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RESEARCH ARTICLE

Arithmetic Transfer for inner forms of GL_{2n}

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Abstract
We formulate Guo–Jacquet type fundamental lemma conjectures and arithmetic transfer conjectures for inner forms of GL_{2n}. Our main results confirm these conjectures for division algebras of invariant $1/4$ and $3/4$.

Contents

1. Introduction

1.1. Background

Even though our paper will be purely local in nature, we begin by describing its global motivation. Let F be a totally real number field, let K/F be a quadratic CM extension, and let D/K be a central simple algebra (CSA) of degree $2n$. Let $* : D \to D$ be an involution of the second kind. Recall that this means that $(xy)^* = y^*x^*$ and that $*$ restricts to the complex conjugation on K. We further assume that $*$ is positive in the sense that $\mathrm{trd}_{D/\mathbb{Q}}(x^*x) > 0$ for all $x \neq 0$. Let $\beta \in D^\times$ be a skew-hermitian element, meaning that $\beta^* = -\beta$. Then the algebraic group (over $\mathbb{Q}$)

$$G := \{x \in D^{\mathrm{op},\times} \mid x^*\beta x = \nu(x)\beta \text{ for some } \nu(x) \in \mathbb{G}_m\} \tag{1.1}$$

is an inner form of a unitary similitude group in $2n$ variables and the above data can be completed into a PEL type Shimura datum (G, X_G). The resulting Shimura variety Sh_G can be described as a moduli space of polarized abelian varieties with D-action and level structure.

For example, we may take $(D, *)$ as the matrix algebra $M_{2n}(K)$ with transpose conjugation $x^* = \bar{x}^t$. Then $\beta \in GL_{2n}(K)$ can be any skew-hermitian matrix and G is the corresponding unitary similitude group $GU(K^{2n}, \beta)$.

Coming back to the general situation, we assume that the signatures of β are $(2n-1,1)$ at a unique archimedean place and $(2n,0)$ at all others. Then Sh_G is of dimension $2n-1$, and our interest lies in algebraic cycles in the arithmetic middle dimension $n-1 = \lfloor (2n-1)/2 \rfloor$. By work of Li–Liu [25, 26] for $D = M_{2n}(K)$, it is known in many cases that the height pairings of such cycles are related to the leading terms of the Taylor expansions of certain L-functions. This relation can be understood as a generalization of the Gross–Zagier formula [14, 43] to higher dimensions. It is moreover parallel to the arithmetic Gan–Gross–Prasad conjectures [12], and it also represents an instance of the Beilinson–Bloch height conjectures [3, 4].

We are thus lead to the problem of constructing algebraic cycles in arithmetic middle dimension on Sh_G, and of relating their height pairings to the Taylor expansions of L-functions. One construction of such cycles, going back to W. Zhang, is given by imposing additional quadratic multiplication. The resulting cycles differ from the ones in [25, 26] which are instead closely related to Kudla–Rapoport divisors. We next describe this construction in more detail:

Let E/F be a totally real quadratic extension and let $E \hookrightarrow D$ be an F-linear embedding such that $x^* = x$ for all $x \in E$. The centralizer $C = \mathrm{Cent}_D(E)$ then has center EK and is preserved by $*$. If we further choose these data such that β lies in $C^\times$, then we can define the algebraic group (over $\mathbb{Q}$)

$$H := \{x \in C^{\mathrm{op},\times} \mid x^*\beta x = \nu(x)\beta \text{ for some } \nu(x) \in \mathbb{G}_m\}. \tag{1.2}$$

This is an inner form of a unitary similitude group in n variables for the quadratic extension EK/E. One may always find a PEL type Shimura datum X_H for H such that $(H, X_H) \to (G, X_G)$ is a morphism of Shimura data. Then we obtain a closed immersion $\mathrm{Sh}_H \hookrightarrow \mathrm{Sh}_G$ of Shimura varieties. Moreover, Sh_H is of the desired middle dimension $n-1$.

The existence of the Beilinson–Bloch height pairing is still conjectural. It is expected, however, that it decomposes into a sum of local heights whose non-archimedean terms are closely related to intersection numbers on integral models. For this reason, we consider integral models $\widetilde{\mathrm{Sh}}_H \hookrightarrow \widetilde{\mathrm{Sh}}_G$ of the given Shimura varieties. Our interest then lies in the intersection numbers $I(f) = \langle \widetilde{\mathrm{Sh}}_H,\ f * \widetilde{\mathrm{Sh}}_H\rangle$ for varying Hecke operators $f \in C_c^\infty(G(\mathbb{A}_f))$, and their relations with leading terms of L-functions.

In this context, there is a relative trace formula (RTF) comparison approach due to Leslie–Xiao–Zhang [24] that goes back to work of Guo [13] and Friedberg–Jacquet [11]: One RTF is formulated for the given pair of groups $(G,\ H)$, and the other one is formulated for the pair $(G',\ H')$ defined as $(GL_{2n},\ GL_n \times GL_n)$. The significance of the pair $(G',\ H')$ is that the L-functions of interest occur on the spectral side of its RTF. The problem thus becomes to relate the intersection numbers $I(f)$ with $(H' \times H')(\mathbb{A})$-orbital integrals on $G'(\mathbb{A})$.

This relation is made precise by factoring the global orbital integral into local ones and analogously decomposing the global intersection number into place-by-place contributions. The latter relies on Rapoport–Zink (RZ) uniformization. One then, finally, arrives at a purely local question – namely, that of expressing intersection numbers on moduli spaces of p-divisible groups (RZ spaces) in terms of local orbital integrals.

These ideas have recently lead to the discovery of new arithmetic fundamental lemma (AFL) identities: the linear AFL of the first author [27], the variants from his joint work with B. Howard [19] and with the second author [30], and the AFL for unitary groups of Leslie–Xiao–Zhang [24]. The terminology 'AFL' here refers to the fact that all these are identities of intersection numbers on RZ spaces with *good reduction* and central derivatives of local orbital integrals for *spherical* Hecke functions. Put differently, they concern the places of good reduction of Sh_H and Sh_G.

We mention that the above ideas were first proposed by W. Zhang in the context of the unitary arithmetic Gan–Gross–Prasad conjecture [45]. He deduced an AFL for unitary groups that he later proved with contributions of the second author and Z. Zhang [48, 31, 32, 49]. We refer to his ICM report [47] for a survey.

Arithmetic transfer (AT) identities extend the realm of AFL identities in the sense that they express intersection numbers on RZ spaces with *bad reduction* in terms of orbital integrals. Their role in

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the global setting is similar to that of AFL identities, but for the places of bad reduction of the Shimura varieties in question. This idea was first studied systematically by Rapoport–Smithling–Zhang [34, 35, 36] in the context of the unitary arithmetic Gan–Gross–Prasad conjecture. Z. Zhang [49] generalized and solved one of their cases by proving AT identities in arbitrary dimension for maximal parahoric level for unramified quadratic extensions. His result has found global applications in the work of Disegni–Zhang [8] who prove a p-adic variant of the arithmetic Gan–Gross–Prasad conjecture. Another application of AT in the global setting can be found in the work of C. Qiu [33], who proved AT identities for all places and level structures to show an analog of the Gross–Zagier formula over function fields.

1.2. *Arithmetic transfer for central simple algebras*

Consider again the above intersection numbers $I(f) = \langle \widetilde{\mathrm{Sh}_H}, f * \widetilde{\mathrm{Sh}_H} \rangle$. The AT identities in the present paper provide an expression for the contributions to $I(f)$ from places of F that are split in K and inert in E. The completions of G and H at such places are essentially *inner* forms of general linear groups. Correspondingly, we consider an intersection problem of EL type RZ spaces for central simple algebras (CSA). It is worth singling out two special cases:

If the CSAs in question split, then the resulting moduli spaces are Lubin–Tate spaces, which leads to the intersection problem of the linear AFL mentioned above. If, however, the Hasse invariants of the CSAs in question are $1/2n$ and $1/n$, then the intersection problem is formulated for Drinfeld's half spaces and the cycles arise from Drinfeld's 'basic construction' [10, §3]. This situation comes up when the two Shimura varieties have p-adic uniformization [38, §6.40].

We will now give a precise formulation of our AT conjecture and our results.

1.3. *The fundamental lemma conjecture*

Let F be a p-adic local field,[1] let E/F be an unramified quadratic field extension and let $\eta : F^\times \longrightarrow \{\pm 1\}$ be the corresponding quadratic character. Let $G' = GL_{2n}(F)$ with subgroup $H' = GL_n(F \times F)$. For a test function $f' \in C_c^\infty(G')$, an element $\gamma \in G'$ that is regular semi-simple with respect to the $(H' \times H')$-action, and for a complex parameter $s \in \mathbb{C}$, we consider Guo's [13] orbital integral

$$\mathrm{Orb}(\gamma, f', s) = \Omega(\gamma, s) \int_{\frac{H' \times H'}{(H' \times H')_\gamma}} f'(h_1^{-1} \gamma h_2) |h_1 h_2|^s \eta(h_2)\, dh_1 dh_2. \tag{1.3}$$

Here, the so-called transfer factor $\Omega(\gamma, s) \in \pm q^{\mathbb{Z}s}$ is defined in a way that ensures that $\mathrm{Orb}(\gamma, f', s)$ only depends on the double coset $H'\gamma H'$; detailed definitions will be given in §3. Our interest lies in the central value and the first derivative which we denote by

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0), \qquad \partial\mathrm{Orb}(\gamma, f') = \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s).$$

Let D/F be a CSA of degree $2n$ and let $E \subset D$ be a fixed embedding. The centralizer $C = \mathrm{Cent}_D(E)$ is again a CSA but over E. The Hasse invariants of D and C are related by $\mathrm{inv}_E(C) = 2\,\mathrm{inv}_F(D)$ (combine [9, Corollary 9.1] with [39, Proposition XIII.7]). Define $(G, H) = (D^{\mathrm{op},\times}, C^{\mathrm{op},\times})$. The reason for passing to the opposite algebra here is that D^{op} is isomorphic to the ring of left endomorphisms of D as left D-module. Given a test function $f \in C_c^\infty(G)$ and an element $g \in G$ that is regular semi-simple with respect to the $(H \times H)$-action, we consider the orbital integral

$$\mathrm{Orb}(g, f) := \int_{\frac{H \times H}{(H \times H)_g}} f(h_1^{-1} g h_2) dh_1 dh_2. \tag{1.4}$$

[1] We allow local function fields in the main text with one exception: Section 13.6 relies on the theory of O_F-displays from [1] which has only been developed for p-adic local fields.

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It evidently only depends on the double coset HgH. Let $O_D \subseteq D$ be a maximal order such that $O_C := C \cap O_D$ is a maximal order of C and put $f_D = 1_{O_D^\times}$. One can show that all such orders O_D form a single H-conjugation orbit (Lemma 3.6), so the orbital integrals $\mathrm{Orb}(g, f_D)$ do not depend on the choice of O_D. In dependence on the order ℓ of D in the Brauer group of F, we will define a specific parahoric subgroup $K' \subseteq GL_{2n}(O_F)$. It corresponds to a parabolic of $(2n/\ell \times 2n/\ell)$-block upper triangular matrices and is chosen compatibly with the subgroup $H' \subseteq G'$. Let $f_D'^{\circ} = \mathrm{vol}(K' \cap H')^{-2} \cdot 1_{K'}$ and define f_D' as a certain H'-translate of $f_D'^{\circ}$; see Definition 3.9 for details. The orbital integrals $O(\gamma, f_D', s)$ have a simple functional equation with respect to $s \longleftrightarrow -s$ (Proposition 3.19).

Two regular semi-simple elements $\gamma \in G'$ and $g \in G$ are said to match if the invariants (in the sense of geometric invariant theory) of the orbits $H'\gamma H'$ and HgH agree. This notion defines an injection $[G_{\mathrm{rs}}] \hookrightarrow [G'_{\mathrm{rs}}]$ of the regular semi-simple orbits of G into those of G', which allows to compare orbital integrals on the two groups. Assuming all Haar measures are chosen compatibly, we conjecture that f_D' is a smooth transfer of f_D in the following sense:

Conjecture 1.1 (Fundamental Lemma, Conjecture 3.10). *Let $\gamma \in G'$ be an element that is regular semi-simple with respect to the $(H' \times H')$-action. Then*

$$\mathrm{Orb}(\gamma, f_D') = \begin{cases} \mathrm{Orb}(g, f_D) & \text{if there exists an element } g \in G \text{ that matches } \gamma \\ 0 & \text{if there is no such } g. \end{cases} \tag{1.5}$$

The theorem about existence of smooth transfer of C. Zhang and H. Xue [44, 41] already states that there exists *some* test function $f' \in C_c^\infty(G')$ such that $\mathrm{Orb}(\gamma, f')$ is given by the right-hand side of (1.5) for all γ. The new aspect of Conjecture 1.1 is that it provides f_D' as a specific candidate. The choice of f_D' is motivated by Z. Zhang's transfer result [49]. We explain this in more detail in Remark 3.14.

Conjecture 1.1 specializes to the Guo–Jacquet fundamental lemma (FL) from [13] if $D = M_{2n}(F)$. By work of N. Hultberg and the second author [20, Theorem A], Conjecture 1.1 is also known if the Hasse invariant of D is $1/2$: The proof in this case is by reduction to the base change FL and to the Guo–Jacquet FL. Our main result on Conjecture 1.1 in the present paper is as follows.

Theorem 1.2 (Theorem 8.2). *Assume that D is a division algebra of degree* 4. *Then Conjecture 1.1 holds.*

Note that if D is a division algebra of degree $2n$, then $\mathrm{Orb}(g, f_D)$ is either 0 or an integer divisor of n (Proposition 3.13). In the case of Theorem 1.2, for example, $\mathrm{Orb}(\gamma, f_D')$ is either 0 or 1 (Theorem 8.2).

1.4. The arithmetic transfer conjecture

Denote by $\breve{F}$ the completion of a maximal unramified extension of F and fix an embedding $E \subset \breve{F}$. Choose an isomorphism $\breve{F} \otimes_F C^{\mathrm{op}} \cong M_n(\breve{F} \times \breve{F})$ that restricts to $(\mathrm{id}, \tau) \otimes 1_n$ on the center E of C, where $\tau : E \to E$ is the Galois conjugation. Consider the conjugacy class of the 'Drinfeld type' minuscule cocharacter

$$\mu_H : \mathbb{G}_m \longrightarrow H_{\breve{F}} \cong (GL_n \times GL_n)_{\breve{F}}, \quad t \longmapsto ((t, \dots, t, 1), (t, \dots, t)).$$

Let $\mu_G : \mathbb{G}_m \to G_{\breve{F}}$ denote its composition with $H \to G$. Then every element $b \in B(H, \mu_H)$ defines a morphism

$$(H, b, \mu_H) \longrightarrow (G, b, \mu_G) \tag{1.6}$$

of local Shimura data in the sense of [37]. We denote by $H_b \to G_b$ the automorphism groups of the H-isocrystal (resp. G-isocrystal) defined by b. In analogy with our definition of f_D as the characteristic function of $O_D^\times$, we consider integral models $\mathcal{M}_C$ and $\mathcal{M}_D$ for the local Shimura varieties for (1.6) at levels $O_C^\times$ and $O_D^\times$. Their definition is as follows.

The elements of $B(H, \mu_H)$ are in bijection with isogeny classes of pairs $(\mathbb{Y}, \iota)$, where $\mathbb{Y}$ is a strict O_F-module of height $2n^2$ and dimension n over the residue field $\mathbb{F}$ of $\breve{F}$, and where $\iota : O_C \to \mathrm{End}(\mathbb{Y})$

is an O_C-action. The set $B(G, \mu_G)$ is similarly in bijection with pairs $(\mathbb{X}, \kappa)$, where $\mathbb{X}/\mathbb{F}$ is a strict O_F-module of height $4n^2$ and dimension $2n$, and where $\kappa : O_D \to \mathrm{End}(\mathbb{X})$ is an O_D-action. Under these bijections, the map $B(H, \mu_H) \to B(G, \mu_G)$ corresponds to the Serre tensor construction

$$(\mathbb{Y}, \iota) \longmapsto (O_D \otimes_{O_C} \mathbb{Y},\ \kappa(x) := x \otimes \mathrm{id}_{\mathbb{Y}}). \tag{1.7}$$

Let $\mathcal{M}_C$ and $\mathcal{M}_D$ be the RZ spaces for $(\mathbb{Y}, \iota)$ and $(\mathbb{X}, \kappa) = O_D \otimes_{O_C} (\mathbb{Y}, \iota)$. These are certain EL type moduli spaces of p-divisible groups with O_C-action resp. O_D-action. They are formal schemes over $\mathrm{Spf}\, O_{\breve{F}}$ that are regular with semi-stable reduction and such that $\dim \mathcal{M}_D = 2 \dim \mathcal{M}_C = 2n$. Furthermore, the groups H_b and G_b act from the right on $\mathcal{M}_C$ resp. $\mathcal{M}_D$ and there is a closed immersion

$$\mathcal{M}_C \hookrightarrow \mathcal{M}_D$$

that is equivariant with respect to $H_b \to G_b$. This closed immersion can be defined by a Serre tensor construction as in (1.7).

Definition 1.3. Let $g \in G_b$ be a regular semi-simple element and let $\Gamma \subseteq H_b \cap g^{-1} H_b g$ be a free, discrete subgroup of covolume 1. Define the intersection number

$$\mathrm{Int}(g) = \langle \Gamma\backslash\mathcal{M}_C,\ \Gamma\backslash g \cdot \mathcal{M}_C \rangle_{\Gamma\backslash\mathcal{M}_D} \in \mathbb{Z}.$$

Taking the quotient by Γ in this definition is the natural analog of taking the quotient by the stabilizer in the orbital integrals (1.3) and (1.4). The restriction to regular semi-simple g ensures that $\Gamma\backslash(\mathcal{M}_C \cap g \cdot \mathcal{M}_C)$ is a proper scheme over $\mathrm{Spec}\, O_{\breve{F}}$ with empty generic fiber. The definition is moreover independent of Γ.

The quantity $\mathrm{Int}(g)$ only depends on the $(H_b \times H_b)$-orbit of g, so we can use orbit matching to view it as a function on a subset of the $(H' \times H')$-orbits in G'. In this context, there is the following uniqueness and vanishing result.

Proposition 1.4 (Proposition 4.6). (1) *Let $\gamma \in G'$ be a regular semi-simple element. There exists at most one isogeny class $b \in B(H, \mu_H)$ such that there exists an element $g \in G_b$ that matches γ.*

(2) *Assume that such a pair (b, g) exists. Then the sign of the functional equation of* $\mathrm{Orb}(\gamma, f'_D, s)$ *is negative, and in particular,* $\mathrm{Orb}(\gamma, f'_D) = 0$.

The second statement already hints that $\mathrm{Int}(g)$ might be related to the first derivative $\partial\mathrm{Orb}(\gamma, f'_D)$ for matching g and γ. This is made precise by the AT Conjecture in its explicit form:

Conjecture 1.5 (ATC – explicit form). *There exists a correction function $f'_{\mathrm{corr}} \in C_c^\infty(G')$ such that for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,*

$$\partial\mathrm{Orb}(\gamma, f'_D) + \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}) = \begin{cases} 2\ \mathrm{Int}(g)\log(q) & \text{if there exists some } b \in B(H, \mu_H) \\ & \text{and some } g \in G_b \text{ that matches } \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Conjecture 1.5 builds on our FL Conjecture. There is also a weaker form that postulates the existence of a test function $f' \in C_c^\infty(G')$ whose orbital integral derivatives $\partial\mathrm{Orb}(\gamma, f')$ agree with the intersection numbers $\mathrm{Int}(g)$ on the nose. We formulate and compare such variants in §4.5.

The AT Conjecture is currently known in some low-dimensional cases: Consider first the case of $D = M_{2n}(F)$. Then $f'_D = 1_{GL_{2n}(O_F)}$, and it is conjectured that one may take $f'_{\mathrm{corr}} = 0$. This is precisely the linear AFL conjecture from [30]. By the main results of [28] and [30], it is known to hold whenever the connected part of $\mathbb{Y}$ has height ≤ 4.

Assume next that the Hasse invariant of D is $1/2$. Then it is again expected that one may take $f'_{\mathrm{corr}} = 0$. The main result of [20] states that Conjecture 1.5 for such D follows from the linear AFL.

The main result of the present paper, to be formulated in the next section, states that Conjecture 1.5 holds for division algebras of degree 4. Here, f'_{corr} can be chosen from the Iwahori Hecke algebra.

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We speculate that this is a general phenomenon, meaning that f'_{corr} can always be chosen as a linear combination of indicator functions of standard parahoric subgroups. In this context, we mention related work of He–Shi–Yang [16] and He–Li–Shi–Yang [17]: There, the authors prove intersection number identities for Kudla–Rapoport divisors in the presence of bad reduction. Their result involves a unique characterization of certain occurring correction terms. It would be interesting to know if similar ideas apply in the context of Conjecture 1.5.

1.5. *Invariant* $1/4$ *and* $3/4$

Assume from now on that $n = 2$ and that D denotes a central division algebra (CDA) of Hasse invariant $\lambda \in \{1/4, 3/4\}$. Then C is a quaternion division algebra over E. The test function $f'_D \in C_c^\infty(G')$ is an H'-translate of a scalar multiple f'_{Iw} of the characteristic function of an Iwahori in $G' = GL_4(F)$. We will also consider a test function f'_{Par} that is the characteristic function of a (2×2)-block parahoric in $GL_4(F)$.

The set $B(H, \mu_H)$ is a singleton in this situation, and the moduli space $\mathcal{M}_C$ is Drinfeld's half plane [10]. If $\lambda = 1/4$, then $\mathcal{M}_D$ is the four-dimensional Drinfeld half space. If $\lambda = 3/4$, however, then no explicit description of $\mathcal{M}_D$ is known. The two groups H_b and G_b are given by

$$H_b \cong GL_2(E), \qquad G_b \cong \begin{cases} GL_4(F) & \text{if } \lambda = 1/4 \\ GL_2(B) & \text{if } \lambda = 3/4. \end{cases}$$

Here, B/F denotes a quaternion division algebra.

Theorem 1.6 (Theorem 9.1). *The AT conjecture holds for D. More precisely, let f'_{corr} be given by*

$$f'_{\mathrm{corr}} = \begin{cases} -4q\log(q)\cdot f'_{\mathrm{Par}} & \text{if } \lambda = 1/4 \\ 0 & \text{if } \lambda = 3/4. \end{cases} \tag{1.9}$$

Then, for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,

$$\partial\mathrm{Orb}(\gamma, f'_D) + \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}) = \begin{cases} 2\,\mathrm{Int}(g)\log(q) & \text{if there exists a } g \in G_b \text{ that matches } \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Why and how it is that the two cases differ by a parahoric type orbital integral is a mystery to us. The difference only emerges during the proof of Theorem 1.6 and is encoded in the 0-dimensional embedded components of the intersection locus.

