bar{\tilde{q}} + (\bar{\psi}\bar{\lambda})q - \tilde{q}(\bar{\lambda}\bar{\tilde{\psi}})\big] \\
& + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\big[\bar{q}\, 2T^a\,(\lambda^a\psi) - (\tilde{\psi}\lambda^a)\, 2T^a\,\bar{\tilde{q}} + (\bar{\psi}\bar{\lambda}^a)\, 2T^a\, q - \tilde{q}\, 2T^a\,(\bar{\lambda}^a\bar{\tilde{\psi}})\big] \\
& + i\,\mathrm{Tr}\big[\tilde{q}(\psi\zeta) + (\tilde{\psi}q\zeta) + (\bar{\psi}\bar{\tilde{q}}\bar{\zeta}) + \bar{q}(\bar{\tilde{\psi}}\bar{\zeta})\big] \\
& + i\,\mathrm{Tr}\big(\tilde{\psi}\psi M + \bar{\psi}\bar{\tilde{\psi}}\bar{M}\big)\Bigg\},
\end{aligned} \tag{5.4.1}
$$

where the matrix color-flavor notation is used for the matter fermions $(\psi^\alpha)^{kA}$ and $(\tilde{\psi}^\alpha)_{Ak}$, and the traces are performed over the color-flavor indices. Moreover, ζ denotes the fermion component of the matrix M superfield,

$$
\zeta^A_B = \frac{1}{2}\delta^A_B\,\zeta^0 + (T^a)^A_B\,\zeta^a\,. \tag{5.4.2}
$$

In order to find the number of the fermion zero modes in the background of the non-Abelian string solution (4.4.4) we have to carry out the following program. Since our string solution depends only on two coordinates x_i $(i = 1, 2)$, we can reduce our theory to two dimensions. Given the theory defined on the (x_1, x_2) plane we have to identify an axial current and derive the anomalous divergence for this current. In two dimensions the axial current anomaly takes the form

$$
\partial_i\, j_{i5} \sim F^*\,, \tag{5.4.3}
$$

where $F^* = (1/2)\varepsilon_{ij}F_{ij}$ is the dual U(1) field strength in two dimensions.

Then, the integral over the left-hand side over the (x_1, x_2) plane gives us the index of the 2D Dirac operator ν coinciding with the number of the 2D left-handed minus 2D right-handed zero modes of this operator in the given background field. The integral over the right-hand side is proportional to the string flux. This will fix the number of the chiral fermion zero modes[4] on the string with the given flux. Note that the reduction of the theory to two dimensions is an important step in this program. The anomaly relation in four dimensions involves the instanton charge F^*F rather than the string flux and is therefore useless for our purposes.

[4] Chirality is understood here as the two-dimensional chirality.

Table 5.1. *The* $\mathrm{U}(1)_R$ *and* $\mathrm{U}(1)_{\tilde{R}}$ *charges of fields of the two-dimensional reduction of the theory.*

Field	ψ_+	ψ_-	$\tilde{\psi}_+$	$\tilde{\psi}_-$	λ_+	λ_-	ζ_+	ζ_-	q	$\tilde{q}$
$\mathrm{U}(1)_R$ charge	-1	1	-1	1	-1	1	-1	1	0	0
$\mathrm{U}(1)_{\tilde{R}}$ charge	-1	1	1	-1	-1	1	1	-1	0	0

The reduction of $\mathcal{N} = 1$ gauge theories to two dimensions is discussed in detail in [157] and here we will be brief. Following [157] we use the rules

$$\begin{aligned} &\psi^\alpha \to (\psi^-, \psi^+), \quad \tilde{\psi}^\alpha \to (\tilde{\psi}^-, \tilde{\psi}^+), \\ &\lambda^\alpha \to (\lambda^-, \lambda^+), \quad \zeta^\alpha \to (\zeta^-, \zeta^+). \end{aligned} \tag{5.4.4}$$

With these rules the Yukawa interactions in (4.4.22) take the form

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} = {} & i\sqrt{2}\,\mathrm{Tr}\big[-\bar{q}(\hat{\lambda}_-\psi_+ - \hat{\lambda}_+\psi_-) + (\tilde{\psi}_-\hat{\lambda}_+ - \tilde{\psi}_+\hat{\lambda}_-)\bar{\tilde{q}} + \text{H.c.}\big] \\ & - i\,\mathrm{Tr}\big[\tilde{q}(\psi_-\zeta_+ - \psi_+\zeta_-) + (\tilde{\psi}_- q\zeta_+ - \tilde{\psi}_+ q\zeta_-) + \text{H.c.}\big], \end{aligned} \tag{5.4.5}$$

where the color matrix $\hat{\lambda} = (1/2)\,\lambda + T^a\lambda^a$.

It is easy to see that $\mathcal{L}_{\text{Yukawa}}$ is classically invariant under the chiral $\mathrm{U}(1)_R$ transformations with the $\mathrm{U}(1)_R$ charges presented in Table 5.1. The axial current associated with this $\mathrm{U}(1)_R$ is not anomalous [157]. This is easy to understand. In two dimensions the chiral anomaly comes from the diagram shown in Fig. 5.2. The $\mathrm{U}(1)_R$ chiral charges of the fields ψ and $\tilde{\psi}$ are the same while their electric charges are opposite. This leads to cancellation of their contributions to this diagram.

It turns out that for the particular string solution we are interested in the classical two-dimensional action has more symmetries than generically, for a general background. To see this, please note that the field $\tilde{q}$ vanishes on the string solution (4.4.4), see (5.1.16). Then the Yukawa interactions (5.4.5) reduce to

$$i\sqrt{2}\,\mathrm{Tr}\big[-\bar{q}(\hat{\lambda}_-\psi_+ - \hat{\lambda}_+\psi_-)\big] - i\,\mathrm{Tr}\big[\tilde{\psi}_- q\zeta_+ - \tilde{\psi}_+ q\zeta_-\big] + \text{H.c.} \tag{5.4.6}$$

The fermion ψ interacts only with λ's while the fermion $\tilde{\psi}$ interacts only with ζ. Note also that the interaction in the last line in (5.4.1) is absent because $M = 0$ on the string solution. This property allows us to introduce another chiral symmetry in the theory, the one which is relevant for the string solution. We will refer to this extra chiral symmetry as $\mathrm{U}(1)_{\tilde{R}}$.

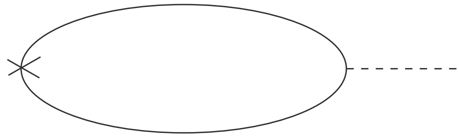


Figure 5.2. Diagram for the chiral anomaly in two dimensions. The solid lines denote fermions ψ, $\tilde{\psi}$, the dashed line denotes the photon, while the cross denotes insertion of the axial current.

The $U(1)_{\tilde{R}}$ charges of our set of fields are also shown in Table 5.1. Note that ψ and $\tilde{\psi}$ have the opposite charges under this symmetry. The corresponding current then has the form

$$\tilde{j}_{i5} = \begin{pmatrix} \bar{\psi}_-\psi_- - \bar{\psi}_+\psi_+ - \bar{\tilde{\psi}}_-\tilde{\psi}_- + \bar{\tilde{\psi}}_+\tilde{\psi}_+ + \cdots \\ -i\,\bar{\psi}_-\psi_- - i\,\bar{\psi}_+\psi_+ + i\,\bar{\tilde{\psi}}_-\tilde{\psi}_- + i\,\bar{\tilde{\psi}}_+\tilde{\psi}_+ + \cdots \end{pmatrix}, \tag{5.4.7}$$

where the ellipses stand for terms associated with the λ and ζ fields which do not contribute to the anomaly relation.

It is clear that the $U(1)_{\tilde{R}}$ symmetry is anomalous in quantum theory. The contributions of the fermions ψ and $\tilde{\psi}$ double in the diagram in Fig. 5.2 rather than cancel. It is not difficult to find the coefficient in the anomaly formula

$$\partial_i \tilde{j}_{i5} = \frac{N^2}{\pi} F^* \,, \tag{5.4.8}$$

which can be normalized e.g. from [202]. The factor N^2 appears due to the presence of $2N^2$ fermions ψ^{kA} and $\tilde{\psi}_{Ak}$.

Now, taking into account the fact that the flux of the Z_N string under consideration is

$$\int d^2x\, F^* = \frac{4\pi}{N} \,, \tag{5.4.9}$$

(see the expression for the U(1) gauge field for the solution (4.2.6) or (4.4.4)) we conclude that the total number of the fermion zero modes in the string background[5]

$$\nu = 4N \,. \tag{5.4.10}$$

This number can be decomposed as

$$\nu = 4N = 4(N-1) + 4 \,, \tag{5.4.11}$$

[5] Equations (5.4.9) and (5.4.10) are very similar in essence to analogous four-dimensional relations connecting the instanton topological charge with the number of the fermion zero modes in the instanton background. For a review see [203].

where 4 is the number of the supertranslational modes while $4(N-1)$ is the number of the superorientational modes. Four supertranslational modes are associated with four fermion fields in the two-dimensional effective theory on the string world sheet, which are superpartners of the bosonic translational moduli x_0 and y_0. Furthermore, $4(N-1)$ corresponds to $4(N-1)$ fermion fields in the $\mathcal{N}=2$ CP$(N-1)$ model on the string world sheet. CP$(N-1)$ describes dynamics of the orientational moduli of the string. For $N=2$ the latter number $(4(N-1)=4)$ counts four fermion fields χ_1^a, χ_2^a in the model (4.4.31).

A similar theorem can be formulated for $\mathcal{N}=1$ theory with the potential (5.1.2) as well; it implies $4(N-1)$ orientational zero modes in this case too, i.e. the doubling of the number of the fermion zero modes on the string as compared with the one which follows from "BPS-ness."

In [189] and [104] four orientational fermion zero modes were found explicitly in $\mathcal{N}=1$ SQCD and the M model, by solving the Dirac equations in the string background. Note that these fermion zero modes in the M model are perfectly normalizable provided we keep the coupling constant h non-vanishing. Instead, in conventional $\mathcal{N}=1$ SQCD without the M field the second pair of the fermion zero modes (proportional to χ_1^a) become non-normalizable in the large-μ limit [189]. This is related to the presence of the Higgs branch and massless bulk states in conventional $\mathcal{N}=1$ SQCD. As was already mentioned more than once, in the M model, Eq. (5.2.8), we have no massless states in the bulk.

Note that in both translational and orientational sectors the number of the fermion zero modes is twice larger than the one dictated by 1/2 "BPS-ness." Fermion supertranslational zero modes of the non-Abelian string in $\mathcal{N}=1$ theory with the potential (5.1.2) were found in [191]. Just like superorientational modes, they acquire long-range tails in the large-μ limit and become non-normalizable.