1.6. *Key aspects*

Our proof of Theorem 1.6 is by determining explicitly and comparing both sides in (1.10). Key aspects are as follows:

(1) Concerning the orbital integral side, we combine three techniques to determine all occurring $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$, $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ and $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. First, we work out these orbital integrals for all hyperbolic orbits. In this situation, the computation can be reduced to a Levi subgroup and hence to GL_2. Second, we establish a germ expansion principle (Proposition 7.4) that allows to write each orbital integral as a linear combination of a principal germ and a unipotent germ. The principal germ can be described explicitly in all situations. Third and finally, we use the results for hyperbolic orbits and the linear relations among the various germs to also determine the remaining orbital integrals. A summary of the final results can be found in §5. In particular, Proposition 5.4 gives a formula for the derivatives $\partial\mathrm{Orb}(\gamma, f'_D)$ when D is a division algebra of degree 4. The basis for all mentioned results is a combinatorial expression for $\mathrm{Orb}(\gamma, f'_D)$ in terms of lattices; see (3.25).

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(2) Concerning the intersection-theoretic side, we first prove a general formula for intersection numbers of surfaces in a 4-dimensional space:

Proposition 1.7 (see Corollary 10.3). *Let X be a regular 4-dimensional formal scheme that is locally formally of finite type over* $\mathrm{Spf}\, O_{\breve{F}}$. *Let $Y_1, Y_2 \subseteq X$ be two regular 2-dimensional closed formal subschemes. Assume that $Z = Y_1 \cap Y_2$ is a proper scheme over* $\mathrm{Spec}\, O_{\breve{F}}$ *with empty generic fiber and of dimension ≤ 1. Let Z^{pure}, $Z^{\mathrm{art}} \subseteq Z$ be its purely 1-dimensional locus and the artinian subscheme of 0-dimensional embedded components. Then*

$$\langle Y_1, Y_2\rangle_X = \mathrm{len}(\mathcal{O}_{Z^{\mathrm{art}}}) - \deg(\det \mathcal{C}_1|_{Z^{\mathrm{pure}}}) - \langle Z^{\mathrm{pure}}, Z^{\mathrm{pure}}\rangle_{Y_2}. \tag{1.11}$$

Here, $\mathcal{C}_1$ is the conormal bundle of $Y_1 \subseteq X$, and the intersection number on the very right is that of divisors on Y_2.

(3) In order to compute the occurring intersection numbers $\mathrm{Int}(g)$, we determine the three quantities on the right-hand side of (1.11). The most important input here is Drinfeld's theorem [10], which provides an explicit description for $\mathcal{M}_C$ and, if $\lambda = 1/4$, for $\mathcal{M}_D$. This in particular enables us to compute the degree of the conormal bundle:

Proposition 1.8 (see Propositions 12.6 and 13.1). *Let $P \subseteq \mathbb{F} \otimes_{O_{\breve{F}}} \mathcal{M}_C$ be an irreducible component of the special fiber of $\mathcal{M}_C$. Let $\mathcal{C}$ be the conormal bundle of $\mathcal{M}_C \subseteq \mathcal{M}_D$. Then, for both Hasse invariants $\lambda \in \{1/4, 3/4\}$,*

$$\deg(\mathcal{C}|_P) = q^2 - 1.$$

Set $\mathcal{I}(g) = \mathcal{M}_C \cap g \cdot \mathcal{M}_C$. It is left to describe $\mathcal{I}(g)^{\mathrm{pure}}$ and $\mathcal{I}(g)^{\mathrm{art}}$, where the notation is meant in the sense of Proposition 1.7.

(4) The description of $\mathcal{I}(g)^{\mathrm{pure}}$ is in terms of the Bruhat–Tits stratification of the special fiber of $\mathcal{M}_C$: Each of its irreducible components is isomorphic to $\mathbb{P}^1$. Restricting to a fixed connected component $\mathcal{M}_C^0$ of $\mathcal{M}_C$, the dual graph of its special fiber is the Bruhat–Tits tree $\mathcal{B}$ for $PGL_{2,E}$. This is a (q^2+1)-regular tree whose vertices are in bijection with homothety classes of O_E-lattices $\Lambda \subseteq E^2$. Thus, we may write

$$\mathcal{M}_C^0 \cap \mathcal{I}(g)^{\mathrm{pure}} = \sum_{\Lambda \in \mathrm{Vert}(\mathcal{B})} m(g, \Lambda)[P_\Lambda]. \tag{1.12}$$

One of our main auxiliary results (Theorem 11.10) describes the coefficient function $\Lambda \mapsto m(g, \Lambda)$ in terms of g.

(5) The artinian locus $\mathcal{I}(g)^{\mathrm{art}}$ is where the two cases $\lambda = 1/4$ and $\lambda = 3/4$ are substantially different. If $\lambda = 1/4$, then $\mathcal{M}_C^0 \cap \mathcal{I}(g)^{\mathrm{art}}$ consists of at most a single point of length 1. If $\lambda = 3/4$, however, then every stratum P_Λ that occurs in (1.12) contributes artinian embedded components of total length q. The individual lengths and positions of these components depend on g and Λ; see Table 2 for all possibilities. The total length of $\Gamma\backslash\mathcal{I}(g)^{\mathrm{art}}$ will, however, always match the orbital integral $4q\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ that makes up the difference of the two analytic sides for $\lambda = 1/4$ and $\lambda = 3/4$ in (1.10).

(6) As mentioned before, there is no explicit description of $\mathcal{M}_D$ when $\lambda = 3/4$ which makes the computation of $\mathcal{I}(g)$ trickier than in the case $\lambda = 1/4$. We use a mix of Dieudonné theory, Cartier theory and display theory, as well as the previous results for the case of invariant $1/4$, to achieve a precise description; see the results in §13.3 and §13.5 as well as Proposition 13.22.

1.7. Open directions

There is, of course, the question of how to prove Conjectures 3.10 and 1.5 resp. the linear AFL [30] in general. We mention here three further problems of interest.

(1) The AT conjecture of the present article concerns inner forms of GL_{2n} and their moduli spaces. An equally interesting question would be to study moduli spaces for GL_{2n} (Lubin–Tate spaces), but with parahoric level structure. The global motivation from §1.1 applies verbatim to that situation.

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(2) Another question would be for an extension of our results to the biquadratic situation in the following sense: Instead of a single quadratic extension E/F, one considers two such extensions $E_1, E_2/F$, fixes embeddings $E_1, E_2 \to D$, and defines intersection numbers from the corresponding cycles on $\mathcal{M}_D$. For the linear AFL, such a biquadratic generalization has been formulated by B. Howard and the first author [19].

(3) Finally, there is the problem of relating the local quantities of the present article with global intersection numbers and L-functions. We hope to return to this question in the future.

1.8. Layout of the paper

We now give an overview of the contents of this paper. The paper consists of three parts.

In Part 1, we first give the group-theoretic setup (invariant theory, matching). We next define the orbital integrals of interest and formulate our FL conjecture. Then we introduce the moduli spaces of strict O_F-modules in question and the intersection numbers $\mathrm{Int}(g)$. We state and compare three variants of the AT conjecture.

In Part 2, we consider the case $G' = GL_4(F)$ and determine the quantities $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$, $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ and $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. A summary of our results can be found in §5, the main one being the proof of the FL conjecture for f'_{Iw}. The further contents of Part 2 have been described at the beginning of §1.6.

In Part 3, we consider the case of a CDA D/F of degree 4 and compute the intersection numbers $\mathrm{Int}(g)$. Our main result is the proof of the AT conjecture in this situation. There are four main sections: In §10, we prove the intersection number formula in Proposition 1.7. In §11, we study the functions $m(g, \Lambda)$ on $\mathcal{B}$ that will later describe the multiplicities of curves in the intersection locus; see (1.12). Then, in §12, we prove the AT conjecture for $\lambda = 1/4$. Because of Drinfeld's description of both $\mathcal{M}_C$ and $\mathcal{M}_D$ in this situation, this does not involve any π-divisible groups at all. Finally, in §13, we extend these results to $\lambda = 3/4$.

Part I
The arithmetic transfer conjecture

2. Invariants

Let F be a field, let E/F be an étale quadratic extension and let D/F be a CSA of degree $2n$. Assume that there exists, and fix, an F-algebra embedding $E \to D$ that makes D into a free E-module. (The latter condition is only relevant if $E \cong F \times F$.) Since E/F is étale, $E \otimes_F E \cong E \times E$. So the left and right multiplication actions of E on D provide an eigenspace decomposition $D = D_+ \oplus D_-$ into E-linear and E-conjugate linear elements. That is, $D_+ = \mathrm{Cent}_D(E)$ which is the E-algebra C from the introduction. Note that $\dim_F(D_+) = \dim_F(D_-)$. We denote the two components of an element $g \in D$ by g_+ and g_-.

Write $G = D^\times$ and $H = D_+^\times$ in the following. We consider the right-action

$$(H \times H) \times G \longrightarrow G, \quad (h_1, h_2) \cdot g = h_1^{-1} g h_2. \tag{2.1}$$

An element $g \in G$ is called regular semi-simple if its $(H \times H)$-orbit is Zariski closed and if its stabilizer is of the minimal possible dimension. We denote these elements by G_{rs}. The regular semi-simple orbits have been classified by Jacquet–Rallis [21] and Guo [13, §1]. We work with the variant of their results that best suits our purposes.

Definition 2.1. Let $g \in G$ be an element such that also g_+ lies in G. Then we define $z_g = g_+^{-1} g_-$ which lies in D_-. It can be thought of as the normalized conjugate-linear part of g. It is easily checked after base change to $\overline{F}$ that the reduced characteristic polynomial[2] of z_g^2 is a square; see (2.3) below. We define

[2]Recall that the reduced characteristic polynomial of an element $x \in D$ is defined by $\mathrm{charred}_{D/F}(x;T) := \mathrm{char}_{D\otimes_F \overline{F}/\overline{F}}(\alpha(x \otimes 1);T)$, where $\overline{F}/F$ is an algebraic closure and $\alpha : D \otimes_F \overline{F} \xrightarrow{\sim} M_{2n}(\overline{F})$ any choice of $\overline{F}$-algebra isomorphism.

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the invariant of g as its unique monic square root:

$$\mathrm{Inv}(g;T) = \mathrm{charred}_{D/F}\left(z_g^2;\ T\right)^{1/2} \in F[T]. \tag{2.2}$$

This is a monic polynomial of degree n. It satisfies $\mathrm{Inv}(g;1) \neq 0$ because $1 + g_+^{-1}g_-$ and $1 - g_+^{-1}g_-$ are both invertible, so $g_+^{-1}g_-$ does not have eigenvalues ± 1. Indeed, $1 + g_+^{-1}g_- = g_+^{-1}g$ is invertible by assumption. Let $\xi \in E^\times$ satisfy $\bar{\xi} = -\xi$. The identity $1 - g_+^{-1}g_- = \xi(1 + g_+^{-1}g_-)\xi^{-1}$ implies that $1 - g_+^{-1}g_-$ is invertible as well.

It is clear by definition that $\mathrm{Inv}(g;T)$ only depends on the orbit HgH. The next lemma provides a converse for regular semi-simple elements.

Lemma 2.2 (Guo [13, §1]). *An element $g \in G$ is regular semi-simple if and only if both g_+, g_- belong to G and if the invariant* $\mathrm{Inv}(g;T)$ *is a separable polynomial. Moreover, two regular semi-simple elements $g_1, g_2 \in G$ lie in the same $(H \times H)$-orbit if and only if their invariants agree.*

Example 2.3. Consider the split quadratic extension $F \times F$, the CSA $M_{2n}(F)$ and the embedding $\iota : F \times F \to D$ that is given by $(a, b) \mapsto \mathrm{diag}(a1_n, b1_n)$. A (2×2)-block matrix $g = \left(\begin{smallmatrix} v & w \\ x & y \end{smallmatrix}\right)$ with blocks $v, w, x, y \in M_n(F)$ can only be regular semi-simple with respect to ι if all its four blocks are invertible. In this case, we have

$$z_g^2 = \begin{pmatrix} v^{-1}wy^{-1}x & \\ & y^{-1}xv^{-1}w \end{pmatrix}. \tag{2.3}$$

The invariant of g is hence $\mathrm{Inv}(g;T) = \mathrm{char}(v^{-1}wy^{-1}x;T)$. Moreover, if g is regular semi-simple and if $w \in GL_n(F)$ is any element with $\mathrm{char}(w;T) = \mathrm{Inv}(g;T)$, then

$$HgH = H\begin{pmatrix} 1 & w \\ 1 & 1 \end{pmatrix}H. \tag{2.4}$$

Remark 2.4. A slightly different definition of invariant is given in [27, Definition 1.1]. It is defined for every $g \in G$ and again a monic polynomial of degree n; let us call it $\mathrm{Inv}'(g;T)$. Assuming that g_+ lies in G, the two definitions are related by the Moebius transformation

$$\mathrm{Inv}'(g;T) = T^n\, \mathrm{Inv}(g;1)^{-1}\, \mathrm{Inv}\left(g; \frac{T-1}{T}\right). \tag{2.5}$$

Definition 2.5. Let $g \in G_{\mathrm{rs}}$ be regular semi-simple. We denote by $B_g \subseteq D$ the F-subalgebra that is generated by E and $g^{-1}Eg$. We denote by $L_g \subset B_g$ its center. Up to isomorphism, these objects as well as z_g only depend on the orbit HgH because

$$B_{(h_1,h_2)g} = h_2^{-1}B_g h_2, \quad L_{(h_1,h_2)g} = h_2^{-1}L_g h_2, \quad z_{(h_1,h_2)g} = h_2^{-1}z_g h_2. \tag{2.6}$$

The next proposition summarizes their most important properties.

Proposition 2.6. *Let $g \in G_{\mathrm{rs}}$ be a regular semi-simple element.*

(1) *The square z_g^2 lies in L_g. In fact, L_g equals $F[z_g^2]$ and is, in particular, an étale F-algebra of degree n that is isomorphic to $F[T]/(\mathrm{Inv}(g;T))$.*

(2) *The composite EL_g of E and L_g in D is isomorphic to $E \otimes_F L_g$ and, in particular, an étale quadratic L_g-algebra.*

(3) *The algebra B_g equals $EL_g[z_g]$ and is, in particular, a quaternion algebra over L_g. It coincides with the centralizer* $\mathrm{Cent}_D(L_g)$.

Proof. This is a special case of [19, Proposition 2.5.4]. Since the argument is short and instructive, and since our notation is slightly different from that in [19], we include a proof for convenience.

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Choose an F-algebra generator $\zeta \in E$ to write $B_g = F[\zeta, g^{-1}\zeta g]$. We make this choice with $\mathrm{tr}_{E/F}(\zeta) = 1$ (i.e. $\overline{\zeta} = 1 - \zeta$.) Then we obtain (put $z = z_g$)

$$\begin{aligned} g^{-1}\zeta g &= (1+z)^{-1}\zeta(1+z) \\ &= \zeta + \frac{z}{1+z}(1-2\zeta). \end{aligned} \tag{2.7}$$

It always holds that $1-2\zeta \in E^\times$: This is clear if E is a field and can be verified directly if $E \cong F \times F$. The fraction $z/(1+z)$ is a Moebius transformation that is defined at z. The inverse Moebius transformation is then defined at $z/(1+z)$, so we obtain $B_g = F[\zeta, z]$. It is evident from definitions that z^2 commutes with both ζ and z, and hence lies in the center L_g of B_g.

Claim: The elements $1, z, \dots, z^{2n-1}, \zeta, \zeta z, \dots, \zeta z^{2n-1}$ *form an F-vector space basis of* B_g. This may be shown after base change to $\overline{F}$ which puts us into the situation of Example 2.3. Then we may assume that g is given as a block matrix $g = \left(\begin{smallmatrix} 1 & w \\ 1 & 1 \end{smallmatrix}\right)$ as in (2.4). In this specific case, we have

$$z_g = \begin{pmatrix} & w \\ 1 & \end{pmatrix} \quad \text{and} \quad B_g = (F \times F)[z_g]. \tag{2.8}$$

The claim then follows from the fact that the characteristic polynomial of w, which equals $\mathrm{Inv}(g;T)$, is separable by Lemma 2.2 and hence agrees with the minimal polynomial. The identities $L_g = F[z_g^2]$ as well as $EL_g \cong E \otimes_F L_g$ and $B_g = EL_g[z_g]$ all follow from the claim.

It remains to show the statement $B_g = \mathrm{Cent}_D(L_g)$. We have already seen that EL_g is an étale F-algebra of degree $2n = (\dim_F D)^{1/2}$. The only possibility for D is then to be free as EL_g-module. This implies that D is free as L_g-module, so the centralizer $\mathrm{Cent}_D(L_g)$ is a quaternion algebra over L_g. It also contains B_g, however, and hence equals B_g. □

We call a polynomial $\delta \in F[T]$ regular semi-simple if it is monic, separable and satisfies $\delta(0)\delta(1) \neq 0$. Example 2.3 shows that the regular semi-simple polynomials of degree n are in bijection with the regular semi-simple $GL_n(F \times F)$-double cosets on $GL_{2n}(F)$. The following construction is taken from [19, Proposition 2.5.6].

Definition 2.7. Let $\delta \in F[T]$ be regular semi-simple. We define the two F-algebras

$$L_\delta := F[z^2]/(\delta(z^2)) \quad \text{and} \quad B_\delta := (E \otimes_F L_\delta)[z]$$

with commutator relation $(a \otimes b)z = z(\bar{a} \otimes b)$ for $a \in E$, $b \in L_\delta$. Note that if $g \in G_{\mathrm{rs}}$ is a regular semi-simple element, then $B_g \cong B_\delta$ by Proposition 2.6. We call B_δ the universal quaternion algebra for invariant δ because it detects orbits of invariant δ in the following sense.

Corollary 2.8. *Let $\delta \in F[T]$ be regular semi-simple of degree n. The following three conditions are equivalent.*

(1) *There exists an element $g \in G_{\mathrm{rs}}$ of invariant δ.*
(2) *There exists an F-algebra embedding $B_\delta \to D$.*
(3) *The identity $[B_\delta] = [L_\delta \otimes_F D]$ holds in the Brauer group of L_δ.*

Proof. Assume that (1) holds and let $g \in G_{\mathrm{rs}}$ be such that $\mathrm{Inv}(g;T) = \delta(T)$. Then Proposition 2.6 states that $B_g \cong B_\delta$, so (2) holds. Conversely, assume that there exists an embedding $\iota : B_\delta \to D$. Then $\iota(E \otimes_F L_\delta) \subset D$ is a commutative F-subalgebra of F-dimension $2n = [D : F]$. It is hence a maximal commutative subalgebra, so D is a free $\iota(E \otimes_F L_\delta)$-module. In particular, D is free both as $\iota(E)$-module and as $\iota(L_\delta)$-module. It then follows from the Skolem–Noether Theorem that the given embedding $E \to D$ and $\iota|_E : E \to D$ are conjugate. We may hence find ι such that $\iota|_E$ agrees with the

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given embedding of E. Then $g = 1 + \iota(z)$ has the property that $g_+ = 1$ and $g_- = \iota(z)$, and consequently that $z_g^2 = \iota(z^2)$. Since D is free over $\iota(L_\delta)$, it holds that

$$\mathrm{charred}_{D/F}(z^2;T) = \mathrm{char}_{L_\delta/F}(z^2;T)^2$$

and hence that $\mathrm{Inv}(g;T) = \delta(T)$. This shows that (2) implies (1).

We now prove the equivalence of (2) and (3), which holds more generally. Let B be a quaternion algebra over an étale F-algebra L of degree n. We claim the equivalence of

(2) There exists an embedding $\iota : B \to D$.

(3) It holds that $[B] = [L \otimes_F D]$ in the Brauer group of L.

Assume that there exists an embedding $\iota : B \to D$. Then D is necessarily free as $\iota(L)$-module and $\iota(B) = \mathrm{Cent}_D(\iota(L))$ for dimension reasons. The identity $[B] = [L \otimes_F D]$ follows from (a mild extension of) the centralizer theorem [9, Theorem 9.6]. This shows that (2) implies (3).

Assume conversely that $[B] = [L \otimes_F D]$ holds. Let M/L be a quadratic étale extension that splits B. Then it also holds that $M \otimes_F D \cong M_{2n}(M)$. Let

$$L = \prod_{i \in I} L_i, \quad M = \prod_{i \in I} M_i, \quad B = \prod_{i \in I} B_i \tag{2.9}$$

denote the factorizations of L, M and B that correspond to the idempotents of L. Also pick an isomorphism $D \cong M_m(D_0)$ for a central division algebra (CDA) D_0. Each factor M_i splits D_0, so $d = \dim_F(D_0)^{1/2}$ divides $\dim_F(M_i) = 2[L_i : F]$ by [9, Corollary 9.4]. For every i, we define $D_i = M_{[M_i:F]/d}(D_0)$. By [9, Corollary 9.3], there exists an F-algebra embedding $M_i \to D_i$. Note that $\sum_{i \in I} [M_i : F]/d = m$, so we can form a block diagonal embedding

$$\iota : M = \prod_{i \in I} M_i \hookrightarrow \prod_{i \in I} D_i \hookrightarrow D.$$

It follows from $[M : F] = \dim_F(D)^{1/2}$ that D is free as $\iota(M)$-module and hence also free over $\iota(L)$. The centralizer $\mathrm{Cent}_D(\iota(L))$ satisfies the identity $[\mathrm{Cent}_D(\iota(L))] = [L \otimes_F D]$ and is hence isomorphic to B. Any choice of such an isomorphism defines an embedding $B \to D$. This shows that (3) implies (2). $\square$

We will also require a definition of invariant for semi-simple F-algebras. Assume in what follows that D/F is a finite-dimensional semi-simple F-algebra with center Z. Write $Z = \prod_{i \in I} Z_i$ as a product of fields and also decompose D accordingly, $D = \prod_{i \in I} D_i$. We assume that D is of total degree $2n$ in the sense that

$$2n = \sum_{i \in I} [Z_i : F] \cdot [D_i : Z_i]. \tag{2.10}$$

Finally, we assume the existence of, and fix, an embedding $E \to D$ such that each component D_i becomes a free E-module. Then there is an eigenspace decomposition $D = D_+ \oplus D_-$ as before, and we continue to write $g = g_+ + g_-$ for the corresponding decomposition of elements $g \in D$. We also write g_i for the i-th component of g.

Definition 2.9. Let $g \in G$ be an element with $g_+ \in G$. The invariant of g is defined as

$$\mathrm{Inv}(g;T) = \prod_{i \in I} \mathrm{Inv}(g_i;T) \in F[T]. \tag{2.11}$$

It is a monic polynomial of degree n. We call g regular semi-simple if $\mathrm{Inv}(g;T)$ is regular semi-simple in the same sense as before.

For example, the element $1 + z \in B_\delta$ from Definition 2.7 has invariant $\mathrm{Inv}(1 + z;T) = \delta$ with respect to the embedding $E \to B_\delta$ that comes by construction.

Definition 2.5 and the statements of Proposition 2.6 apply and remain true without change for semi-simple D. In Corollary 2.8, the equivalence of (1) and (2) remains true as well.

3. Fundamental lemma

3.1. Setting

We maintain the following setting throughout the paper.

(1) We denote by F a non-archimedean local field with ring of integers O_F, uniformizer π and residue cardinality q. We let E/F be an unramified quadratic field extension with ring of integers O_E.

(2) We let $K = F \times F$ denote the split quadratic extension of F. We view it as a subring of $M_{2n}(F)$ by the diagonal embedding $(a, b) \mapsto \mathrm{diag}(a1_n, b1_n)$, and we define

$$G' = GL_{2n}(F), \quad H' = GL_n(K).$$

Given $\gamma \in M_{2n}(F)$, we write $\gamma = \gamma_+ + \gamma_-$ for its decomposition into K-linear and conjugate-linear components. Whenever we speak of regular semi-simple elements of G' or of their invariants, then this is meant with respect to the $(H' \times H')$-action. In fact, this is precisely the setting from Example 2.3, albeit in different terminology.

(3) We denote by D a CSA of degree $2n$ over F. We fix an embedding $E \longrightarrow D$ and use the the notations $D = D_+ \oplus D_-$ as well as $g = g_+ + g_-$ like before. We interchangeably write $C = D_+$, and define

$$G = D^\times, \quad H = C^\times.$$

(We will switch to opposed CSAs in §4.) Whenever we speak of regular semi-simple elements of G or of their invariants, then this is meant with respect to the $(H \times H)$-action.

(4) Our normalization of the Hasse invariant is as follows. Let F_{2n}/F be an unramified field extension of degree $2n$ with Frobenius σ. Then there is a unique integer $0 \le r < 2n$ such that D is isomorphic to the cyclic F-algebra

$$F_{2n}[\Pi]/(\Pi^{2n} = \pi^r,\ \Pi a = \sigma(a)\Pi \text{ for } a \in F_{2n}).$$

The Hasse invariant of D is defined as $r/2n \in \mathbb{Q}/\mathbb{Z}$. For $\lambda \in \mathbb{Q}/\mathbb{Z}$, we write D_λ to denote a CDA of Hasse invariant λ over F.

3.2. Orbital integrals

Let $\eta : F^\times \longrightarrow \{\pm 1\}$ be the nontrivial unramified quadratic character. In this section, we define and compare two kinds of orbital integrals. The first kind are η-twisted orbital integrals on $[G'] = H'\backslash G'/H'$. The second kind are orbital integrals on $[G] = H\backslash G/H$. In fact, the orbital integrals will only be defined on the regular semi-simple orbits $[G'_{\mathrm{rs}}]$ and $[G_{\mathrm{rs}}]$.

3.2.1. Orbital integrals on $[G']$

Given $\gamma \in G'$, we denote its stabilizer by

$$(H' \times H')_\gamma := \{(h_1, h_2) \mid h_1^{-1}\gamma h_2 = \gamma\}. \tag{3.1}$$

The stabilizer of a regular semi-simple element γ is isomorphic to the torus $L_\gamma^\times$, where $L_\gamma = L[z_\gamma^2]$ is the étale F-algebra of degree n from Proposition 2.6 (1). Indeed, we may rewrite (3.1) as

$$(H' \times H')_\gamma = \{(\gamma h \gamma^{-1}, h) \mid h \in H' \cap \gamma^{-1} H' \gamma\}.$$

The intersection $H' \cap \gamma^{-1} H' \gamma$ is by definition the centralizer of $B_\gamma = F[K \cup \gamma^{-1} K \gamma]$ in G' (see Definition 2.5). Since $[L_\gamma : F] = n$ and since B_γ/L_γ is a quaternion algebra, this centralizer equals the units of $L_\gamma \subset B_\gamma$.

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We endow $(H' \times H')_\gamma$ with the Haar measures such that $O_{L_\gamma}^\times$ has volume 1. We also normalize the Haar measure on $H' \times H'$ such that a maximal compact subgroup has volume 1.

Let $|\cdot| : F^\times \to \mathbb{R}$, $x \mapsto q^{-\nu(x)}$ be the normalized absolute value on F. We define η and $|\cdot|$ on H' in the following way,

$$\eta,\ |\cdot| : H' \longrightarrow \mathbb{R}, \quad \eta(\mathrm{diag}(a,b)) = \eta(\det(ab^{-1})), \quad |\mathrm{diag}(a,b)| = |\det(ab^{-1})|. \tag{3.2}$$

Definition 3.1. For $\gamma \in G'_{\mathrm{rs}}$, a test function $f' \in C_c^\infty(G')$ and $s \in \mathbb{C}$, we define the orbital integral

$$O(\gamma, f', s) := \int_{\frac{H'\times H'}{(H'\times H')_\gamma}} f'(h_1^{-1}\gamma h_2)|h_1h_2|^s\eta(h_2)\, dh_1dh_2. \tag{3.3}$$

The support of the integrand in (3.3) is compact because the $(H'\times H')$-orbit of a regular semi-simple element is Zariski closed. This ensures convergence, and the resulting expression $O(\gamma, f', s)$ lies in $\mathbb{C}[q^s, q^{-s}]$. However, as a function of γ, the orbital integral does not yet descend to the orbit space $[G'_{\mathrm{rs}}]$ because it transforms by the character $\eta(\,\cdot\,)|\cdot|^s$ under the $(H' \times H')$-action. We next modify it in the simplest possible way that makes it $H' \times H'$-invariant.

Definition 3.2. Let $s \in \mathbb{C}$. Define the transfer factor $\Omega(\,\cdot\,, s) : G'_{\mathrm{rs}} \to \pm q^{\mathbb{Z}s}$ by

$$\Omega\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), s\big) = \eta(\det(cd^{-1})) \cdot |\det(b^{-1}c)|^s.$$

It satisfies $\Omega(h_1^{-1}\gamma h_2, s) = |h_1h_2|^s\eta(h_2)\Omega(\gamma, s)$, so we can modify and rewrite (3.3) as

$$\mathrm{Orb}(\gamma, f', s) := \Omega(\gamma, s) \cdot O(\gamma, f', s) = \int_{\frac{H'\times H'}{(H'\times H')_\gamma}} f'(h_1^{-1}\gamma h_2) \cdot \Omega(h_1^{-1}\gamma h_2, s)\, dh_1dh_2. \tag{3.4}$$

Then $\mathrm{Orb}(\gamma, f', s)$ is $(H' \times H')$-invariant and descends to the orbit space $[G'_{\mathrm{rs}}]$. Note that we still have

$$\mathrm{Orb}(\gamma, (h_1,h_2)^*(f'), s) = |h_1h_2|^{-s}\eta(h_2)\,\mathrm{Orb}(\gamma, f', s) \tag{3.5}$$

for all $(h_1, h_2) \in H' \times H'$. We will mostly be interested in the central value and the central derivative of $\mathrm{Orb}(\gamma, f', s)$ which we denote by

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0), \qquad \partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\bigg|_{s=0} \mathrm{Orb}(\gamma, f', s). \tag{3.6}$$

3.2.2. Orbital integrals on $[G]$

The definition of orbital integrals on $[G_{\mathrm{rs}}]$ is more straightforward because it does not involve any characters. Given $g \in G$, we denote its stabilizer by

$$(H \times H)_g = \{(h_1, h_2) \mid h_1^{-1}gh_2 = g\}.$$

As before, the stabilizer of a regular semi-simple element g is isomorphic to the torus $L_g^\times$. Again, we endow $H\times H$ and $(H\times H)_g$ with the Haar measures such that a maximal compact subgroup has volume 1. Note that all maximal compact subgroups of $H \times H$ are conjugate, so this Haar measure is well defined.

Definition 3.3. For $g \in G$ regular semi-simple and $f \in C_c^\infty(G)$, we define the orbital integral

$$\mathrm{Orb}(g, f) = \int_{\frac{H\times H}{(H\times H)_g}} f(h_1^{-1}gh_2)\, dh_1dh_2. \tag{3.7}$$

This function evidently descends to $[G_{\mathrm{rs}}]$.

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3.2.3. Transfer of orbital integrals

We now compare orbital integrals on $[G'_{\mathrm{rs}}]$ and $[G_{\mathrm{rs}}]$.

Definition 3.4 [41]. (1) Two regular semi-simple elements $\gamma \in G'_{\mathrm{rs}}$ and $g \in G_{\mathrm{rs}}$ (resp. their orbits) are said to match if $\mathrm{Inv}(\gamma) = \mathrm{Inv}(g)$. Note that in this case also $L_\gamma \cong L_g$ so that we have chosen compatible Haar measures on the stabilizers $(H' \times H')_\gamma$ and $(H \times H)_g$.

(2) A test function $f' \in C_c^\infty(G')$ is called a transfer of $f \in C_c^\infty(G)$ if, for all $\gamma \in G'_{\mathrm{rs}}$,

$$\mathrm{Orb}(\gamma, f') = \begin{cases} \mathrm{Orb}(g, f) & \text{if there is a matching } g \in G_{\mathrm{rs}} \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Transfers in this sense exist by a result of C. Zhang [44]; also see [41, Proposition 2.9].

We note that Definition 3.4 is analogous to that of transfer in the context of the Jacquet–Rallis relative trace formula comparison; see, for example, [46, §2.4] or [35, Definition 2.2]. However, a difference in our setting is that the matching relation does not yield a partitioning of $[G'_{\mathrm{rs}}]$ into sets of the form $[G_{\mathrm{rs}}]$. We illustrate this for $n = 2$:

Example 3.5. Assume that $G' = GL_4(F)$ and let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element with invariant δ. Our aim is to describe all possibilities for the CSA D such that there exists a matching element $g \in G$. To this end, set $L_\delta = F[z^2]/(\delta(z^2))$ and let $B_\delta = (E \otimes_F L_\delta)[z]$ be the universal quaternion algebra for invariant δ (with respect to E/F) that was constructed in Definition 2.7. By Corollary 2.8, there exists an element $g \in G_{\mathrm{rs}}$ of invariant δ if and only if there exists an embedding $B_\delta \to D$. The following lists all the possibilities for this situation, each of which can occur.

3.3. The fundamental lemma conjecture

Recall that $C = D_+ = \mathrm{Cent}_D(E)$.

Lemma 3.6. *Let $O_1, O_2 \subseteq D$ be two maximal orders that have the property that $O_1 \cap C$ and $O_2 \cap C$ are maximal orders in C. Then $O_1 \cap C = O_2 \cap C$ implies $O_1 = O_2$. In particular, the maximal orders $O \subset D$ such that $O \cap C$ is also a maximal order form a single $C^\times$-conjugation orbit.*

Proof. The second statement follows directly from the first one because all maximal orders in C are $C^\times$-conjugate. We focus on the first statement from now on.

Let $O_C = O_1 \cap C = O_2 \cap C$. Note that $O_E \subset O_C$ because it is the ring of integers of the center of C. Choose a suitable skew-field Q and an isomorphism $D \cong M_m(Q)$. Let $\Pi \in Q$ be a uniformizer. Recall that Q has a unique maximal order O_Q and that the maximal orders in D are precisely the subrings of the form $O_\Lambda = \mathrm{End}_{O_Q^{\mathrm{op}}}(\Lambda)$ where $\Lambda \subset Q^m$ is an O_Q^{op}-lattice. Moreover, $O_\Lambda = O_{\Lambda'}$ if and only if $\Lambda' \in \Lambda\Pi^{\mathbb{Z}}$. In this way, classifying maximal orders in D that contain O_E is equivalent to classifying $R := O_E \otimes_{O_F} O_Q^{\mathrm{op}}$-stable lattices in Q^m, up to scaling by $\Pi^{\mathbb{Z}}$.

There are two cases for the ring R, depending on the parity of the degree $\ell = 2n/m$ of Q over F. Let M/F be an unramified field extension of degree ℓ. With a suitable choice of Π and for a suitable generator $\tau \in \mathrm{Gal}(L/F)$, we may find a presentation of O_Q as

$$O_Q^{\mathrm{op}} \cong O_M[\Pi], \quad \Pi^\ell = \pi, \quad \Pi a = \tau(a)\Pi \text{ for all } a \in M. \tag{3.9}$$

If ℓ is odd, then $O_E \otimes_{O_F} O_M$ is the maximal order in an unramified field extension of F of degree 2ℓ. Then $R \cong (O_E \otimes_{O_F} O_M)[\Pi]$ is again of the form (3.9) and hence the maximal order in the skew field $E \otimes_F Q^{\mathrm{op}}$. The Q^{op} vector space Q^m is isomorphic to $(E \otimes_F Q^{\mathrm{op}})^{m/2}$ as $(E \otimes_F Q^{\mathrm{op}})$-module; hence, its R-stable lattices form a single orbit under $GL_R(Q^m) = C^\times$. This shows that any two maximal orders in D that contain O_E are $C^\times$-conjugate. In particular, we have proven the lemma whenever ℓ is odd.

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The situation is a little different when ℓ is even. To simplify notation in the following, we make the further assumption that E is contained in M. (This is meant with respect to the inclusions $M \subseteq Q \subseteq D$.) Starting from the presentation (3.9) again, we then obtain

$$\begin{aligned} R &\xrightarrow{\sim} (O_M[\Pi^2] \times O_M[\Pi^2])[\Pi], \quad \Pi(a,b) = (\tau(b), \tau(a))\Pi \text{ for all } a, b \in O_M[\Pi^2]. \\ a \otimes m &\longmapsto (am, \bar{a}m) \end{aligned} \tag{3.10}$$

Here, we have extended τ to $M[\Pi^2]$ by $\tau(\Pi^2) = \Pi^2$. The ring $O_M[\Pi^2]$ is the maximal order in the skew field $M[\Pi^2]$, and we can decompose $Q^m \cong V_0 \times V_1$ as $M[\Pi^2] \times M[\Pi^2]$-module. The operator Π is homogeneous of degree 1 with respect to that decomposition, so V_0 and V_1 are both of dimension m over $M[\Pi^2]$. Moreover, every R-stable lattice $\Lambda \subset Q^m$ is of the form $\Lambda = \Lambda_0 \times \Lambda_1$, where $\Lambda_i \subset V_i$ is an $O_M[\Pi^2]$-stable lattice and where $\Lambda_i \Pi \subset \Lambda_{i+1}$ for both $i = 0, 1$. Conversely, the direct sum of any such pair (Λ_0, Λ_1) is an R-stable lattice in Q^m.

After this general description, we now prove the statement of the lemma. The centralizer C acts diagonally on $V_0 \times V_1$. Given an R-stable lattice $\Lambda = \Lambda_0 \times \Lambda_1$, we find that

$$\mathrm{Stab}_C(\Lambda) = \mathrm{Stab}_C(\Lambda_0) \cap \mathrm{Stab}_C(\Lambda_1).$$

Moreover, since the C-action commutes with Π, we may write $\mathrm{Stab}_C(\Lambda_1) = \mathrm{Stab}_C(\Lambda_1\Pi)$. We see that $\mathrm{Stab}_C(\Lambda)$ is a maximal order in C if and only if $\Lambda_0 \in \Lambda_1\Pi \cdot \Pi^{2\mathbb{Z}}$, which holds if and only if

$$\Lambda_0 = \Lambda_1\Pi \quad \text{or} \quad \Lambda_1 = \Lambda_0\Pi. \tag{3.11}$$

The set of $O_M[\Pi^2]$-lattices in V_0 form a single $C^\times$-orbit. Thus, the set of R-stable lattices Λ such that $\mathrm{Stab}_C(\Lambda)$ is a maximal order form two $C^\times$-orbits that are distinguished by (3.11). However, they are interchanged by multiplication by $\Pi \in Q^{\mathrm{op}}$ and, in particular, define the same $C^\times$-conjugation orbit of maximal orders in C and D. The proof of the lemma is now complete. □

Example 3.7. One byproduct of the above proof is the following statement: Assume that ℓ is odd and that $O \subset D$ is a maximal order that contains O_E. Then the intersection $O \cap C$ is a maximal order in C.

Consider, for example, an embedding $E \to M_{2n}(F)$. If $O_E \subset \mathrm{End}(\Lambda)$ for some O_F-lattice $\Lambda \subset F^{2n}$, then Λ is an O_E-lattice and $\mathrm{End}_{O_F}(\Lambda) \cap \mathrm{End}_E(V) = \mathrm{End}_{O_E}(\Lambda)$ is a maximal order.

The statement does not hold true if ℓ is even: Let $Q = D_{1/2}$ be a quaternion division algebra over F with uniformizer Π and let $E \to Q$ be a fixed embedding. Let $D = M_2(Q)$ and let $E \to D$ be the diagonal embedding. The centralizer $C = \mathrm{Cent}_D(E)$ is then simply $M_2(E)$. Both the maximal orders $O_1 = M_2(O_Q)$ and $O_2 = \mathrm{diag}(\Pi, 1)^{-1} O_1 \,\mathrm{diag}(\Pi, 1)$ contain O_E. However, they do not both intersect C in a maximal order:

$$O_1 \cap M_2(E) = M_2(O_E), \quad O_2 \cap M_2(E) = \begin{pmatrix} O_E & O_E \\ (\pi) & O_E \end{pmatrix}.$$

Fix some maximal order $O_D \subset D$ such that $O_C = O_D \cap C$ is a maximal order in C and consider the indicator function $f_D = 1_{O_D^\times}$. Lemma 3.6 implies that the orbital integrals $O(g, f_D)$ are independent of the choice of O_D. The purpose of the next definition is to provide a (conjectural) transfer f'_D of f_D in the sense of Definition 3.4.

Definition 3.8. Let $\lambda = k/\ell$, with $(k, \ell) = 1$, be the Hasse invariant of D. Let $\mathcal{L}$ be the set of O_K-stable lattice chains in F^{2n} that have the form

$$\Lambda_\bullet = \left[\Lambda_0 \supset \Lambda_1 \supset \ldots \supset \Lambda_{\ell-1} \supset \Lambda_\ell = \pi\Lambda_0\right] \tag{3.12}$$

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and that furthermore satisfy the following property:

- If ℓ is odd, then we demand that each quotient Λ_i/Λ_{i+1} is a free $O_K/(\pi)$-module of rank n/ℓ.
- If ℓ is even, then we instead require Λ_i/Λ_{i+1} is an $O_F/(\pi)$-vector space of dimension $2n/\ell$ and that the $O_K/(\pi)$-action on Λ_i/Λ_{i+1} factors through the first projection $O_K = O_F \times O_F \to O_F$ if i is even, and through the second projection if i is odd.

The stabilizer in G' of a lattice chain $\Lambda_\bullet \in \mathcal{L}$ is by definition the subgroup

$$\mathrm{Stab}_{G'}(\Lambda_\bullet) = \{\gamma \in G' \mid \gamma\Lambda_i = \Lambda_i \text{ for all } i\}.$$

The group H' acts transitively on $\mathcal{L}$ by translation, so these stabilizers form a single H'-conjugation orbit. However, because of the character $\eta(\,\cdot\,)|\cdot|^s$ in the definition of orbital integrals on G', we have to be more specific about our desired test function than in the case of G.

Definition 3.9. Pick any lattice chain $\Lambda_\bullet^{\mathrm{std}} \in \mathcal{L}$ such that $\Lambda_0^{\mathrm{std}} = O_F^{2n}$ and define

$$f_D'^{\circ} = \mathrm{vol}(\mathrm{Stab}_{H'}(\Lambda_\bullet^{\mathrm{std}}))^{-2} 1_{\mathrm{Stab}_{G'}(\Lambda_\bullet^{\mathrm{std}})}. \tag{3.13}$$

Any two choices for $\Lambda_\bullet^{\mathrm{std}}$ differ by $\mathrm{Stab}(O_F^{2n}) \cap H' = GL_n(O_F) \times GL_n(O_F)$, so $f_D'^{\circ}$ is defined up to conjugation by $GL_n(O_F) \times GL_n(O_F)$. Also note that if $G = GL_{2n}(F)$, then $\ell = 1$ and the only possible standard chain is $O_F^{2n} \supset \pi O_F^{2n}$ – we recover $f_D'^{\circ} = 1_{GL_{2n}(O_F)}$ as in Guo's case. Let $h_1 \in H'$ be any element with $|h_1|^{-s} = q^{-2ns/\ell}$ and define $f_D' \in C_c^\infty(G')$ by

$$f_D'(\gamma) := \begin{cases} f_D'^{\circ} & \text{if } \ell \text{ is odd} \\ f_D'^{\circ}(h_1^{-1}\gamma) & \text{if } \ell \text{ is even.} \end{cases} \tag{3.14}$$

By (3.5), the orbital integrals of $f_D'^{\circ}$ and f_D' are related by

$$\mathrm{Orb}(\gamma, f_D', s) = \mathrm{Orb}(\gamma, f_D'^{\circ}, s) \cdot \begin{cases} 1 & \text{if } \ell \text{ is odd} \\ q^{-2ns/\ell} & \text{if } \ell \text{ is even} \end{cases} \tag{3.15}$$

and, in particular, have the same central value. The advantage of the normalized function f_D' is that its functional equation is completely symmetric; cf. Proposition 3.19 below. Examples for $\Lambda_\bullet^{\mathrm{std}}$ and $f_D'^{\circ}$ when $n = 2$ and $\ell \in \{2, 4\}$ can be found at the beginning of §5.

Conjecture 3.10 (Fundamental lemma for CSAs). *The function f_D' is a transfer of f_D in the sense of Definition 3.4. That is, for regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, f_D') = \begin{cases} \mathrm{Orb}(g, f_D) & \text{if there exists a matching } g \in G_{\mathrm{rs}} \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Conjecture 3.10 complements the Guo–Jacquet Fundamental Lemma which is formulated for the case $D = M_{2n}(F)$ and for the full Hecke algebra. We recall it here for comparison:

Conjecture 3.11 (Guo–Jacquet Fundamental Lemma [13, (1.12)]). *Assume that $D = M_{2n}(F)$ and that the embedding $E \to D$ satisfies $O_E \subset M_{2n}(O_F)$. Then every $\mathrm{GL}_{2n}(O_F)$-biinvariant compactly supported function f is a transfer of itself: For regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, f) = \begin{cases} \mathrm{Orb}(g, f) & \text{if there exists a matching } g \in G_{\mathrm{rs}} \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

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Conjecture 3.10 and Conjecture 3.11 precisely overlap for the unit Hecke function $1_{GL_{2n}(O_F)}$. This is also the case that was proved by Guo; see [13, (1.12)]. We mention that Guo's formulation does not involve the transfer factor. Instead, he works with an orbit representative of the form

$$\gamma = \begin{pmatrix} 1 & w \\ 1 & 1 \end{pmatrix},$$

where each entry is an $(n \times n)$-matrix (see the line after [13, (1.10)]). Such a representative satisfies $\Omega(\gamma, 0) = 1$ which gives the link of his result with our formulation.

For Hasse invariant $\lambda = 1/2$, the CSA D is isomorphic to $M_n(D_{1/2})$. In this case, Conjecture 3.10 can be reduced to Guo's result and is hence known; we refer to [20].

The following is our main result in this setting. Its proof will be given as Theorem 8.2 below.

Theorem 3.12. *Conjecture 3.10 holds whenever D is a division algebra of degree* 4.

Note that the orbital integrals on the right-hand side of (3.16) have a particularly simple form if D is a division algebra:

Proposition 3.13. *Assume that D is a division algebra; denote by $v_D : D^\times \to \mathbb{Z}$ its normalized valuation. Assume that $g \in G_{\mathrm{rs}}$ is regular semi-simple and denote by $f(L_g/F)$ the inertia degree of L_g/F. Then*

$$\mathrm{Orb}(g, f_D) = \begin{cases} f(L_g/F) & \text{if } v_D(g) \in 2\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

Proof. In the given situation, f_D is the indicator function of the units $O_D^\times$ of the unique maximal order in D. The centralizer $C = \mathrm{Cent}_D(E)$ is a CDA of degree n over E, so $v_D(C^\times) = 2\mathbb{Z}$. It moreover holds that $v_D(D_- \backslash \{0\}) = 2\mathbb{Z} + 1$ and hence follows that $(O_C^\times\, g\, O_C^\times) \cap O_D^\times \neq \emptyset$ if and only if $v_D(g) \in 2\mathbb{Z}$. By the triangle inequality, this is equivalent to $v_D(g_-) > v_D(g_+)$, which is equivalent to $v_D(1 + z_g) = 0$. We obtain from Definition 3.3 that

$$\mathrm{Orb}(g, f_D) = \int_{L_g^\times \backslash C^\times} 1_{O_D^\times}(c(1 + z_g)c^{-1})\, dc = \mathrm{vol}(L_g^\times \backslash C^\times) 1_{O_D}(z_g).$$

The Haar measures were defined such that $\mathrm{vol}(O_C^\times) = \mathrm{vol}(O_{L_g}^\times) = 1$ and, in particular, satisfy $\mathrm{vol}(L_g^\times \backslash C^\times) = f(L_g/F)$. This proves (3.18). □

It is possible that the FL conjecture for D a division algebra is related to Kottwitz's Euler–Poincaré functions [22]. Our proof of Theorem 1.2 is not along such lines, however, but rather a byproduct of our calculation of $\partial\mathrm{Orb}\, O(\gamma, f_D')$ when $n = 2$.

Remark 3.14. The original motivation for our definition of f_D' was the following. Let $\breve{F}$ be the completion of a maximal unramified field extension of F. Denote by $O_{\breve{F}}$ its ring of integers and by $\mathbb{F}$ its residue field. The scalar extension $\breve{F} \otimes_F D$ is isomorphic to $M_{2n}(\breve{F})$, and under any such isomorphism, $\breve{O}_D = O_{\breve{F}} \otimes_{O_F} O_D$ gets identified with the stabilizer of a lattice chain

$$\breve{\Lambda}_\bullet = [\breve{\Lambda}_0 \supset \breve{\Lambda}_1 \supset \ldots \supset \breve{\Lambda}_{\ell-1} \supset \breve{\Lambda}_\ell = \pi\breve{\Lambda}_0]$$

such that $\dim_{\mathbb{F}}(\breve{\Lambda}_i/\breve{\Lambda}_{i+1}) = 2n/\ell$. The action of $R = O_{\breve{F}} \otimes_{O_F} O_E \subset \breve{O}_D$ on the quotients $\breve{\Lambda}_i/\breve{\Lambda}_{i+1}$ has the characteristics from Definition 3.8: If ℓ is odd, then every quotient $\breve{\Lambda}_i/\breve{\Lambda}_{i+1}$ is free over $R/(\pi)$ of rank n/ℓ. If ℓ is even, then the R-action on $\breve{\Lambda}_i/\breve{\Lambda}_{i+1}$ alternatingly factors through one of the two projections $R \to O_{\breve{F}}$.

A similar phenomenon occurs for the parahoric level fundamental lemma of Z. Zhang [49, Theorem 4.1] in the Gan–Gross–Prasad setting (also see [35, Conjecture 10.3] for an earlier formulation in a special case). The two group-theoretic data there that define the two test functions also have the property that they become isomorphic after scalar extension to $\breve{F}$.

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3.4. Functional equation for $\mathrm{Orb}(\gamma, f'_D, s)$

Our aim in this section is to prove a functional equation for $\mathrm{Orb}(\gamma, f'_D, s)$. The motivation for this is twofold: First, it will imply the vanishing part of the fundamental lemma (Conjecture 3.10) in many cases. Second, it will imply that the derivatives that will occur in our AT conjecture are indeed the leading terms of the Taylor expansion of the orbital integral in question. We begin by defining the sign of the functional equation.

Definition 3.15. Let $\lambda \in (2n)^{-1}\mathbb{Z}/\mathbb{Z}$ be the Hasse invariant of D and let $\delta \in F[T]$ be regular semi-simple of degree n. Let $L_\delta = F[z^2]/(\delta(z^2))$ and $B_\delta = (E \otimes_F L_\delta)[z]$ be the universal algebras for E and δ from Definition 2.7. Write $L_\delta = \prod_{i\in I} L_i$ for the decomposition of L_δ into fields and let $B_\delta = \prod_{i\in I} B_i$ be the corresponding decomposition of B_δ. Denoting by $\beta_i \in 2^{-1}\mathbb{Z}/\mathbb{Z}$ the Hasse invariant of B_i/L_i, we define

$$\varepsilon_D(\delta) := n\lambda + \sum_{i\in I} \beta_i \in 2^{-1}\mathbb{Z}/\mathbb{Z} \cong \{\pm 1\}. \tag{3.19}$$

We define $\varepsilon_D(\gamma) = \varepsilon_D(\mathrm{Inv}(\gamma))$ and $\varepsilon_D(g) = \varepsilon_D(\mathrm{Inv}(g))$ whenever $\gamma \in G'$ and $g \in G$ are regular semi-simple.

Lemma 3.16. *An equivalent description of $\varepsilon_D(\delta)$ is given as follows. Let $\delta_0 \in F^\times$ be the constant coefficient of δ and let $\varepsilon'_D = n\lambda \in \{\pm 1\}$. Then*

$$\varepsilon_D(\delta) = \eta(\delta_0) \cdot \varepsilon'_D. \tag{3.20}$$

In particular, if $\delta = \mathrm{Inv}(\gamma; T)$ for some $\gamma \in G'_{\mathrm{rs}}$, then $\varepsilon_D(\gamma) = \eta(\det_F(z_\gamma))\varepsilon'_D$.

Proof. With notation as in Definition 3.15, we need to see that $\sum_{i\in I} \beta_i = \eta(\delta_0)$. Let z_i denote the component of $z \in B_\delta$ in the factor B_i. Then $(-1)^n\delta_0 = \prod_{i\in I} N_{L_i/F}(z_i^2)$, so it suffices to show that $\beta_i = \eta(N_{L_i/F}(z_i^2))$. This follows directly from the compatibility of the local reciprocity map with the norm of field extensions; see, for example, [39, §2.4].

If $\delta = \mathrm{Inv}(\gamma; T)$, then δ_0 is (by definition) a square root of $\det_F(z_\gamma^2)$, and hence, the last formula holds. □

Lemma 3.17. *Let $g \in G_{\mathrm{rs}}$ be a regular semi-simple element. Then $\varepsilon_D(g) = 1$.*

Proof. Put $\delta = \mathrm{Inv}(g)$. By Corollary 2.8, the existence of an element $g \in G_{\mathrm{rs}}$ of invariant δ is equivalent to the identity $[B_\delta] = [L_\delta \otimes_F D]$ in the Brauer group of L_δ. Writing $L_\delta = \prod_{i\in I} L_i$ as a product of fields as before and taking the sum of the Hasse invariants on both sides, we obtain

$$\sum_{i\in I} \beta_i = \sum_{i\in I} [L_i : F]\lambda = n\lambda.$$ □

Remark 3.18. Table 1 illustrates that the converse to Lemma 3.17 does not hold. Its rows 3 and 4 show cases where the sign $\varepsilon_D(\delta)$ is positive for $D = M_4(F)$ or $M_2(D_{1/2})$, but where there is no $g \in G_{\mathrm{rs}}$ of invariant δ. Row 5 shows the existence of such cases when D is a division algebra of degree 4.

Proposition 3.19. *The orbital integrals of f'_D satisfy the functional equation*

$$\mathrm{Orb}(\gamma, f'_D, -s) = \varepsilon_D(\gamma)\,\mathrm{Orb}(\gamma, f'_D, s). \tag{3.21}$$

The proof will be given at the end of this section. We first establish some auxiliary results that provide a combinatorial expression for $\mathrm{Orb}(\gamma, f'_D, s)$. Everything relies on the following simple observation: Assume that $h_1, h_2 \in H'$ are two elements and that $\Lambda_{i,\bullet} = h_i\Lambda^{\mathrm{std}}_\bullet$ are the two corresponding lattice chains in $\mathcal{L}$. Then

$$h_1^{-1}\gamma h_2 \in \mathrm{Stab}_{G'}(\Lambda^{\mathrm{std}}_\bullet) \iff \gamma\Lambda_{2,\bullet} = \Lambda_{1,\bullet}. \tag{3.22}$$

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Table 1. *Matching to $[G_{\mathrm{rs}}]$ for $n = 2$. Here, D_λ denotes a CDA of Hasse invariant λ over F. Moreover, $\varepsilon_\lambda(\delta)$ denotes $\varepsilon_D(\delta)$ for D a CSA of degree 4 and Hasse invariant λ. These are the signs of the functional equation for f'_D and will be defined in §3.4.*

L_δ	B_δ	D s.th. there is some $g \in G$ with $\mathrm{Inv}(g) = \delta$	$\varepsilon_0(\delta)$ $\varepsilon_{1/2}(\delta)$	$\varepsilon_{1/4}(\delta)$ $\varepsilon_{3/4}(\delta)$
Field	$M_2(L)$	$M_4(F)$ and $M_2(D_{1/2})$	+	−
Field	Division	$D_{1/4}$ and $D_{3/4}$	−	+
$F \times F$	$M_2(F) \times M_2(F)$	$M_4(F)$	+	−
$F \times F$	$D_{1/2} \times D_{1/2}$	$M_2(D_{1/2})$	+	−
$F \times F$	$M_2(F) \times D_{1/2}$	none	−	+

Definition 3.20. Motivated by (3.22), we make the following two definitions. First, we let

$$\mathcal{L}(\gamma) := \{\Lambda_\bullet \in \mathcal{L} \mid \gamma\Lambda_\bullet \in \mathcal{L}\}. \tag{3.23}$$

Second, for every lattice $\Lambda \subset F^{2n}$ such that both Λ and $\gamma\Lambda$ are $(O_F \times O_F)$-stable, we put

$$\Omega(\gamma, \Lambda, s) := \Omega(h_1^{-1}\gamma h_2, s), \tag{3.24}$$

where $h_1, h_2 \in H'$ are chosen such that $\Lambda = h_2 \cdot O_F^{2n}$ and and $\gamma\Lambda = h_1 \cdot O_F^{2n}$. (The lattice O_F^{2n} comes up here because we have normalized the test function f'_D by the requirement $\Lambda_0^{\mathrm{std}} = O_F^{2n}$.)

Assume that $\gamma \in G'_{\mathrm{rs}}$ is regular semi-simple. The torus $L_\gamma^\times \subset G'$ acts on $\mathcal{L}(\gamma)$ by multiplication and we write

$$\mathrm{Stab}(\Lambda_\bullet) = \{x \in L_\gamma^\times \mid x\Lambda_i = \Lambda_i \text{ for all } i = 0, \dots, \ell - 1\}$$

for the stabilizer a lattice chain $\Lambda_\bullet \in \mathcal{L}(\gamma)$. Taking into account the volume factor in the definition of f'_D (see Definition 3.9) as well as the normalization in (3.15), we can then write the orbital integral of f'_D as

$$\mathrm{Orb}(\gamma, f'_D, s) = q^{-ms} \sum_{\Lambda_\bullet \in L_\gamma^\times \backslash \mathcal{L}(\gamma)} [O_{L_\gamma}^\times : \mathrm{Stab}(\Lambda_\bullet)]\, \Omega(\gamma, \Lambda_0, s), \tag{3.25}$$

where $m = 0$ if ℓ is odd and $m = 2n/\ell$ if ℓ is even. The next few lemmas study this expression in more detail.

Lemma 3.21. *Let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element and let $z = z_\gamma$.*

(1) *Let Λ be an O_K-lattice. Then $\gamma\Lambda$ is an O_K-lattice as well if and only if $z/(1+z) \cdot \Lambda \subseteq \Lambda$. If z is topologically nilpotent, then this is furthermore equivalent to $z\Lambda \subset \Lambda$.*

(2) *Assume that ℓ is odd and that z is topologically nilpotent. Then*

$$\mathcal{L}(\gamma) = \{\Lambda_\bullet \in \mathcal{L} \mid z\Lambda_i \subseteq \Lambda_i \textit{ for all } i = 0, \dots, \ell - 1\}. \tag{3.26}$$

(3) *Assume that ℓ is even. Then $\mathcal{L}(\gamma) \neq \emptyset$ only for γ such that z is topologically nilpotent. More precisely,*

$$\mathcal{L}(\gamma) = \{\Lambda_\bullet \in \mathcal{L} \mid z\Lambda_i \subseteq \Lambda_{i+1} \textit{ for all } i = 0, \dots, \ell - 1\}. \tag{3.27}$$

Proof. (1) Write $O_K = O_F[\zeta]$, where ζ satisfies $\bar{\zeta} = 1 - \zeta$. Then $\gamma^{-1}\zeta\gamma = \zeta + z/(1+z) \cdot (1 - 2\zeta)$ as in (2.7). Hence, given an O_K-lattice Λ, the lattice $\gamma\Lambda$ is ζ-stable if and only if

$$(\zeta + z/(1+z) \cdot (1 - 2\zeta))\Lambda \subseteq \Lambda.$$

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It is checked directly that $1 - 2\zeta \in O_K^\times$, so this inclusion holds if and only if $z/(1+z) \cdot \Lambda \subseteq \Lambda$. If z is moreover topologically nilpotent, then $O_F[z] = O_F[z/(1+z)]$, and this condition becomes equivalent to $z\Lambda \subset \Lambda$.

(2) Assume that ℓ is odd. Any lattice chain $\Lambda_\bullet \in \mathcal{L}(\gamma)$ has the property that each Λ_i is both O_K-stable and $\gamma^{-1}O_K\gamma$-stable. By Part (1) and under our assumption that z is topologically nilpotent, this is equivalent to $z \cdot \Lambda_i \subseteq \Lambda_i$ which proves the relation $\subseteq$ in (3.26). Assume conversely that $\Lambda_\bullet \in \mathcal{L}$ has the property that each Λ_i is z-stable. We need to show and claim that each $(\gamma\Lambda_i)/(\gamma\Lambda_{i+1})$ is a free $O_K/(\pi)$-module. Since γ_+ is O_K-linear, this is equivalent to each quotient

$$(\gamma_+^{-1}\gamma\Lambda_i)/(\gamma_+^{-1}\gamma\Lambda_{i+1}) = ((1+z)\Lambda_i)/((1+z)\Lambda_{i+1})$$

being a free $O_K/(\pi)$-module. But it was assumed that z is topologically nilpotent and that each Λ_i is z-stable, so $(1+z)\Lambda_i = \Lambda_i$ for all i, and the claim follows because $\Lambda_\bullet \in \mathcal{L}$.

(3) Assume that ℓ is even and that $\Lambda_\bullet \in \mathcal{L}(\gamma)$ is any lattice chain. Then by definition of $\mathcal{L}(\gamma)$, the action of O_K on $(\gamma\Lambda_i)/(\gamma\Lambda_{i+1})$ factors over the first factor of $O_K = O_F \times O_F$ if i is even and over the second factor if i is odd. Equivalently (apply the isomorphism γ), the γ-conjugated action of O_K on Λ_i/Λ_{i+1} factors over the first factor if i is even and over the second factor if i is odd. This is yet equivalent to ζ and $\gamma^{-1}\zeta\gamma = \zeta + z/(1+z)\cdot(1-2\zeta)$ defining the same endomorphism of Λ_i/Λ_{i+1}. Since $1 - 2\zeta \in O_K^\times$, this happens if and only if $z/(1+z) \cdot \Lambda_i \subseteq \Lambda_{i+1}$. Given that this holds for all i and that $\Lambda_\ell = \pi\Lambda_0$, we deduce that $z/(1+z)$ is topologically nilpotent. Then z is topologically nilpotent as well, as claimed in the lemma. Identity (3.27) follows easily from the given arguments. □

Lemma 3.22. (1) *The following operator Z_γ, defined on lattice chains in F^{2n}, defines an automorphism of $\mathcal{L}(\gamma)$:*

$$Z_\gamma \cdot [\Lambda_0 \supset \Lambda_1 \supset \ldots \supset \Lambda_\ell] := [z_\gamma\Lambda_1 \supset z_\gamma\Lambda_2 \supset \ldots \supset z_\gamma\Lambda_\ell \supset \pi z_\gamma\Lambda_1]. \tag{3.28}$$

(2) *Moreover, Z_γ commutes with the $L_\gamma^\times$-action on $\mathcal{L}(\gamma)$ and satisfies*

$$\Omega(\gamma, (Z_\gamma\Lambda_\bullet)_0, s) = \varepsilon_D(\gamma)\Omega(\gamma, \Lambda_0, -s) \cdot \begin{cases} 1 & \textit{if } \ell \textit{ is odd} \\ q^{4ns/\ell} & \textit{if } \ell \textit{ is even.} \end{cases} \tag{3.29}$$

Proof. (1) A direct computation shows that $\gamma z_\gamma \gamma^{-1} = \gamma_+ z_\gamma \gamma_+^{-1}$, so both elements z_γ and $\gamma z_\gamma \gamma^{-1}$ are K-conjugate linear elements of G'. It follows that if a lattice Λ has the property that both Λ and $\gamma\Lambda$ are O_K-stable, then also $z_\gamma\Lambda$ and $\gamma z_\gamma\Lambda$ are O_K-stable. Thus, given any $\Lambda_\bullet \in \mathcal{L}(\gamma)$, the new chains $Z_\gamma\Lambda_\bullet$ and $\gamma Z_\gamma\Lambda_\bullet$ are again chains of O_K-lattices. Taking into account the shift by one in (3.28), both $Z_\gamma\Lambda_\bullet$ and $\gamma Z_\gamma\Lambda_\bullet$ again satisfy the eigenvalue condition in the definition of $\mathcal{L}$ (see Definition 3.8). Hence, $Z_\gamma\Lambda_\bullet \in \mathcal{L}(\gamma)$ as claimed.

(2) Proposition 2.6 states that $L_\gamma = F[z_\gamma^2]$, which implies that multiplication by z_γ and by elements from $L_\gamma^\times$ commute. It is left to prove Identity (3.29). It is easily checked that both sides of that identity are invariant under left-multiplication of H' on γ. So we may assume that $\gamma = 1+z$ with $z = z_\gamma$. This implies that γ and z commute which will simplify some expressions below. It furthermore allows for a more convenient description of $\Omega(\gamma, \Lambda, s)$. In its formulation, we write $\Lambda = \Lambda_+ \oplus \Lambda_-$ for the decomposition of an O_K-lattice Λ into its O_K-eigenspaces.

Lemma 3.23. *Assume that $\gamma = 1 + z$ with $z = z_\gamma$. Assume that Λ is an O_K-lattice such that also $\gamma\Lambda$ is an O_K-lattice. Then*

$$\Omega(\gamma, \Lambda, s) = (-1)^{[(\gamma\Lambda)_-:z\Lambda_+]+[(\gamma\Lambda)_-:\Lambda_-]} q^{([(\gamma\Lambda)_+:z\Lambda_-]-[(\gamma\Lambda)_-:z\Lambda_+])s}. \tag{3.30}$$

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Proof. This follows directly from the definition of $\Omega(\gamma, \Lambda, s)$: Assume that $\Lambda = h_2 O_F^{2n}$ and that $\gamma\Lambda = h_1 O_F^{2n}$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = h_1^{-1}\gamma h_2$. Recall that now by (3.24) and by Definition 3.2,

$$\Omega(\gamma, \Lambda, s) = \Omega\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), s\big) = (-1)^{v(c)+v(d)} q^{(v(b)-v(c))\cdot s}, \tag{3.31}$$

where $v : F^\times \to \mathbb{Z}$ is the normalized valuation. Translating to Λ, we have

$$\begin{aligned} v(a) &= [(\gamma\Lambda)_+ : \Lambda_+] & v(b) &= [(\gamma\Lambda)_+ : z\Lambda_-] \\ v(c) &= [(\gamma\Lambda)_- : z\Lambda_+] & v(d) &= [(\gamma\Lambda)_- : \Lambda_-]. \end{aligned} \tag{3.32}$$

Substituting (3.32) in (3.31) proves the lemma. □

Let Λ be an O_K-lattice such that also $\gamma\Lambda$ is an O_K-lattice. Since z is K-conjugate linear and furthermore commutes with γ, it holds that

$$(z\Lambda)_\pm = z\Lambda_\mp, \quad (\gamma z\Lambda)_\pm = z(\gamma\Lambda)_\mp.$$

We obtain from (3.30) that

$$\Omega(\gamma, z\Lambda, s) = (-1)^{[z(\gamma\Lambda)_+ : z^2\Lambda_-] + [z(\gamma\Lambda)_+ : z\Lambda_+]} q^{([z(\gamma\Lambda)_- : z^2\Lambda_+] - [z(\gamma\Lambda)_+ : z^2\Lambda_-])s}. \tag{3.33}$$

The exponents of the signs of (3.30) and (3.33) are related by

$$[z(\gamma\Lambda)_+ : z^2\Lambda_-] + [z(\gamma\Lambda)_+ : z\Lambda_+] = [\gamma\Lambda : z\Lambda] - [(\gamma\Lambda)_- : z\Lambda_+] + [\gamma\Lambda : \Lambda] - [\gamma\Lambda_- : \Lambda_-], \tag{3.34}$$

those for the q-powers by

$$[z(\gamma\Lambda)_- : z^2\Lambda_+] - [z(\gamma\Lambda)_+ : z^2\Lambda_-] = -[(\gamma\Lambda)_+ : z\Lambda_-] + [(\gamma\Lambda)_- : z\Lambda_+]. \tag{3.35}$$

Note that $[\gamma\Lambda : z\Lambda] + [\gamma\Lambda : \Lambda] \equiv [\Lambda : z\Lambda] \bmod 2$ in (3.34), so we obtain

$$\Omega(\gamma, z\Lambda, s) = \eta(\det(z))\Omega(\gamma, \Lambda, -s). \tag{3.36}$$

It is left to take care of the shift in (3.28). Assume that $\Lambda' \subseteq \Lambda$ is a sublattice that also has the property that both Λ' and $\gamma\Lambda'$ are O_K-stable. Define integers $a_\pm$ and $b_\pm$ by the identities

$$a_\pm = [\Lambda_\pm : \Lambda'_\pm] \quad \text{and} \quad b_\pm = [(\gamma\Lambda)_\pm : (\gamma\Lambda')_\pm].$$

Then we obtain from (3.30) that

$$\Omega(\gamma, \Lambda', s) = (-1)^{a_+ + a_-} q^{(a_- - b_+ - a_+ + b_-)s}\Omega(\gamma, \Lambda, s). \tag{3.37}$$

Apply this to the two lattices $z\Lambda_1 \subset z\Lambda_0$ that arise from $z\Lambda_\bullet$ with $\Lambda_\bullet \in \mathcal{L}(\gamma)$. Depending on the parity of ℓ, the following two cases occur. If ℓ is odd, then $z\Lambda_0/z\Lambda_1$ is free over $O_K/(\pi)$ so $a_+ = a_-$ and $b_+ = b_-$. We obtain that

$$\Omega(\gamma, z\Lambda_1, s) = \Omega(\gamma, z\Lambda_0, s). \tag{3.38}$$

If ℓ is even, then $z\Lambda_0/z\Lambda_1$ and $\gamma z\Lambda_0/\gamma z\Lambda_1$ are both free over $O_F/(\pi)$ of rank $2n/\ell$ with O_K acting via the second projection. (Indeed, O_K acts via the first projection on Λ_0/Λ_1 and $(\gamma\Lambda_0)/(\gamma\Lambda_1)$, but z is O_K-conjugate linear.) In particular, $a_+ = b_+ = 0$ and $a_- = b_- = 2n/\ell$. Identity (3.37) then specializes to

$$\Omega(\gamma, z\Lambda_1, s) = (-1)^{2n/\ell} q^{4ns/\ell}\Omega(\gamma, z\Lambda_0, s). \tag{3.39}$$

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Combining (3.36), (3.38) and (3.39), it follows that

$$\Omega(\gamma, (Z_\gamma\Lambda_\bullet)_0, s) = (-1)^{2n/\ell}\eta(\det(z))\Omega(\gamma, \Lambda_0, -s)\cdot\begin{cases}1 & \text{if } \ell \text{ is odd}\\ q^{4ns/\ell} & \text{if } \ell \text{ is even.}\end{cases} \tag{3.40}$$

Recall that the constant coefficient of $\mathrm{Inv}(\gamma)$ is a square root of $\det(z^2)$. The sign $(-1)^{2n/\ell}\eta(\det(z))$ from (3.40) hence equals $\varepsilon_D(\gamma)$ by Lemma 3.16, and the proof of (3.29) is complete. □

Proof of the functional equation (Proposition 3.19). Let $m = 0$ if ℓ is odd and $m = 2n/\ell$ if ℓ is even. Using the combinatorial description (3.25) together with Lemma 3.22, we have

$$\begin{aligned}\mathrm{Orb}(\gamma, f'_D, -s) &= q^{ms}\sum_{\Lambda_\bullet\in L_\gamma^\times\backslash\mathcal{L}(\gamma)}[O_{L_\gamma}^\times : \mathrm{Stab}(\Lambda_\bullet)]\,\Omega(\gamma, \Lambda_0, -s)\\ &= \varepsilon_D(\gamma)q^{ms}q^{-2ms}\sum_{\Lambda_\bullet\in L_\gamma^\times\backslash\mathcal{L}(\gamma)}[O_{L_\gamma}^\times : \mathrm{Stab}(\Lambda_\bullet)]\,\Omega(\gamma, (Z_\gamma\Lambda_\bullet)_0, s)\\ &= \varepsilon_D(\gamma)\,\mathrm{Orb}(\gamma, f'_D, s),\end{aligned}$$

as was to be shown. □

4. Arithmetic transfer

The setting is the same as in §3.1 except that, from now on, we take

$$G = D^{\mathrm{op},\times}, \quad H = C^{\mathrm{op},\times}. \tag{4.1}$$

Note that G and G^{op} have the same underlying topological space, which implies that $C_c^\infty(G) = C_c^\infty(G^{\mathrm{op}})$. Moreover, H and H^{op} have the same underlying topological space as well, and the definitions of $g \in G$ being regular semi-simple, of the invariant $\mathrm{Inv}(g; T)$, and of the orbital integral $\mathrm{Orb}(f, g)$ are all unchanged when taking them for opposed CSAs.

4.1. Local Shimura data

Let $\breve{F}$ be the completion of a maximal unramified extension of F. Let $O_{\breve{F}}$ denote its ring of integers and let $\mathbb{F}$ be its residue field. The Frobenius automorphism of $\breve{F}$ is the unique F-automorphism inducing q-Frobenius $x \mapsto x^q$ on $\mathbb{F}$; we denote it by $\sigma : \breve{F} \to \breve{F}$. Let $v_{\breve{F}} : \breve{F}^\times \to \mathbb{Z}$ be the normalized valuation.

By F-isocrystal, or simply isocrystal, we mean a pair $\mathbf{N} = (N, \mathbf{F})$ that consists of a finite-dimensional $\breve{F}$-vector space N and a σ-linear automorphism $\mathbf{F}$. The Verschiebung of $\mathbf{N}$ is defined as $\mathbf{V} = \pi\mathbf{F}^{-1}$. Height, dimension and slope of $\mathbf{N}$ are all meant in the relative sense with respect to F: The height $\mathrm{ht}(\mathbf{N})$ is the $\breve{F}$-dimension of N, the dimension $\dim(\mathbf{N})$ is the integer $v_{\breve{F}}(\det\mathbf{V})$, and the slope is their ratio $\dim(\mathbf{N})/\mathrm{ht}(\mathbf{N})$. Note that $\dim(\mathbf{N})$ might be negative.

The Dieudonné–Manin classification [7] states that the (F-linear) category of isocrystals is semi-simple and that the isomorphism classes of its simple objects are in bijection with $\mathbb{Q}$: For every $\mu = r/s$, where $(r, s) = 1$, there is a unique (up to isomorphism) simple isocrystal $\mathbf{N}_\mu$ of height s and dimension r. The endomorphism ring $\mathrm{End}(\mathbf{N}_\mu)$ is a CDA over F of Hasse invariant μ.

Definition 4.1. (1) By C-isocrystal, we mean a pair $(\mathbf{N}_+, \iota)$ that consists of an isocrystal $\mathbf{N}_+$ and an F-linear C-action $\iota : C \to \mathrm{End}(\mathbf{N}_+)$ with the following numerical conditions: The height of $\mathbf{N}_+$ is $2n^2 = \dim_F(C)$, the dimension of $\mathbf{N}_+$ is n, and the slopes of all subisocrystals of $\mathbf{N}_+$ lie in the interval $[0, 1]$.

(2) By D-isocrystal, we mean a pair $(\mathbf{N}, \kappa)$ that consists of an isocrystal $\mathbf{N}$ and an F-linear D-action $\kappa : D \to \mathrm{End}(\mathbf{N})$ with the following numerical conditions: The height of $\mathbf{N}$ is $4n^2 = \dim_F(D)$, the dimension of $\mathbf{N}$ is $2n$, and the slopes of all subisocrystals of $\mathbf{N}$ lie in the interval $[0, 1]$.

Remark 4.2. Recall that by covariant Dieudonné theory p-divisible groups over $\mathbb{F}$ together with quasi-homomorphisms are equivalent to $\mathbb{Q}_p$-isocrystals that have the slopes of all subisocrystals in the interval $[0, 1]$. Under this equivalence, height and dimension of the p-divisible equal height and dimension of the corresponding $\mathbb{Q}_p$-isocrystal. The analogous statement holds for strict O_F-modules over $\mathbb{F}$ (see Definition 4.8) and F-isocrystals. This motivates the slope condition in Definition 4.1.

The Serre tensor construction defines a functor

$$\begin{aligned} \{C\text{-isocrystals}\} &\longrightarrow \{D\text{-isocrystals}\} \\ (\mathbf{N}_+, \iota) &\longmapsto (\mathbf{N} = D \otimes_C \mathbf{N}_+,\ \kappa(x) = x \otimes \mathrm{id}_{\mathbf{N}_+}). \end{aligned} \tag{4.2}$$

Lemma 4.3. (1) *Two C-isocrystals (resp. two D-isocrystals) are isomorphic if and only if the underlying isocrystals are isomorpic. In particular, the functor (4.2) defines an injective map on isomorphism classes.*

(2) *A D-isocrystal $(\mathbf{N}, \kappa)$ lies in the essential image of (4.2) if and only if there exists an F-algebra map $E \to \mathrm{End}_D(\mathbf{N}, \kappa)$.*

Proof. (1) Let $\mathbf{N}$ be any isocrystal. By the Dieudonné–Manin classification, $\mathrm{End}(\mathbf{N})$ is a product of CSAs over F. It then follows from the Skolem–Noether Theorem applied factor by factor that any two F-algebra homomorphisms $C \to \mathrm{End}(\mathbf{N})$ are conjugate. In other words, there is at most one way (up to C-linear isomorphism) to define a C-action on $\mathbf{N}$. The same argument applies to D-isocrystals. The injectivity of (4.2) on isomorphism classes follows directly because $D \otimes_C \mathbf{N}_+ \cong \mathbf{N}_+^{\oplus 2}$ as isocrystal.

(2) The category of C-isocrystals is E-linear because the center of C is E. The Serre tensor construction is functorial, so every object in its image has a D-linear E-action. Explicitly, E acts on $(\mathbf{N}, \kappa) = D \otimes_C (\mathbf{N}_+, \iota)$ by

$$\iota|_E : E \longrightarrow \mathrm{End}_D(\mathbf{N}, \kappa), \quad a \longmapsto 1 \otimes \iota(a).$$

Assume conversely that there exists an embedding $\iota : E \to \mathrm{End}_D(\mathbf{N}, \kappa)$. Then $\mathbf{N}$ has an action by $\kappa(E) \otimes_F \iota(E)$ and thus decomposes into C-stable eigenspaces $\mathbf{N} = \mathbf{N}_+ \oplus \mathbf{N}_-$. Let us write ι for the resulting C-action $C \to \mathrm{End}(\mathbf{N}_+)$ on the first factor. The natural D-linear map $D \otimes_C (\mathbf{N}_+, \iota) \to \mathbf{N}$ is the desired isomorphism. □

We next place the above definitions into a group-theoretic context, following the EL formalism in [38, Definition 3.18]. For this, we consider H and G as algebraic groups over F. Our convention is that $\mathrm{End}_D(D)$ acts on the left of D and is hence isomorphic to D^{op}. In light of (4.1), we have an isomorphism

$$G \xrightarrow{\cong} \mathrm{End}_D(D)^\times, \quad g \longmapsto [x \longmapsto xg].$$

Recall that the Kottwitz set of G is defined as the set of σ-conjugacy classes in $G(\breve{F})$:

$$B(G) = G(\breve{F})/\{b \sim gb\sigma(g)^{-1}\}.$$

It is a standard fact (see [38, §1.7]) that this set is in bijection with isomorphism classes of isocrystals of height $\dim_F(D)$ with D-action:

$$\begin{aligned} B(G) &\xrightarrow{\cong} \left\{ (\mathbf{N}, \kappa) \,\middle|\, \begin{matrix} \mathbf{N} \text{ an } F\text{-isocrystal of height } \dim_F(D) \\ \kappa : D \to \mathrm{End}(\mathbf{N}) \text{ a } D\text{-action} \end{matrix} \right\} /\cong \\ [b] &\longmapsto (\mathbf{N}_b, \kappa) := (\breve{F} \otimes_F D, \sigma \otimes b). \end{aligned} \tag{4.3}$$

After scalar extension to $\overline{F}$, there is an isomorphism $\alpha : \overline{F} \otimes_F D^{\mathrm{op}} \cong M_{2n}(\overline{F})$ of $\overline{F}$-algebras from which one obtains an isomorphism $G_{\overline{F}} \cong GL_{2n,\overline{F}}$. Consider the $\overline{F}$-conjugacy class of the cocharacter

$$\mu_G : \mathbb{G}_m \longrightarrow G_{\overline{F}} \xrightarrow[\alpha]{\cong} GL_{2n,\overline{F}}, \quad t \longmapsto \mathrm{diag}(t, \ldots, t, 1). \tag{4.4}$$

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Via 4.1, the subset $B(G, \mu_G) \subset B(G)$ of μ-admissible elements is in bijection with the isomorphism classes of D-isocrystals from Definition 4.1. (For the purposes of our article, the reader may take that as the definition of $B(G, \mu_G)$.) Namely, by the Dieudonné classification from Remark 4.2, every D-isocrystal $(\mathbf{N}, \kappa)$ is the isocrystal of a strict O_F-module X over $\mathbb{F}$ together with a rational action $\kappa : D \to F \otimes_{O_F} \mathrm{End}(X)$. By [38, §3.19] which also holds for F-isocrystals, this implies that $(\mathbf{N}, \kappa)$ lies in $B(G, \mu_G)$. Conversely, an F-isocrystal with D-action $(\mathbf{N}, \kappa) \in B(G, \mu_G)$ is necessarily μ-weakly admissible. By definition (see [38, Definition 1.18 and §1.3]), the latter is equivalent to $\mathbf{N}$ being of dimension $2n$ and with the slopes of all subisocrystals in $[0, 1]$.

To make the analogous definitions for H, we need to fix an embedding $E \subset \overline{F}$. Also choose an isomorphism $\beta : \overline{F} \otimes_F C^{\mathrm{op}} \cong M_n(\overline{F}) \times M_n(\overline{F})$ such that $\beta(a) = (a, \bar{a})$ for all $a \in E$. Consider the $H(\overline{F})$-conjugacy class of

$$\mu_H : \mathbb{G}_m \longrightarrow H_{\overline{F}} \xrightarrow[\beta]{\cong} GL_{n,\overline{F}} \times GL_{n,\overline{F}}, \quad t \longmapsto \mathrm{diag}((t, \dots, t, 1),\ (t, \dots, t)). \tag{4.5}$$

Then $B(H, \mu_H)$ is in bijection with the isomorphism classes of C-isocrystals in the sense of Definition 4.1. Moreover, the natural map $B(H, \mu_H) \to B(G, \mu_G)$ is given by (4.2).

Definition 4.4. For $b \in H(\breve{F})$, we denote by $(\mathbf{N}_{b,+}, \iota)$ the C-isocrystal given by $(\breve{F} \otimes_F C, \sigma \otimes b)$ with its natural C-action. We write $C_b = \mathrm{End}_C(\mathbf{N}_{b,+}, \iota)$ and $H_b = C_b^\times$. We further define

$$(\mathbf{N}_b, \kappa) := D \otimes_C (\mathbf{N}_{+,b}, \iota)$$

as well as $D_b = \mathrm{End}_D(\mathbf{N}_b, \kappa)$ and $G_b = D_b^\times$. Note that there is an inclusion $H_b \to G_b$, $g \mapsto \mathrm{id}_D \otimes g$ by functoriality of the Serre tensor construction.

It follows from the Dieudonné–Manin classification that D_b is a semi-simple F-algebra of total degree $2n$ in the sense of (2.10). By construction, there is an embedding $E \to D_b$ and $C_b = \mathrm{Cent}_{D_b}(E)$. The more precise description of D_b is as follows: Let $\mathbf{N}_b \cong \bigoplus_{\mu \in [0,1]} \mathbf{N}_\mu^{n_\mu}$ be the slope decomposition, where $\mathbf{N}_\mu$ denotes a simple isocrystal of slope μ. Then

$$D_b \cong \prod_{\mu \in [0,1]} M_{m_\mu}(D_{\mu-\lambda}),$$

where λ is the Hasse invariant of D, where $D_{\mu-\lambda}$ denotes a CDA over F of Hasse invariant $\mu - \lambda$, and where m_μ is characterized by

$$M_{m_\mu}(D_{\mu-\lambda}) \otimes_F D_\lambda \cong M_{n_\mu}(D_\mu).$$

It is a well-known and curious phenomenon that the number of elements of $B(H, \mu_H)$ and $B(G, \mu_G)$ strongly depends on λ. We give some examples:

Example 4.5. (1) Assume that $D = M_m(D_0)$, where D_0 is a CDA over F. Then, by Morita equivalence, the category of isocrystals with D-action is equivalent to that of isocrystals with D_0-action: To a pair $(\mathbf{N}_0,\ \kappa_0 : D_0 \to \mathrm{End}(\mathbf{N}_0))$, one associates the m-th power $\mathbf{N}_0^m$ with its natural extension of κ_0 to $M_m(D_0)$. Under this equivalence, D-isocrystals in the sense of Definition 4.1 correspond to isocrystals with D_0-action $(\mathbf{N}_0, \kappa_0)$ such that $\mathbf{N}_0$ is of height $\dim_F(D)/m = m \cdot \dim_F(D_0)$, of dimension $[D : F]/m = [D_0 : F]$, and has all its slopes within $[0, 1]$. (Note that $[D : F] = m \cdot [D : F_0]$, explaining why $\mathbf{N}_0$ is required to have dimension $[D_0 : F]$.)

(2) Consider the special case $D = M_{2n}(F)$. By (1), the Kottwitz set $B(G, \mu_G)$ is in bijection with isomorphism classes of isocrystals of height $2n$, dimension 1 and with all slopes within $[0, 1]$. The slope vector of such an isocrystal is of the form $(0^{(2n-n_0)}, 1/n_0)$ for a unique integer $1 \leq n_0 \leq 2n$, and every such n_0 can occur. The endomorphism ring in this case is isomorphic to $M_{2n-n_0}(F) \times D_{1/n_0}$, which admits an embedding of E if and only if n_0 is even. This characterizes the image of the map $B(H, \mu_H) \to B(G, \mu_G)$ by Lemma 4.3.

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(3) Assume that $\lambda \in \{1/2n, (n+1)/2n\}$. Then the Hasse invariant (over E) of C is $2\lambda = 1/n$. In this case, $B(H, \mu_H)$ consists of a single element $[b]$ (cf. [38, Lemma 3.60]), which is known as the Drinfeld case. The corresponding isocrystals $\mathbf{N}_{+,b}$ and $\mathbf{N}_b$ are isoclinic of slope $1/2n$. This applies in particular when $n = 2$ and $\lambda \in \{1/4, 3/4\}$ which is the main case of interest of the paper.

(4) Assume that $n = 3$ and $\lambda \in \{1/3, 5/6\}$. Then the Hasse invariant of C is $2/3$ and $B(H, \mu_H)$ consists of two elements. By Lemma 4.3, they may be characterized uniquely by the slope vector of the underlying isocrystal. One possibility is $(1/6, 1/6, 1/6)$, which is the basic case; the other is $(1/12, 1/3, 1/3)$.

Recall that we have given a definition of invariant for double cosets $H_b \backslash G_b / H_b$; see (2.11).

Proposition 4.6. *Let $\delta \in F[T]$ be a regular semi-simple invariant of degree n. Then there is at most one $[b] \in B(H, \mu_H)$ such that there exists an element $g \in G_b$ of invariant δ. In case of existence, all such elements g form a single $H_b \times H_b$-orbit. Furthermore, in this case, $\varepsilon_D(\delta) = -1$.*

Note that the statement about the set of such g forming a single orbit is nontrivial because D_b may not be simple, so the Skolem–Noether Theorem does not immediately apply.

Corollary 4.7. *Let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element such that there exists an isogeny class $[b] \in B(H, \mu_H)$ and an element $g \in G_b$ that matches γ. Then* $\mathrm{Orb}(\gamma, f'_D) = 0$.

Proof of the corollary. Proposition 4.6 states that the sign in the functional equation of $\mathrm{Orb}(\gamma, f'_D, s)$ is negative. (See Proposition 3.19 for that functional equation.) □

Proof of Proposition 4.6. We write $B = B_\delta$ and $L = L_\delta$ in the following. Let $L = \prod_{i \in I} L_i$ denote the decomposition of L into fields and let $B = \prod_{i \in I} B_i$ be the corresponding decomposition of B. The tensor product $P = D \otimes_F B$ has center L and a similar decomposition $P = \prod_{i \in I} P_i$. The i-th factor P_i is a CSA of degree $4n$ over L_i. Let $\rho_i \in (4n)^{-1}\mathbb{Z}/\mathbb{Z}$ be its Hasse invariant, and write $n_i = [L_i : F]$. Let β_i be the Hasse invariant of B_i/L_i. These invariants are related by $\rho_i = n_i\lambda + \beta_i$ because

$$\begin{aligned} \mathrm{inv}_{L_i}((D \otimes_F L_i) \otimes_{L_i} B_i) &= \mathrm{inv}_{L_i}(D \otimes_F L_i) + \mathrm{inv}_{L_i}(B_i) \\ &= [L_i : F] \cdot \lambda + \beta_i. \end{aligned} \tag{4.6}$$

Let $b \in B(G, \mu_G)$ be any isomorphism class; denote by $(\mathbf{N}_b, \kappa)$ the corresponding D-isocrystal. Giving an embedding $E \longrightarrow D_b$ and an orbit $H_b g H_b \subset G_b$ of invariant δ is the same as lifting κ to a faithful action $\widetilde{\kappa} : P \longrightarrow \mathrm{End}(\mathbf{N})$ up to G_b-conjugacy by (2.6) and because necessarily $B_g = B_\delta$ (via $z_g \mapsto z$) if such an orbit exists (Proposition 2.6). The uniqueness of the pair $(b,\ H_b g H_b)$ is thus equivalent to the uniqueness (up to P-linear isomorphism) of an isocrystal $\mathbf{N}$ of height $4n^2$, of dimension $2n$, with all slopes within $[0, 1]$ and with a faithful P-action $\widetilde{\kappa}$.

Assume $(\mathbf{N}, \widetilde{\kappa})$ is such a pair. Then $\mathbf{N}$ decomposes, $\mathbf{N} = \prod_{i \in I} \mathbf{N}_i$, into a product of isocrystals with faithful P_i-action. The i-th factor P_i is a CSA over L_i of degree $4n$ so the height of $\mathbf{N}_i$ has to be an integer multiple of $4nn_i$. Since $\sum_{i \in I} 4nn_i = 4n^2 = \mathrm{ht}(\mathbf{N})$, the height of $\mathbf{N}_i$ has to be exactly $4nn_i$. Furthermore, $\mathbf{N}_i$ is necessarily isoclinic, say of slope $\mu_i = d_i/4nn_i$. The uniqueness of $(\mathbf{N}, \widetilde{\kappa})$ up to P-linear isomorphism is then equivalent to the vector $(d_i)_{i \in I}$ being uniquely determined by P.

The i-th endomorphism ring $\mathrm{End}(\mathbf{N}_i)$ is a CSA over F of Hasse invariant μ_i and degree $4nn_i$. Since $[P_i : F][L_i : F] = 4nn_i$, it follows from the centralizer theorem that $\widetilde{\kappa}(P_i)$ equals the centralizer of $\widetilde{\kappa}(L_i)$ in $\mathrm{End}(\mathbf{N}_i)$. This implies that

$$\rho_i = n_i \mu_i = d_i / 4n. \tag{4.7}$$

Here, the integer d_i is the dimension of $\mathbf{N}_i$ and we know that $\sum_{i \in I} d_i = 2n$. Because all slopes are assumed to lie within the interval $[0, 1]$, we in particular obtain that $0 \leq d_i \leq 2n$ and hence see that $(d_i)_{i \in I}$ is uniquely determined by P. This shows the uniqueness of b and the orbit $H_b g H_b$.

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It is still left to prove that $\varepsilon_D(\delta) = -1$ if b and such an orbit exist. To this end, we take up the identity $\rho_i = \beta_i + n_i\lambda$ from the beginning of the proof. Combining with (4.7), we see that $\beta_i + n_i\lambda = d_i/4n$. Taking the sum over all $i \in I$, it follows that

$$\varepsilon_D(\delta) = \sum_{i\in I} \frac{d_i}{4n} = \frac{1}{2} \in 2^{-1}\mathbb{Z}/\mathbb{Z} \cong \{\pm 1\}, \tag{4.8}$$

as claimed. □

4.2. Moduli spaces

Let $[b] \in B(H, \mu_H)$ be an isogeny class. Then (H, b, μ_H) and (G, b, μ_G) are local Shimura data triples. Our aim in this section is to define integral models of the corresponding local Shimura varieties (over $\breve{F}$) at maximal level.

Fix an embedding $E \to \breve{F}$ as well as maximal orders $O_C \subset C$ and $O_D \subset D$. By definition, a Spf $O_{\breve{F}}$-scheme is an $O_{\breve{F}}$-scheme S such that $\pi \in \mathcal{O}_S$ is locally nilpotent.

Definition 4.8. Let S be a Spf $O_{\breve{F}}$-scheme.

(1) Assume that F is of characteristic 0. A strict O_F-module over S is a pair (X, α) that consists of a p-divisible group X over S and an action $\alpha : O_F \to \mathrm{End}(X)$ that is strict in the sense that $\mathrm{Lie}(\alpha(a)) = a$ for all $a \in O_F$. Height and slope of a strict O_F-module are meant in the relative sense, meaning $[F : \mathbb{Q}_p] \cdot \mathrm{ht}(X)$ is the height of X as p-divisible group.

(2) Assume that $F \cong \mathbb{F}_q((\pi))$ is of characteristic p. A strict O_F-module over S is a π-divisible group (X, α) over S in the sense of [15, Definition 7.1] such that $\mathrm{Lie}(\alpha(a)) = a$ for all $a \in O_F$. (In other words, we demand $d = 1$ in part (iv) of [15, Definition 7.1].) Height, dimension and slope are defined as in [15, §7].

Definition 4.9. Let S be a Spf $O_{\breve{F}}$-scheme.

(1) A special O_C-module over S is a pair (Y, ι) that consists of a strict O_F-module Y and an O_C-action $\iota : O_C \to \mathrm{End}(Y)$ such that the following conditions are satisfied. The height of Y is $2n^2$, its dimension is n, and the O_C-action is special in the sense that for all $x \in O_C$,

$$\mathrm{char}(\iota(x) \mid \mathrm{Lie}(Y); T) = \mathrm{charred}_{C/E}(x; T). \tag{4.9}$$

Here, the right-hand side is considered as an element of $\mathcal{O}_S[T]$ via the fixed embedding $E \subset \breve{F}$ and the structure map $O_{\breve{F}} \to \mathcal{O}_S$.

(2) A special O_D-module over S is a pair (X, κ) that consists of a strict O_F-module X and an O_D-action $\kappa : O_D \to \mathrm{End}(X)$ such that the following conditions are satisfied. The height of X is $4n^2$, its dimension is $2n$, and the O_D-action is special in the sense that for all $x \in O_D$,

$$\mathrm{char}(\kappa(x) \mid \mathrm{Lie}(X); T) = \mathrm{charred}_{D/F}(x; T). \tag{4.10}$$

Here, the right-hand side is considered as an element of $\mathcal{O}_S[T]$ via the structure map $O_{\breve{F}} \to \mathcal{O}_S$.

Remark 4.10. By [38, 3.58], an equivalent way to formulate (4.9) and (4.10) is as follows. Let L/E be an unramified field extension of degree n and fix an embedding $O_L \to O_C$. Then, given an action $\iota : O_C \to \mathrm{End}(Y)$ or $\kappa : O_D \to \mathrm{End}(X)$, the Lie algebra $\mathrm{Lie}(Y)$ resp. $\mathrm{Lie}(X)$ becomes an $O_L \otimes_{O_F} \mathcal{O}_S$-module. Since S was assumed to be an $O_{\breve{F}}$-scheme, there is an eigenspace decomposition

$$\mathrm{Lie}(Y) = \bigoplus_{\varphi\in\mathrm{Hom}_E(L,\breve{F})} \mathrm{Lie}(Y)_\varphi \quad \text{resp.} \quad \mathrm{Lie}(X) = \bigoplus_{\varphi\in\mathrm{Hom}_F(L,\breve{F})} \mathrm{Lie}(X)_\varphi.$$

Then (4.9) resp. (4.10) holds for all $x \in O_C$ (resp. all $x \in O_D$) if and only if each summand $\mathrm{Lie}(Y)_\varphi$ (resp. each summand $\mathrm{Lie}(X)_\varphi$) is locally free of rank 1 as O_S-module.

Remark 4.11 (Morita equivalence). It is possible to reformulate Definition 4.9 in terms of division algebras only. For brevity, we only consider the case of D: Assume that $O_D = M_m(O_{D_0})$, where O_{D_0} denotes the maximal order in a CDA D_0. Then special O_D-modules over S are equivalent to pairs (X_0, κ_0), where X_0 is a strict O_F-module over S of height $m \cdot \dim_F(D_0)$, dimension $[D_0 : F]$, and where $\kappa_0 : O_{D_0} \to \mathrm{End}(X_0)$ is special in the sense that for all $x \in O_{D_0}$,

$$\mathrm{char}(\kappa_0(x) \,|\, \mathrm{Lie}(X_0); T) = \mathrm{charred}_{D_0/F}(x; T).$$

The equivalence is given by $(X_0, \kappa_0) \mapsto (X_0^m, M_m(\kappa_0))$.

We fix a special O_C-module $(\mathbb{Y}, \iota)$ and a special O_D-module $(\mathbb{X}, \kappa)$ over $\mathbb{F}$ for the next definition. These are the so-called framing objects.

Definition 4.12. The RZ moduli space $\mathcal{M}_C$ is defined as the following functor on the category of schemes over $\mathrm{Spf}\, O_{\breve{F}}$,

$$\mathcal{M}_C(S) = \left\{ (Y, \iota, \rho) \middle| \begin{array}{c} (Y, \iota) \text{ a special } O_C\text{-module over } S \\ \rho : \bar{S} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{Y} \longrightarrow \bar{S} \times_S Y \text{ an } O_C\text{-linear quasi-isogeny} \end{array} \right\}.$$

We define a moduli space of $\mathcal{M}_D$ in the exact same way,

$$\mathcal{M}_D(S) = \left\{ (X, \kappa, \rho) \middle| \begin{array}{c} (X, \kappa) \text{ a special } O_D\text{-module over } S \\ \rho : \bar{S} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{X} \longrightarrow \bar{S} \times_S X \text{ an } O_D\text{-linear quasi-isogeny} \end{array} \right\}.$$

Proposition 4.13. *The functors $\mathcal{M}_C$ and $\mathcal{M}_D$ are representable by formal schemes that are locally formally of finite type over* $\mathrm{Spf}\, O_{\breve{F}}$. *The irreducible components of the maximal reduced subschemes of $\mathcal{M}_C$ and $\mathcal{M}_D$ are projective over* $\mathrm{Spec}\,\mathbb{F}$. *Both formal schemes are regular with semi-stable reduction over* $\mathrm{Spf}\, O_{\breve{F}}$. *Moreover, $\mathcal{M}_C$ has dimension n, and $\mathcal{M}_D$ has dimension $2n$.*

Proof. The representability of $\mathcal{M}_C$ and $\mathcal{M}_D$ by a locally formally finite type formal scheme, and the fact that the irreducible components of their reduced loci are projective over $\mathrm{Spec}\,\mathbb{F}$, are general properties of (P)EL type RZ spaces; see [38, Theorem 3.25]. (The analogous result in the equal characteristic setting is [2, Theorem 4.18].) The regularity follows with the standard local model argument: [38, Proposition 3.33] states that regularity of the two spaces follows from that of the local models for the data $(H, \mu_H, O_C^\times)$ and $(G, \mu_G, O_D^\times)$. After base extension to $O_{\breve{F}}$, these local models are isomorphic to parahoric type local models for GL_n resp. GL_{2n} with cocharacter $\mu = (1, \ldots, 1, 0)$. It is well known that these are of dimension n (resp. $2n$) with semi-stable reduction [18, Theorem 5.6]. (The assumption $p \neq 2$ is not needed for this part of the theorem. The result is originally due to Drinfeld [10].) □

4.3. Quadratic CM cycles

Assume from now on that the maximal orders O_C and O_D are chosen such that $O_C = C \cap O_D$. (See Lemma 3.6 for a uniqueness statement in this context.) Let S be a scheme over $\mathrm{Spf}\, O_{\breve{F}}$ and let (Y, ι) be a strict O_F-module with O_C-action ι over S. Then $O_D \otimes_{O_C} Y$ is a strict O_F-module over S with a natural O_D-action. It will be useful to have a more explicit description of this construction.

The ring O_D is an $O_E \otimes_{O_F} O_E$-module via left and right multiplication and decomposes into eigenspaces with respect to this action: $O_D = O_C \oplus (D_- \cap O_D)$. We have used here that E is unramified over F. The space of conjugation linear element $D_- \cap O_D$ is an O_C-module via left multiplication and takes the form $O_C \cdot \Pi$ for some generator Π. With respect to the O_C-basis $(1, \Pi)$, there is then the presentation

$$O_D \otimes_{O_C} Y = Y \oplus \Pi Y, \tag{4.11}$$

where ΠY is our notation for the summand $\Pi \otimes Y$, which we identify with Y. Note that $\Pi^2 \in O_C$ and that $\Pi^{-1} O_C \Pi = O_C$. The O_D-action on $O_D \otimes_{O_C} Y$ has the matrix description

$$\begin{aligned} O_D &\longrightarrow \mathrm{End}(Y \oplus \Pi Y) \\ a + b\Pi &\longmapsto \begin{pmatrix} a & b\Pi^2 \\ \Pi^{-1} b \Pi & \Pi^{-1} a \Pi \end{pmatrix}. \end{aligned} \tag{4.12}$$

Lemma 4.14. *Let (Y, ι) be a special O_C-module over a $(\mathrm{Spf}\, O_{\breve{F}})$-scheme S. Then the Serre tensor construction $(O_D \otimes_{O_C} Y,\ \kappa(x) = x \otimes \mathrm{id}_Y)$ is a special O_D-module.*

Proof. Let L denote an unramified extension of degree $2n$ of F; fix an embedding $E \to L$. We claim that for any choice of two E-linear embeddings $i_1, i_2 : O_L \to O_C$, the two images $i_1(O_L)$ and $i_2(O_L)$ are $O_C^\times$-conjugate. To prove this, we consider the decomposition

$$O_C = \bigoplus_{\varphi \in \mathrm{Gal}(L/E)} \Lambda_\varphi$$

into eigenspaces with respect to the action $i_1 \otimes i_2$ of $O_L \otimes_{O_F} O_L$ by left and right multiplication. Our task is to show that there exists an index φ and an element $x \in O_C^\times \cap \Lambda_\varphi$. Namely, any such element satisfies $x^{-1} \circ i_1 \circ x = i_2 \circ \varphi$ and hence $x^{-1} i_1(O_L) x = i_2(O_L)$.

Each Λ_φ is an $i_1(O_L)$-module of rank 1 and $\Lambda_\varphi \Lambda_\psi \subseteq \Lambda_{\varphi+\psi}$. It follows that every nonzero homogeneous element $x_\varphi \in \Lambda_\varphi$ lies in $C^\times$, and that if $x_\varphi \in \Lambda_\varphi$ is topologically nilpotent and $0 \neq x_\psi \in \Lambda_\psi$ any other homogeneous element, then $x_\psi x_\varphi {x_\psi}^{-1}$ is again topologically nilpotent. Thus, given any two topologically nilpotent homogeneous elements x_φ, x_ψ in degrees φ and ψ, say, their product $x_\varphi x_\psi$ is again topologically nilpotent. Since O_C contains elements that are not topologically nilpotent, it follows that there also exists a homogeneous element $x_\varphi \in \Lambda_\varphi$ that is not topologically nilpotent. Then $x_\varphi^{n+1} \in i_1(O_L)^\times x_\varphi$ implies that $x_\varphi \in O_C^\times$, and we have proved the claim.

We now come to the main arguments. Let $X = O_D \otimes_{O_C} Y$. The above claim implies that there exists an embedding $i : O_L \to O_C$ and a choice of Π in (4.11) such that $\Pi i(O_L) = i(O_L)\Pi$. Let $\psi \in \mathrm{Gal}(L/F)$ be defined by $\Pi^{-1} \circ i \circ \Pi = i \circ \psi$. Note that ψ satisfies $\psi|_E \neq \mathrm{id}_E$ because Π is E-conjugate linear and consider the decompositions of $\mathrm{Lie}(Y)$ and $\mathrm{Lie}(X)$ into O_L-eigenspaces,

$$\mathrm{Lie}(Y) = \bigoplus_{\varphi \in \mathrm{Hom}_E(L, \breve{F})} \mathrm{Lie}(Y)_\varphi, \quad \mathrm{Lie}(X) = \bigoplus_{\varphi \in \mathrm{Hom}_F(L, \breve{F})} \mathrm{Lie}(X)_\varphi.$$

Then (4.11) and (4.12) for our specific choices of i and Π imply that

$$\mathrm{Lie}(X)_\varphi \cong \begin{cases} \mathrm{Lie}(Y)_\varphi & \text{if } \varphi|_E = \mathrm{id}_E \\ \mathrm{Lie}(Y)_{\varphi\psi^{-1}} & \text{if } \varphi|_E \neq \mathrm{id}_E. \end{cases} \tag{4.13}$$

It follows that if $\mathrm{Lie}(Y)_\varphi$ is of rank 1 for all $\varphi \in \mathrm{Hom}_E(L, \breve{F})$, then $\mathrm{Lie}(X)_\varphi$ is of rank 1 for all $\varphi \in \mathrm{Hom}_F(L, \breve{F})$. By Remark 4.11, this precisely means that X as a special O_D-module if Y is a special O_C-module. □

From now on, we assume that the two framing objects are related by the Serre tensor construction

$$(\mathbb{X}, \kappa) = O_D \otimes_{O_C} (\mathbb{Y}, \iota). \tag{4.14}$$

In this situation, Lemma 4.14 states that there is a morphism of formal schemes given by

$$\begin{aligned} \mathcal{M}_C &\longrightarrow \mathcal{M}_D \\ (Y, \iota, \rho) &\longmapsto (O_D \otimes_{O_C} Y,\ \kappa(x) = x \otimes \mathrm{id}_Y,\ \mathrm{id}_{O_D} \otimes \rho). \end{aligned} \tag{4.15}$$

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Given any subset $T \subseteq \mathrm{End}^0(\mathbb{Y})$ of the quasi-endomorphisms of $\mathbb{Y}$, we define a subfunctor $\mathcal{Z}(T) \subseteq \mathcal{M}_C$ by

$$\mathcal{Z}(T)(S) := \{(Y, \iota, \rho) \in \mathcal{M}_C(S) \mid \rho T \rho^{-1} \subseteq \mathrm{End}(Y)\}. \tag{4.16}$$

Here, the condition is meant in the sense that $\rho T \rho^{-1}$ is always a subset of the *quasi*-endomorphisms $\mathrm{End}^0(Y)$ because ρ is a quasi-isogeny. The functor $\mathcal{Z}(T)$ is representable by a closed formal subscheme of $\mathcal{M}_C$; see [38, Proposition 2.9]. In exactly the same way, we define a closed formal subscheme $\mathcal{Z}(T) \subseteq \mathcal{M}_D$ whenever $T \subseteq \mathrm{End}^0(\mathbb{X})$. We apply this construction to the subring $\iota(O_E) \subseteq \mathrm{End}(\mathbb{X})$ to obtain the closed formal subscheme $\mathcal{Z}(\iota(O_E)) \subset \mathcal{M}_D$.

Consider an S-valued point $(X, \kappa, \rho) \in \mathcal{Z}(\iota(O_E))(S)$. Then X is equipped with the two commuting O_E-actions $\kappa|_{O_E}$ and $\rho \circ \iota \circ \rho^{-1}$. Since E/F is unramified, $(\mathrm{id}\cdot\mathrm{id}, \mathrm{id}\cdot\tau) : O_E \otimes_{O_F} O_E \xrightarrow{\sim} O_E \times O_E$. We denote by $X = X_+ \oplus X_-$ the resulting eigenspace decomposition of X. In particular, X_+ is the summand on which the two O_E-actions agree.

The purpose of the above definitions was that we can now give a description of the image of $\mathcal{M}_C \to \mathcal{M}_D$.

Proposition 4.15. *The morphism $\mathcal{M}_C \to \mathcal{M}_D$ is a closed immersion. Its image consists of all those points $(X, \kappa, \rho) \in \mathcal{Z}(\iota(O_E))$ with the following two additional properties. Let $X = X_+ \oplus X_-$ be the eigenspace decomposition as explained before.*

(1) *The $\kappa(O_C)$-action on X_+ is special in the sense of (4.9).*

(2) *The endomorphism $\kappa(\Pi)$ defines an isomorphism $\kappa(\Pi) : X_+ \xrightarrow{\cong} X_-$.*

Proof. Let $\mathcal{Z} \subseteq \mathcal{Z}(\iota(O_E))$ be the subfunctor defined by the conditions (1) and (2). Condition (1) is a Zariski closed condition on $\mathcal{Z}(\iota(O_E))$ because it is given by the equality of the two polynomials in (4.9). Condition (2) is an open and closed condition: The map $\kappa(\Pi) : X_+ \to X_-$ is always an isogeny because $\rho^{-1}\kappa(\Pi)\rho : \mathbb{X}_+ \to \mathbb{X}_-$ is an isogeny. Condition (2) then describes the locus where the height of $\kappa(\Pi)$ is 0, which is open and closed. We conclude that $\mathcal{Z}$ is a closed formal subscheme of $\mathcal{Z}(\iota(O_E))$.

It is clear from definitions that the map $\mathcal{M}_C \to \mathcal{M}_D$ factors through $\mathcal{Z}$. Conversely, given a point $(X, \kappa, \rho) \in \mathcal{Z}(S)$, let $(X_+, \kappa|_{O_C}, \rho_+)$ be the direct summand where the $\kappa(O_E)$ and $(\rho\iota(O_E)\rho^{-1})$-actions coincide. Then $(X_+, \kappa|_{O_C}, \rho_+) \in \mathcal{M}_C(S)$ because of Condition (1). Condition (2) ensures that

$$O_D \otimes_{O_C} (X_+, \kappa|_{O_C}, \rho_+) \xrightarrow{\cong} (X, \kappa, \rho)$$

via the natural O_D-linear map $O_D \otimes_{O_C} X_+ \to X$. This constructs an inverse $\mathcal{Z} \to \mathcal{M}_C$. □

4.4. Intersection numbers

Let $[b] \in B(H, \mu_H)$ be the isogeny class defined by the framing object $(\mathbb{Y}, \iota)$. After a suitable choice of identification, we may simply write (or redefine) $H_b = \mathrm{End}^0_C(\mathbb{Y}, \iota)^\times$ and $G_b = \mathrm{End}^0_D(\mathbb{X}, \kappa)$. Then H_b and G_b act from the right on $\mathcal{M}_C$ resp. $\mathcal{M}_D$ by composition in the framing. The closed immersion $\mathcal{M}_C \to \mathcal{M}_D$ is equivariant with respect to $H_b \to G_b$.

Definition 4.16. Let $g \in G_{b,\mathrm{rs}}$ be a regular semi-simple element. The intersection locus for g is

$$\mathcal{I}(g) := \mathcal{M}_C \cap (g \cdot \mathcal{M}_C).$$

Let $g \in G_{b,\mathrm{rs}}$ be regular semi-simple. Recall from Definition 2.5 that $B_g \subset D_b$ denotes the subring $F[\iota(E), g^{-1}\iota(E)g]$ and that $L_g \subset B_g$ denotes its center. In particular, it holds that $L_g = C_b \cap g^{-1}C_b g$, and we obtain the following lemma.

Lemma 4.17. *The action of $L_g^\times \subset G_b$ on $\mathcal{M}_D$ preserves both $\mathcal{M}_C$ and $g \cdot \mathcal{M}_C$. In particular, it preserves $\mathcal{I}(g)$.*

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One consequence is that $\mathcal{I}(g)$ is never quasi-compact if it is nonempty. (Consider, for example, the action of $\pi^{\mathbb{Z}} \subset L_g^\times$.) However, taking the quotient by $L_g^\times$ solves this issue:

Proposition 4.18. *Assume that F is p-adic. Let $g \in G_{b,\mathrm{rs}}$ be regular semi-simple and let $\Gamma \subset L_g^\times$ be a discrete cocompact subgroup with $L_g^\times = \Gamma \times O_{L_g}^\times$. Then $\mathcal{I}(g)$ is a scheme and the quotient $\Gamma\backslash\mathcal{I}(g)$ is proper over* $\operatorname{Spec} O_{\breve{F}}$.

Remark 4.19. The only reason for the restriction to p-adic F is that our proof relies on [6, Lemma 4.3.15], which is only stated for p-divisible groups. The statement should also be true when $F = \mathbb{F}_q(\!(\pi)\!)$, however, and we assume this for later definitions.

Note that any Γ as in Proposition 4.18 acts without fixed points on $\mathcal{M}_D$ by [38, Corollary 2.35]. The quotient $\Gamma\backslash\mathcal{I}(g)$ can be constructed in the following way. First choose a finite index subgroup $\Gamma' \subset \Gamma$ that acts properly discontinuously on $\mathcal{M}_D$. The quotient $\Gamma'\backslash\mathcal{I}(g)$ can be constructed in the Zariski topology. Then pass to $(\Gamma/\Gamma')\backslash(\Gamma'\backslash\mathcal{I}(g))$, which is a quotient of a formal scheme by a finite group that acts without fixed points.

In the following, we write $L = L_g$ and $B = B_g$.

Proof. The proof will even show that $\mathcal{Z}(\iota(O_E)) \cap g\mathcal{Z}(\iota(O_E))$ is a scheme and that the quotient $\Gamma\backslash(\mathcal{Z}(\iota(O_E)) \cap g \cdot \mathcal{Z}(\iota(O_E)))$ is quasi-compact. It is based on the observation that

$$\mathcal{Z}(\iota(O_E) \cap g \cdot \mathcal{Z}(\iota(O_E)) = \mathcal{Z}(\iota(O_E) \cup g^{-1}\iota(O_E)g) = \mathcal{Z}(R), \tag{4.17}$$

where R is defined as the ring $R = O_F[\iota(O_E), g^{-1}\iota(O_E)g] \otimes_{O_F} O_D$. This is an order in the semi-simple F-algebra $P = B \otimes_F D$.

It follows that the generic fiber of $\mathcal{I}(g)$ is empty: The algebra P is a CSA of degree $4n$ over L. It can only act faithfully on an étale π-divisible O_F-module of height $4n^2$ if $P \cong M_{4n}(L)$. But this would mean that B splits D, which would imply that $\varepsilon_D(g) = 1$ by Lemma 3.17. However, this is excluded by Proposition 4.6.

We next prove that $\Gamma\backslash\mathcal{I}(g)$ is quasi-compact. This is equivalent to proving that the set of closed points $\mathcal{Z}(R)(\mathbb{F})$ is bounded modulo Γ in the following sense. The set $\mathcal{Z}(R)(\mathbb{F})$ identifies with the set of R-stable Dieudonné lattices $\mathbf{M}$ in the isocrystal $\mathbf{N} = (N, \mathbf{F})$ of $\mathbb{X}$ that are special. Let $\mathbf{M}(\mathbb{X}) \subset \mathbf{N}$ be the Dieudonné lattice defined by $\mathbb{X}$. We need to see that there exists an integer $c \geq 0$ such that for every $\mathbf{M} \in \mathcal{Z}(R)(\mathbb{F})$, there is some $x \in \Gamma$ with $\pi^c\mathbf{M}(\mathbb{X}) \subseteq x\mathbf{M} \subseteq \pi^{-c}\mathbf{M}(\mathbb{X})$. (The condition of points in $\mathcal{Z}(R)(\mathbb{F})$ being special will not play a role for the argument.)

Let $L = \prod_{i\in I} L_i$ and $P = \prod_{i\in I} P_i$ be the decompositions that correspond to the idempotents in L. Then P_i is a CSA over L_i of degree $4n$, and the corresponding summand $\mathbf{N}_i$ of $\mathbf{N}$ has height $4n[L_i : F]$. In this situation, the set of Dieudonné lattices $\mathbf{M}_i \subseteq \mathbf{N}_i$ that are stable under some choice of maximal order $O_{P_i} \subset P_i$ is bounded modulo $\pi^{\mathbb{Z}}$. (Indeed, $\breve{O}_{P_i} = O_{\breve{F}} \otimes_{O_{L_i}} O_{P_i}$ is an order in $M_{4n}(\breve{F})$. The set of $\breve{O}_{P_i}$-stable lattices in $\breve{F}^{4n}$ is bounded.) Thus, the set of $O_P = \prod_{i\in I} O_{P_i}$-stable Dieudonné lattices in $\mathbf{N}$ is bounded modulo Γ. For every R-stable Dieudonné lattice $\mathbf{M}$, the lattice $O_P \cdot \mathbf{M}$ is O_P-stable and the index $[O_P \cdot \mathbf{M} : \mathbf{M}]$ is bounded in terms the index $[O_P : R]$. It follows that $\Gamma\backslash\mathcal{Z}(R)(\mathbb{F})$ is bounded as claimed.

It is left to show that $\mathcal{I}(g)$ is a scheme. A priori, it is known to be a locally noetherian formal scheme. We thus need to see that for every noetherian, adic, π-adically complete $O_{\breve{F}}$-algebra A, every morphism $f : \operatorname{Spf} A \to \mathcal{I}(g)$ extends to a map $\operatorname{Spec} A \to \mathcal{I}(g)$. Equivalently, we need to see that for every such f, the ideal $J(f) = f^{-1}(\mathcal{O}_{\mathcal{I}(g)}^{\circ\circ})A$ that is generated by the inverse images of all topologically nilpotent elements in $\mathcal{O}_{\mathcal{I}(g)}$ is nilpotent. We claim that it suffices to consider the case of a DVR: Indeed, assume that there exists a prime ideal $\mathfrak{p} \subset A$ such that $J(f) \not\subset \mathfrak{p}$ and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. It necessarily holds that $J(f) \subseteq \mathfrak{m}$ because $J(f)$ is nilpotent. Since A is noetherian, there exists a complete DVR B and a map $g : \operatorname{Spec} B \to \operatorname{Spec} A$ such that the generic point of $\operatorname{Spec} B$ maps to $\mathfrak{p}$ and the special point to $\mathfrak{m}$. In particular, $J(f \circ g) = g^{-1}(J(f))B$ would be a nontrivial ideal of B. This proves the claim and allows us to henceforth assume that A is a DVR. We will even assume that A is complete with algebraically closed residue field. We already know that $\mathcal{I}(g)$ has empty generic fiber, so it holds that $\pi A = 0$. We write $\operatorname{Spec} A = \{s, \eta\}$, where s is the special and η the generic point.

Let (X, κ, ρ) be the point that defines the morphism $f : \operatorname{Spf} A \to \mathcal{I}(g)$. The datum X algebraizes to a strict O_F-module over $\operatorname{Spec} A$ with R-action. Our task is to show that ρ algebraizes as well. The key observation for this is that, by Proposition 4.6, the geometric isogeny class of the generic fiber X_η is uniquely determined by $\operatorname{Inv}(g; T)$ and hence equal to that of $\mathbb{X}$. In other words, the point-wise slope vector of X on $\operatorname{Spec} A$ is constant. The perfection $A^{\mathrm{perf}} = \operatorname{colim}_{x \mapsto x^p} A$ of A is again strictly henselian.

Lemma 4.20 [6, Lemma 4.3.15]. *Let k be the residue field of A^{perf}. The functor $Y \mapsto k \otimes_{A^{\mathrm{perf}}} Y$ from strict O_F-modules Y up to isogeny over A^{perf} to strict O_F-modules up to isogeny over k is an equivalence.*

Proof. The cited lemma states this when $F = \mathbb{Q}_p$. The general case follows immediately because strict O_F-modules are nothing but p-divisible groups with strict O_F-action. □

By Lemma 4.20, there exists a quasi-isogeny $\rho' : A^{\mathrm{perf}} \otimes_k \mathbb{X} \to X$ such that $k \otimes_{A^{\mathrm{perf}}} \rho' = k \otimes_A \rho$. By the rigidity of quasi-isogenies [38, (2.1)], this implies $\rho' = A^{\mathrm{perf}} \otimes_A \rho$, which shows that $A^{\mathrm{perf}} \otimes_A \rho$ is algebraic. The map $A \to A^{\mathrm{perf}}$ is faithfully flat, so this implies that ρ is algebraic. □

Definition 4.21. For $g \in G_{b,\mathrm{rs}}$, we define

$$\operatorname{Int}(g) := \chi\Big(\Gamma\backslash\mathcal{I}(g),\ \ \mathcal{O}_{\Gamma\backslash\mathcal{M}_C} \otimes^{\mathbb{L}}_{\mathcal{O}_{\Gamma\backslash\mathcal{M}_D}} \mathcal{O}_{\Gamma\backslash g\cdot\mathcal{M}_C}\Big) \in \mathbb{Z}.$$

Here, by the regularity of $\mathcal{M}_C$ and $\mathcal{M}_D$ from Proposition 4.13, the complex on the right-hand side is perfect. It is supported on $\Gamma\backslash\mathcal{I}(g)$, which is a projective $O_{\breve{F}}$-scheme with $\pi^N \mathcal{O}_{\mathcal{I}(g)} = 0$ for $N \gg 0$ by Proposition 4.18. This explains why $\operatorname{Int}(g)$ is well defined. Note that the passage to the quotient by Γ is completely analogous to taking the quotient by the stabilizer in the definition of the orbital integrals in §3.1.

We end this section with some auxiliary results about the intersection locus $\mathcal{I}(g)$ that will be useful in later sections.

Lemma 4.22. *Let $\mathbf{N}$ be an isocrystal with E-action and let $g = g_+ + g_- \in \operatorname{End}(\mathbf{N})^\times$ be an automorphism such that g_+ lies again in $\operatorname{End}(\mathbf{N})^\times$, where g_+ and g_- denote the E-linear resp. E-conjugate linear components of g. Assume that there exists an O_E-stable Dieudonné lattice $\mathbf{M} \subset \mathbf{N}$ such that $g\mathbf{M}$ is O_E-stable as well. Assume furthermore that the Verschiebung $\mathbf{V}$ on $\mathbf{N}$ is topologically nilpotent and that the O_E-action on both $\mathbf{M}/\mathbf{V}\mathbf{M}$ and $g\mathbf{M}/\mathbf{V}(g\mathbf{M})$ is strict. Then $z_g = g_+^{-1}g_-$ is topologically nilpotent.*

Proof. Considering $g_+^{-1}g\mathbf{M}$ instead, we may assume that g is of the form $g = 1 + z$ with $z = z_g$. Let $\mathbf{M} = \mathbf{M}_+ \oplus \mathbf{M}_-$ and $g\mathbf{M} = \mathbf{M}'_+ \oplus \mathbf{M}'_-$ be the bigradings that come from the O_E-action. *Claim: It holds that $z\mathbf{M}_+ \subseteq \mathbf{M}_-$.* Assume this claim holds. The strictness condition for $\mathbf{M}$ means that $\mathbf{V}\mathbf{M}_+ = \mathbf{M}_-$. Thus, we obtain $z\mathbf{M}_+ \subseteq \mathbf{V}\mathbf{M}_+$ and hence $z^{2n}\mathbf{M}_+ \subseteq \mathbf{V}^{2n}\mathbf{M}_+$ for every $n \geq 0$. The Verschiebung is topologically nilpotent by assumption, so it follows that z is topologically nilpotent, as claimed. It is only left to prove the claim.

Proof of the Claim. First note that

$$\mathbf{M}'_+ = \mathbf{M}_+ + z\mathbf{M}_- \quad \text{and} \quad \mathbf{M}'_- = \mathbf{M}_- + z\mathbf{M}_+. \tag{4.18}$$

Moreover, the O_E-action on $g\mathbf{M}$ was assumed to be strict as well, meaning that $\mathbf{V}\mathbf{M}'_+ = \mathbf{M}'_-$. Substituting this in (4.18), it follows that

$$\mathbf{M}_- + z\mathbf{M}_+ = \mathbf{M}_- + zV\mathbf{M}_-.$$

In particular, $z\mathbf{M}_+ \subseteq \mathbf{M}_- + z\mathbf{V}^2\mathbf{M}_+$ and hence, for all $i \geq 1$,

$$z\mathbf{V}^{2i-2}\mathbf{M}_+ \subseteq \mathbf{M}_- + z\mathbf{V}^{2i}\mathbf{M}_+. \tag{4.19}$$

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Since $\mathbf{V}$ is topologically nilpotent by assumption, there exists an integer i such that $z\mathbf{V}^{2i}\mathbf{M}_+ \subset \mathbf{M}_-$. Descending induction on i with the help of (4.19) proves that $z\mathbf{M}_+ \subseteq \mathbf{M}_-$, as claimed. $\square$

By definition, for every regular semi-simple $g \in G_b$, the element $z_g = g_+^{-1}g_-$ lies in $\mathrm{End}^0_D(\mathbb{X}, \kappa)$. So the definition in (4.16) applies and defines a closed formal subscheme $\mathcal{Z}(z_g) \subseteq \mathcal{M}_D$.

Proposition 4.23. (1) *Assume that* $[b] \in B(H, \mu_H)$ *is such that* $\mathbf{N}_b$ *has no étale part and assume that* $g \in G_{b,\mathrm{rs}}$ *is regular semi-simple. If* $\mathcal{I}(g) \neq \emptyset$*, then* z_g *is topologically nilpotent.*

(2) *Let* $[b] \in B(H, \mu_H)$ *be any and let* $g \in G_{b,\mathrm{rs}}$ *be an element such that* z_g *is topologically nilpotent. Then* $\mathcal{I}(g) = \mathcal{M}_C \cap \mathcal{Z}(z_g)$.

Proof. (1) The assumption that $\mathbf{N}_b$ has no étale part precisely says that $\mathbf{V}$ is topologically nilpotent. Then any point $(X, \kappa, \rho) \in \mathcal{I}(g)(\mathbf{M})$ defines a Dieudonné lattice $\mathbf{M} \subset \mathbf{N}_b$ that satisfies the conditions of Lemma 4.22.

(2) Let $\zeta \in O_E^\times$ be an O_F-algebra generator of trace 1. It always holds that $1-2\zeta \in O_E^\times$ because E/F is unramified. Using that z_g is topologically nilpotent, identity (2.7) then implies the following equality of subrings of $\mathrm{End}^0_D(\mathbb{X})$:

$$R = O_F[\iota(O_E), g^{-1}\iota(O_E)g] = O_E[z_g]. \tag{4.20}$$

We obtain from (4.17) that $\mathcal{M}_C \cap g \cdot \mathcal{M}_C \subseteq \mathcal{M}_C \cap \mathcal{Z}(z_g)$. Assume conversely that $(X, \kappa, \rho) \in \mathcal{M}_C \cap \mathcal{Z}(z_g)$. We need to show that $(X, \kappa, \rho) \in g \cdot \mathcal{M}_C$. Equivalently, by the H_b-equivariance of the embedding $\mathcal{M}_C \to \mathcal{M}_D$, we need to show that $(X, \kappa, \rho) \in (g_+^{-1}g)\mathcal{M}_C = (1 + z_g)\mathcal{M}_C$. So we may assume that $g = 1 + z_g$ from now on. Using that $(X, \kappa, \rho) \in \mathcal{Z}(z_g)$ and also that z_g is topologically nilpotent by assumption, $\rho g \rho^{-1} = 1 + \rho z_g \rho^{-1}$ defines an automorphism of X. Thus, (X, κ, ρ) is a g-fixed point of $\mathcal{M}_D$ that also lies in $\mathcal{M}_C$, and hence lies in $g \cdot \mathcal{M}_C$. $\square$

4.5. The Arithmetic Transfer Conjecture

We can now formulate our AT conjecture. Recall that $f_D = 1_{O_D^\times} \in C_c^\infty(G)$ denotes the standard test function on the CSA side; see §3.3.

Conjecture 4.24 (ATC). *There exists a transfer* $f_D'' \in C_c^\infty(G')$ *of* f_D *in the sense of Definition 3.4 with the following additional property. For every regular semi-simple element* $\gamma \in G'_{\mathrm{rs}}$,

$$\partial \mathrm{Orb}(\gamma, f_D'') = \begin{cases} 2\,\mathrm{Int}(g)\log(q) & \begin{array}{l}\textit{if there exists some } [b] \in B(H,\mu_H)\\ \textit{and some } g \in G_{b,\mathrm{rs}} \textit{ that matches } \gamma\end{array} \\ 0 & \textit{otherwise.}\end{cases} \tag{4.21}$$

Conjecture 4.25 (ATC – equivalent form). *For every transfer* $f_D'' \in C_c^\infty(G')$ *of* f_D *in the sense of Definition 3.4, there exists a correction function* $f''_{\mathrm{corr}} \in C_c^\infty(G')$ *such that for every regular semi-simple element* $\gamma \in G'_{\mathrm{rs}}$,

$$\partial \mathrm{Orb}(\gamma, f_D'') + \mathrm{Orb}(\gamma, f''_{\mathrm{corr}}) = \begin{cases} 2\,\mathrm{Int}(g)\log(q) & \begin{array}{l}\textit{if there exists some } [b] \in B(H,\mu_H)\\ \textit{and some } g \in G_{b,\mathrm{rs}} \textit{ that matches } \gamma\end{array} \\ 0 & \textit{otherwise.}\end{cases} \tag{4.22}$$

Proof of the equivalence of Conjectures 4.24 and 4.25. One direction is completely elementary: Let $\phi' \in C_c^\infty(G')$ be any test function. Fix some $h \in H'$ that satisfies $|h|^{-s} = q^s$. The function $\theta(\phi') := \phi' - (h,1)^*(\phi')$ then satisfies $\mathrm{Orb}(\gamma, \theta(\phi'), s) = (1-q^s)\,\mathrm{Orb}(\gamma, \phi', s)$ for all $\gamma \in G'_{\mathrm{rs}}$ (see (3.5)) and hence

$$\mathrm{Orb}(\gamma, \theta(\phi')) = 0, \quad \partial \mathrm{Orb}(\gamma, \theta(\phi')) = -\mathrm{Orb}(\gamma, \phi')\log(q).$$

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So if f''_D and f''_{corr} are as in Conjecture 4.25, then $f''_D - \theta(f''_{\mathrm{corr}})/\log(q)$ has all the properties required in Conjecture 4.24.

The converse direction relies on the density principle for orbital integrals on G', which is due to H. Xue [42, Theorem 8.3]. It states that any test function $f' \in C_c^\infty(G')$ such that $\mathrm{Orb}(\gamma, f') = 0$ for all $\gamma \in G'_{\mathrm{rs}}$ lies in the space

$$V = \{\phi' - \eta(h_2)(h_1, h_2)^*(\phi') \mid \phi' \in C_c^\infty(G'),\ h_1, h_2 \in H'\}.$$

We apply this as follows: Assume that f''_D has all the properties that are required in Conjecture 4.24 and assume that $f' \in C_c^\infty(G')$ is any transfer of f_D. Then $f''_D - f'$ lies in V. Since

$$\partial\mathrm{Orb}(\gamma, \phi' - \eta(h_2)(h_1, h_2)^*(\phi')) = \mathrm{Orb}(\gamma, \phi') \log|h_1 h_2|,$$

we deduce that there exists a correction function f''_{corr} with $\mathrm{Orb}(\gamma, f''_{\mathrm{corr}}) = \partial\mathrm{Orb}(\gamma, f''_D - f')$ for all $\gamma \in G'_{\mathrm{rs}}$. Then $(f', f''_{\mathrm{corr}})$ has all the properties that are required in Conjecture 4.25. □

Taking into account our FL (Conjecture 3.10), we have the following explicit form of the AT:

Conjecture 4.26 (ATC – explicit form). *Let f'_D be the test function from Definition 3.9. There exists a correction function $f'_{\mathrm{corr}} \in C_c^\infty(G')$ such that for every regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$,*

$$\partial\mathrm{Orb}(\gamma, f'_D) + \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}) = \begin{cases} 2\ \mathrm{Int}(g)\log(q) & \text{if there exists some } [b] \in B(H, \mu_H) \\ & \text{and some } g \in G_{b,\mathrm{rs}} \text{ that matches } \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

The status of Conjecture 4.26 is as follows:

(1) Consider the case that $D \cong M_{2n}(F)$. Then it is conjectured that one may take $f'_{\mathrm{corr}} = 0$ (AFL conjecture). The AFL conjecture first appeared in [27][3] and has been verified for $n = 1$ and $n = 2$ in [27] and [29].

For general n, at least the vanishing part of (4.23) is known by [30, Corollary 2.14]. Furthermore, [30, Theorem 1.2] states that it is enough to consider (4.23) for all basic isogeny classes.

(2) Consider next the case that $D \cong M_n(D_{1/2})$. Then [20, Theorem B] reduces Conjecture (4.26) to the linear AFL conjecture for $M_{2n}(F)$. In particular, the case $D \cong M_2(D_{1/2})$ is known by [29].

(3) The main result of the present paper is a verification of Conjecture 4.26 for $D \cong D_{1/4}$ and $D \cong D_{3/4}$. In particular, Conjecture 4.26 is known in all cases with $n \leq 2$.

Part II
Orbital integrals for GL_4

5. Main results

We now specialize to the case $n = 2$ (i.e., $G' = GL_4(F)$). Consider the following two subgroups of $GL_4(O_F)$:

$$\mathrm{Par} := \begin{pmatrix} GL_2(O_F) & \pi M_2(O_F) \\ M_2(O_F) & GL_2(O_F) \end{pmatrix}, \qquad \mathrm{Iw} := \begin{pmatrix} O_F^\times & (\pi) & (\pi) & (\pi) \\ O_F & O_F^\times & O_F & (\pi) \\ O_F & (\pi) & O_F^\times & (\pi) \\ O_F & O_F & O_F & O_F^\times \end{pmatrix}. \tag{5.1}$$

The first is the stabilizer of the lattice chain

$$O_F^{\oplus 2} \oplus O_F^{\oplus 2} \supset (\pi)^{\oplus 2} \oplus O_F^{\oplus 2},$$

[3] The version in [30] includes a correction that is related to the counting of connected components of $\Gamma\backslash\mathcal{I}(g)$.

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and the second is the stabilizer of

$$\begin{aligned} O_F^{\oplus 2} \oplus O_F^{\oplus 2} &\supset (\pi) \oplus O_F \oplus O_F \oplus O_F \\ &\supset (\pi) \oplus O_F \oplus (\pi) \oplus O_F \\ &\supset (\pi) \oplus (\pi) \oplus (\pi) \oplus O_F. \end{aligned}$$

These are standard lattice chains in the sense of Definition 3.9. Set $f'_{\mathrm{Par}} = 1_{\mathrm{Par}}$ and $f'_{\mathrm{Iw}} = (q+1)^4\, 1_{\mathrm{Iw}}$. These define the functions f'°_D from Definition 3.9 – that is,

$$f'^{\circ}_D = \begin{cases} 1_{GL_4(O_F)} & \text{if } D \cong M_4(F) \\ f'_{\mathrm{Par}} & \text{if } D \cong M_2(D_{1/2}) \\ f'_{\mathrm{Iw}} & \text{if } D \text{ division.} \end{cases} \tag{5.2}$$

In this section, D will always be a division algebra of degree 4 and f'_D the corresponding test function. The relation of f'_D and f'°_D from (3.15) specializes to

$$\mathrm{Orb}(\gamma, f'_D, s) = q^{-s}\,\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s). \tag{5.3}$$

The aim of this chapter is to compute the central values and the central derivatives $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$, $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ and $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. Our results on $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ will, in particular, prove the FL conjecture for D. The results about $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ and $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ in turn will be used to verify the AT conjecture later.

We now define the so-called *numerical invariant* of an element $\gamma \in G'_{\mathrm{rs}}$ or $g \in G_{\mathrm{rs}}$. It simplifies the invariant $\mathrm{Inv}(\gamma; T)$ resp. $\mathrm{Inv}(g; T)$ in the sense that it only records a certain valuation and a certain conductor. Its significance lies in the fact that all orbital integrals and all intersection numbers in this article only depend on the numerical invariant.

Recall the definition of the conductor: Assume L/F is an étale quadratic extension and $O \subset L$ an O_F-order. The conductor $\mathrm{cond}(O)$ is the unique integer $c \geq 0$ such that $O = O_F + \pi^c O_L$.

Definition 5.1. Let $\delta = T^2 + \delta_1 T + \delta_0 \in F[T]$ be a regular semi-simple invariant of degree 2. Recall that this means that δ is separable with $\delta(0)\delta(1) \neq 0$. The numerical invariant of δ is the triple (L, r, d), where

$$L := F[T]/(\delta(T)), \quad r = v(\delta_0), \quad d = \mathrm{cond}(O_F[\pi^k \cdot t]) - r/2 - k. \tag{5.4}$$

Here, $t := T \bmod (\delta(T))$ is the image of T in L. The étale quadratic F-algebra L is only considered up to isomorphism. In fact, everything will only depend on whether L/F is inert, ramified or split. Moreover, the integer k in (5.4) is chosen sufficiently large so that $\pi^k t \in O_L$; the definition of d is independent of this choice.

The numerical invariant of a regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$ or $g \in G_{\mathrm{rs}}$ is the numerical invariant of $\mathrm{Inv}(\gamma; T)$ resp. $\mathrm{Inv}(g; T)$. For example, the numerical invariant of an element $\gamma \in G'_{\mathrm{rs}}$ may also be written as

$$\Big(L_\gamma,\ v(\det(z_\gamma)),\ \mathrm{cond}(O_F[\pi^k z_\gamma^2]) - v(\det(z_\gamma))/2 - k\Big), \quad k \gg 0. \tag{5.5}$$

Note that Lemma 3.16 expresses the sign of the functional equations of f'_{Par} and f'_{Iw} directly in terms of r:

$$\varepsilon_{M_2(D_{1/2})}(\gamma) = (-1)^r, \quad \varepsilon_D(\gamma) = (-1)^{r+1}. \tag{5.6}$$

The following three are our main results in this chapter and will all be proved in §8.

Proposition 5.2. *Let $\gamma \in G'_{\mathrm{rs}}$ be regular semi-simple with numerical invariant (L, r, d). The parahoric orbital integral* $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ *vanishes if r is odd, or if $r \leq 0$, or if $r/2 + d \leq 0$. In all other cases, it is given by*

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$$\begin{cases} 1+q^2+\ldots+q^{r/2-2} & \text{if } L \text{ ramified and } r \in 4\mathbb{Z} \\ (1+q^2+\ldots+q^{r/2-3})+ \ \ (q^{r/2-1}+q^{r/2}+\ldots+q^{r/2+d-1}) & \text{if } L \text{ ramified and } r \in 2+4\mathbb{Z} \\ 2(1+q^2+\ldots+q^{r/2-2}) & \text{if } L \text{ inert and } r \in 4\mathbb{Z} \\ 2(1+q^2+\ldots+q^{r/2-3})+2(q^{r/2-1}+q^{r/2}+\ldots+q^{r/2+d-2})+q^{r/2+d-1} & \text{if } L \text{ inert and } r \in 2+4\mathbb{Z} \\ 0 & \text{if } L \text{ split and } r \in 4\mathbb{Z} \\ q^{r/2+d-1} & \text{if } L \text{ split and } r \in 2+4\mathbb{Z}. \end{cases} \tag{5.7}$$

Theorem 5.3. *The fundamental lemma (Conjecture 3.10) holds. In other words, for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \begin{cases} \mathrm{Orb}(g, f_D) & \text{if there exists a matching } g \in G \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Let (L, r, d) be the numerical invariant of an element $\gamma \in G'_{\mathrm{rs}}$ and let $\delta = \mathrm{Inv}(\gamma; T)$. We remark that by Corollary 2.8, the matching element g in (5.8) exists if and only if B_δ (constructed for E/F) is a division algebra, which is if and only if $L_\delta \otimes E$ is a field and $z^2 \in L_\delta$ not a norm from $L_\delta \otimes_F E$, which is if and only if L/F is a ramified field extension and r odd. (Recall that $L \cong L_\delta$.)

Proposition 5.4. *Let $\gamma \in G'_{\mathrm{rs}}$ be regular semi-simple with numerical invariants (L, r, d). Assume first that r is odd, meaning that the sign $\varepsilon_D(\gamma)$ of the functional equation of $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ is positive. Then*

$$\partial\mathrm{Orb}(\gamma, f'_D) = 0 \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) \log(q). \tag{5.9}$$

Assume now that r is even, which implies $\partial\mathrm{Orb}(\gamma, f'_D) = \partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. If $r \leq 0$, then $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = 0$. If $r > 0$, it is given by

$$\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = 4q\log(q)\ \mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) + \log(q) \begin{cases} r & \text{if } L \text{ ramified} \\ 2r & \text{if } L \text{ inert} \\ 0 & \text{if } L \text{ split.} \end{cases} \tag{5.10}$$

6. Hyperbolic orbits

We call a regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$ hyperbolic if $L_\gamma \cong F \times F$. In this situation, the orbital integrals $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ and $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}, s)$ can be expressed in terms of much simpler orbital integrals for the Levi that is defined by L_γ.

In the following, we fix a hyperbolic element $\gamma \in G'_{\mathrm{rs}}$ of the form $\gamma = 1 + z_\gamma$; set $z = z_\gamma$. We also fix an isomorphism $L_\gamma \cong F \times F$. Recall from §3.1 that $K = \{\mathrm{diag}(a, a, b, b) \in M_4(F) \mid a, b \in F\}$ denotes the diagonal copy of $F \times F$, and recall from Proposition 2.6 that $V := F^4$ is free as $K \otimes_F L_\gamma$-module.

Let $V = V^0 \oplus V^1$ be the eigenspace decomposition as L_γ-module. It is preserved by γ because γ and z commute under our assumption $\gamma = 1 + z$. It also has the property that both V^0 and V^1 are free K-modules of rank 1. Thus, we are precisely in the setting of the Levi reduction formula from [30] and we begin by recalling the relevant results from [30].

6.1. Lattice decomposition

The reduction to the Levi is based on the fact that there is a bijection of lattices $X \subset V$ and the set

$$\left\{ (X^0, X^1, s) \,\middle|\, \begin{array}{l} X^0 \subset V^0,\ X^1 \subset V^1 \text{ both } O_F\text{-lattices} \\ s : X^1 \to V^0/X^0 \text{ any } O_F\text{-linear map} \end{array} \right\}. \tag{6.1}$$

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It is given by sending X to (X^0, X^1, s), where

$$X^0 = X \cap V^0, \quad X^1 = (X + V^0)/V^0, \quad s = [X^1 \to X \to V^0/X^0]. \tag{6.2}$$

Here, the map $X \to V^0/X^0$ is the projection to the first component, and the map $X^1 \to X$ is defined by any choice of splitting for $X \twoheadrightarrow X^1$. Moreover, there is a criterion for lattice inclusions. Assume that $X^0 \subseteq Y^0 \subset V^0$ and $X^1 \subseteq Y^1 \subset V^1$ are sublattices and that $s_Y : Y^1 \to V^0/Y^0$ resp. $s_X : X^1 \to V^0/X^0$ are maps as in (6.1). Let $X, Y \subset V$ be the corresponding lattices in V. Then

$$X \subseteq Y \iff \text{the diagram} \begin{array}{ccc} Y^1 & \xrightarrow{s_Y} & V^0/Y^0 \\ \uparrow & & \uparrow \\ X^1 & \xrightarrow{s_X} & V^0/X^0 \end{array} \text{commutes.} \tag{6.3}$$

The following lemma is immediately clear and stated here for later application.

Lemma 6.1. *Consider two lattices $X^0 \subset Y^0$ and $X^1 \subset Y^1$ as in Diagram (6.3).*

(1) *Assume that $X^0 = Y^0$. Then for every map $s_Y : Y^1 \to V^0/Y^0$, there is a unique map s_X such that (6.3) commutes.*

(2) *Assume that $X^1 = Y^1$. Then for every map $s_X : X^1 \to V^0/X^0$, there is a unique map s_Y such that (6.3) commutes.*

In the situation of the fixed hyperbolic element γ, there is the following numerical result. Write $\gamma^0 = \gamma|_{V^0}$ and $\gamma^1 = \gamma|_{V^1}$ for the two components. Then γ^0 and γ^1 are regular semi-simple (in the sense of §2) as endomorphisms of the K-modules V^0 and V^1, respectively, and

$$\mathrm{Inv}(\gamma; T) = \mathrm{Inv}(\gamma^0; T)\mathrm{Inv}(\gamma^1; T).$$

Thus, if we define $\alpha^j \in F$ by $\mathrm{Inv}(\gamma^j; T) = T - \alpha^j$, then $\alpha^0, \alpha^1 \notin \{0, 1\}$ and $\alpha^0 \neq \alpha^1$ by regular semi-simpleness of γ. We also define z^j as the j-component of z. Equivalently, $z^j = z_{\gamma^j}$.

Proposition 6.2. *Assume that $X^0 \subset V^0$ and $X^1 \subset V^1$ are two O_K-lattices that are z^j-stable. Then there are $|\alpha^0 - \alpha^1|^{-1}$ many lattices $X \subset V$ such that*

(1) *$X \cap V^0 = X^0$ and $(X + V^0)/V^0 = X^1$,*

(2) *X is O_K-stable and z-stable.*

This is essentially a very special case of [30, Proposition 4.5]. There are, however, some boundary cases which are not covered by that result (especially if the residue cardinality is 2), which is why we include a short proof.

Proof. Fix O_K-linear isomorphisms $O_K \cong X^j$ for both $j = 0, 1$. Via these coordinates, we understand z^0 and z^1 as O_K-conjugate linear endomorphisms of O_K. By (6.3), the lattices $X \subset V$ that satisfy the conditions (1) and (2) are in bijection with the set

$$\left\{ s \in \mathrm{Hom}_{O_K}(O_K, K/O_K) \mid z^0 s = s z^1 \right\}. \tag{6.4}$$

Also fix an isomorphism $O_K \cong O_F \times O_F$. In this basis, z^0 and z^1 are given by anti-diagonal matrices because they are O_K-conjugate linear, say

$$z^0 = \begin{pmatrix} & a \\ b & \end{pmatrix}, \quad z^1 = \begin{pmatrix} & c \\ d & \end{pmatrix}.$$

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Here, a, b, c and d all lie in O_F while $ab = \alpha^0$ and $cd = \alpha^1$. An element $(s_+, s_-) \in \mathrm{Hom}_{O_F}(O_F, F/O_F)^2$ lies in the set (6.4) if and only if

$$as_+ = cs_-, \quad bs_- = ds_+. \tag{6.5}$$

Thus, we need to count the solutions $(s_+, s_-) \in (F/O_F)^2$ of (6.5). By symmetry of the expression, we may assume that a is the coefficient with minimal valuation. Dividing (6.5) by a, we first note that the solutions to

$$s_+ = a^{-1}cs_-, \quad a^{-1}bs_- = a^{-1}ds_+$$

are precisely the pairs of the form $(a^{-1}cs_-, s_-)$ with $(a^{-1}b - a^{-2}cd)s_- = 0$. There are $|a^{-1}b - a^{-2}cd|^{-1}$ many such pairs. It follows that the solution count for (6.5) is

$$|a^2|^{-1}|a^{-1}b - a^{-2}cd|^{-1} = |ab - cd|^{-1} = |\alpha^0 - \alpha^1|^{-1},$$

and the proposition is proved. □

6.2. Lattice chains

We now extend Proposition 6.2 to the lattice chains from Definitions 3.8 and 3.20 when $n \leq 2$.

Definition 6.3. Define the following sets of lattice chains.

(1) Let $\mathcal{P}$ be the set of chains of O_K-lattices in V of the form

$$\Lambda_0 \supset (\pi, 1)\Lambda_0 \supset \pi\Lambda_0.$$

Here, $(\pi, 1)$ is meant as the element in O_K. Moreover, define

$$\mathcal{P}(\gamma) = \{\Lambda_\bullet \in \mathcal{P} \mid z\Lambda_i \subseteq \Lambda_{i+1} \text{ for } i = 0, 1\}. \tag{6.6}$$

By Lemma 3.21 (3), the set $\mathcal{P}(\gamma)$ agrees (up to notation) with the set defined in Definition 3.20 for $(n, \ell) = (2, 2)$.

(2) Let $\mathcal{L}$ be the set of chains of O_K-lattices in V of the form

$$\Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \Lambda_3 \supset \pi\Lambda_0$$

and such that O_K acts on Λ_i/Λ_{i+1} via the first projection $O_K \to O_F$ if $i = 0, 2$, resp. via the second projection if $i = 1, 3$. Furthermore, let

$$\mathcal{L}(\gamma) = \{\Lambda_\bullet \in \mathcal{L} \mid z\Lambda_i \subseteq \Lambda_{i+1} \text{ for all } i = 0, \ldots, 3\}. \tag{6.7}$$

By Lemma 3.21 (3), the set $\mathcal{L}(\gamma)$ is the set defined in Definition 3.20 for $(n, \ell) = (2, 4)$.

(3) For $j = 0, 1$, let $\mathcal{L}_+^j$ be the set of chains of O_K-lattices in V^j of the form

$$\Lambda_0^j \supset (\pi, 1)\Lambda_0^j \supset \pi\Lambda_0^j.$$

Let $\mathcal{L}_-^j$ be the set of chains of O_K-lattices in V^j of the form

$$\Lambda_0^j \supset (1, \pi)\Lambda_0^j \supset \pi\Lambda_0^j.$$

Denote by z^0 and z^1 the two components of z and define

$$\mathcal{L}_\pm^j(\gamma^j) = \{\Lambda_\bullet^j \in \mathcal{L}_\pm^j(\gamma^j) \mid z^j\Lambda_i^j \subseteq \Lambda_{i+1}^j \text{ for } i = 0, 1\}. \tag{6.8}$$

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By Lemma 3.21 (3), the set $\mathcal{L}_+^j(\gamma^j)$ agrees (up to notation) with the set defined in Definition 3.20 for $(n,\ell) = (1,2)$. The set $\mathcal{L}_-^j(\gamma^j)$ is a variant.

Next, let $\Lambda_\bullet$ lie in $\mathcal{P}$ or $\mathcal{L}$. Applying the map (6.2) to each term, we construct a pair of lattice chains in V^0 and V^1 by

$$\Lambda_\bullet^0 := \Lambda_\bullet \cap V^0 \quad \text{and} \quad \Lambda_\bullet^1 := (\Lambda_\bullet + V^0)/V^0. \tag{6.9}$$

The situation is straightforward for $\mathcal{P}$: If $\Lambda_\bullet \in \mathcal{P}$, then in particular, $\Lambda_1 = (\pi,1)\Lambda_0$ and hence also $\Lambda_1^j = (\pi,1)\Lambda_0^j$ for both $j = 0,1$. It follows that $(\Lambda_\bullet^0, \Lambda_\bullet^1) \in \mathcal{L}_+^0 \times \mathcal{L}_+^1$.

The situation is more subtle for $\mathcal{L}$: For each index $i = 0,\dots,3$, precisely one out of the following two possibilities occurs:

$$\begin{cases} [\Lambda_i^0 : \Lambda_{i+1}^0] = 1 & \text{and} \quad \Lambda_i^1 = \Lambda_{i+1}^1 \\ \Lambda_i^0 = \Lambda_{i+1}^0 & \text{and} \quad [\Lambda_i^1 : \Lambda_{i+1}^1] = 1. \end{cases} \tag{6.10}$$

We define the type of $\Lambda_\bullet$ as the vector $t(\Lambda_\bullet) \in \{0,1\}^4$ with $t(\Lambda_\bullet)_i = 0$ precisely if the first case occurs in (6.10). Since $\Lambda_4 = \pi\Lambda_0$ and since Λ_0, Λ_0^0 and Λ_0^1 are all free over O_K, the type $t(\Lambda_\bullet)$ can take the four values

$$(0,0,1,1),\ (1,1,0,0),\ (0,1,1,0) \text{ and } (1,0,0,1). \tag{6.11}$$

In particular, for each $\Lambda_\bullet \in \mathcal{L}$, each case in (6.10) occurs precisely twice. So there is a natural way to view $\Lambda_\bullet^0$ and $\Lambda_\bullet^1$ as 2-term lattice chains. With this indexing convention,

$$(\Lambda_\bullet^0, \Lambda_\bullet^1) \in \begin{cases} \mathcal{L}_+^0 \times \mathcal{L}_+^1 & \text{if } t(\Lambda_\bullet) = (0,0,1,1) \text{ or } (1,1,0,0) \\ \mathcal{L}_+^0 \times \mathcal{L}_-^1 & \text{if } t(\Lambda_\bullet) = (0,1,1,0) \\ \mathcal{L}_-^0 \times \mathcal{L}_+^1 & \text{if } t(\Lambda_\bullet) = (1,0,0,1). \end{cases} \tag{6.12}$$

Lemma 6.4. *The map in (6.9) restricts to a surjection*

$$\mathcal{P}(\gamma) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L}_+^0(\gamma^0) \times \mathcal{L}_+^1(\gamma^1) \tag{6.13}$$

all of whose fibers have cardinality $q^{-1}|\alpha^0 - \alpha^1|^{-1}$. Similarly, the map in (6.12) restricts to a surjection

$$\mathcal{L}(y) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L}_+^0(\gamma^0) \times \mathcal{L}_+^1(\gamma^1) \sqcup \mathcal{L}_-^0(\gamma^0) \times \mathcal{L}_+^1(\gamma^1) \sqcup \mathcal{L}_+^0(\gamma^0) \times \mathcal{L}_-^1(\gamma^1) \tag{6.14}$$

such that fibers over $\mathcal{L}_+^0(\gamma^0) \times \mathcal{L}_+^1(\gamma^1)$ have cardinality $2\,|\alpha^0 - \alpha^1|^{-1}$ and fibers over its complement have cardinality $|\alpha^0 - \alpha^1|^{-1}$. Moreover, both (6.13) and (6.14) commute with the action of $L_\gamma^\times$.

Proof. It is clear that if $\Lambda_\bullet$ is z-stable in the sense of (6.6) or (6.7), then $\Lambda_\bullet^j$ is z^j-stable in the sense of (6.8). In other words, the two maps (6.13) and (6.14) are defined as claimed. Moreover, the inclusion and projection maps $V^0 \hookrightarrow V \twoheadrightarrow V/V^0$ are L_γ-linear by definition, so it is clear that both maps (6.13) and (6.14) commute with the $L_\gamma^\times$-action. Our main task is to prove the claims on their fiber cardinalities. We will assume $v(\alpha^0), v(\alpha^1) > 0$ for this because otherwise both the sources and targets in (6.13) and (6.14) are empty.

We begin with the case of $\mathcal{P}(\gamma)$. Define the auxiliary operator $\widetilde{z} = (\pi,1)^{-1}z$. Then the pairs $(\Lambda_\bullet^0, \Lambda_\bullet^1) \in \mathcal{L}_+^0(\gamma^0) \times \mathcal{L}_+^1(\gamma^1)$ are in bijection with pairs of $O_K[\widetilde{z}]$-lattices $(\Lambda_0^0, \Lambda_0^1)$ in V^0 and V^1. (Indeed, an $O_K[\widetilde{z}]$-lattice Λ_0^j in V^j can be uniquely extended to the lattice chain $(\Lambda_0^j, (\pi,1)\Lambda_0^j) \in \mathcal{L}_+^j(\gamma^j)$.)

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Note that $\widetilde{z}^2 = \pi^{-1}z^2$, so the eigenvalues of $\widetilde{z}^2$ are $\pi^{-1}\alpha^0$ and $\pi^{-1}\alpha^1$. By Proposition 6.2, there are $q^{-1}|\alpha^0 - \alpha^1|^{-1}$ many $O_K[\widetilde{z}]$-stable lattices Λ_0 such that

$$\Lambda_0 \cap V^0 = \Lambda_0^0 \quad \text{and} \quad (\Lambda_0 \cap V^0)/V^0 = \Lambda_0^1.$$

For any of these possibilities, $\Lambda_0 \supset (\pi, 1)\Lambda_0 \supset \pi\Lambda_0$ defines a unique extension to an element $\Lambda_\bullet \in \mathcal{P}(\gamma)$. This $\Lambda_\bullet$ is then a preimage of $(\Lambda_\bullet^0, \Lambda_\bullet^1)$, and all claims about (6.13) are proved.

Now we turn to $\mathcal{L}(\gamma)$. Assume we are given a pair $(\Lambda_\bullet^0, \Lambda_\bullet^1)$ in the right-hand side of (6.14) as well as a type $t \in \{0,1\}^4$ that is compatible with the pair in the sense of (6.12). We claim that there are $|\alpha^0 - \alpha^1|^{-1}$ many lattice chains $\Lambda_\bullet \in \mathcal{L}(\gamma)$ of type t that map to $(\Lambda_\bullet^0, \Lambda_\bullet^1)$ under (6.14). We first prove this for the type $t = (0,0,1,1)$. By (6.3), the set of four term $O_K[z]$-lattice chains $\Lambda_\bullet$ such that $\Lambda_\bullet \cap V^0 = \Lambda_\bullet^0$ and $(\Lambda_\bullet + V^0)/V^0 = \Lambda_\bullet^1$ is in bijection with the set of tuples (s_0, s_1, s_2, s_3) of z-linear maps that give rise to a commutative ladder of the form

$$\begin{tikzcd}
\Lambda_0^1 \arrow[r, equal] \arrow[d, "s_0"] & \Lambda_0^1 \arrow[r, equal] \arrow[d, "s_1"] & \Lambda_0^1 \arrow[d, "s_2"] & \Lambda_1^1 \arrow[l, hook'] \arrow[d, "s_3"] & \pi\Lambda_0^1 \arrow[l, hook'] \arrow[d, "s_0"] \\
V^0/\Lambda_0^0 & V^0/\Lambda_1^0 \arrow[l, two heads] & V^0/(\pi\Lambda_0^0) \arrow[l, two heads] \arrow[r, equal] & V^0/(\pi\Lambda_0^0) \arrow[r, equal] & V^0/(\pi\Lambda_0^0).
\end{tikzcd} \tag{6.15}$$

By Lemma 6.1 (or by direct inspection), these tuples are in bijection with just the datum of the $O_K[z]$-linear map s_2. By Proposition 6.2, there are precisely $|\alpha^0 - \alpha^1|^{-1}$ many of those. Moreover, the corresponding chain $\Lambda_\bullet$ necessarily lies in $\mathcal{L}(\gamma)$ because for every i, there are j and k with $\Lambda_i/\Lambda_{i+1} \cong \Lambda_k^j/\Lambda_{k+1}^j$. So the condition $z\Lambda_i/\Lambda_{i+1} = 0$ follows from the assumption that the same kind of condition holds for $\Lambda_\bullet^0$ and $\Lambda_\bullet^1$. Our claim is now proved for $t = (0,0,1,1)$. For the other three possibilities for t, the argument is completely identical. Namely, all of the involved lattice chains may be extended (uniquely) to a $\pi^{\mathbb{Z}}$-periodic chain, and the four possibilities for the type t differ by rotation permutations; see (6.11). □

6.3. *Orbital integrals*

We next define suitably normalized orbital integrals on $GL(V^0)$ and $GL(V^1)$. Choose K-bases $V^j \cong F \times F$ such that

$$O_F^4 \cap V^0 = O_F^2 \quad \text{and} \quad (O_F^4 + V^0)/V^0 = O_F^2.$$

Assume that $\Lambda^j \subset V^j$ is a lattice such that both Λ^j and $\gamma^j\Lambda^j$ are O_K-stable. Specializing the definition in (3.24) to this situation, we have defined

$$\Omega(\gamma^j, \Lambda^j, s) = \Omega((h_1^j)^{-1}\gamma h_2^j, s),$$

where $h_1^j, h_2^j \in K^\times$ are such that $h_2^j O_F^2 = \Lambda^j$ and $h_1^j O_F^2 = \gamma^j\Lambda^j$. It is immediate that whenever $(\Lambda^0, \Lambda^1) = (\Lambda \cap V^0, (\Lambda + V^0)/V^0)$ for some O_K-lattice $\Lambda \subset F^4$ such that also $\gamma\Lambda$ is O_K-stable, then

$$\Omega(\gamma, \Lambda, s) = \Omega(\gamma^0, \Lambda^0, s)\Omega(\gamma^1, \Lambda^1, s). \tag{6.16}$$

Consider the two standard parahoric subgroups of $GL_2(F)$,

$$I_+ = \begin{pmatrix} O_F^\times & (\pi) \\ O_F & O_F^\times \end{pmatrix} \quad \text{and} \quad I_- = \begin{pmatrix} O_F^\times & O_F \\ (\pi) & O_F^\times \end{pmatrix}. \tag{6.17}$$

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Let $\phi_\pm$ denote the indicator function of $I_\pm$. These two functions are related by the conjugation $I_- = \mathrm{diag}(\pi,1)^{-1}\cdot I_+\cdot \mathrm{diag}(\pi,1)$, so their orbital integrals satisfy

$$\mathrm{Orb}(\gamma',\phi_-,s) = (-q^{-2s})\,\mathrm{Orb}(\gamma',\phi_+,s). \tag{6.18}$$

Here, $\gamma'\in GL_2(F)_{\mathrm{rs}}$ denotes a regular semi-simple element. Let us write $\mathcal{L}_\pm(\gamma')$ for the analog of (6.8) for the vector space F^2. The combinatorial interpretation of orbital integrals (3.25) specializes to

$$\mathrm{Orb}(\gamma',\phi_\pm,s) = \sum_{\Lambda_\bullet\in F^\times\backslash\mathcal{L}_\pm(\gamma')} \Omega(\gamma',\Lambda_0,s). \tag{6.19}$$

Proposition 6.5. *Let $\gamma'\in GL_2(F)_{\mathrm{rs}}$ be a regular semi-simple element of invariant $T-\alpha$. Put $X=-q^{-2s}$. Then the orbital integrals of ϕ_+ is given by*

$$\mathrm{Orb}(\gamma',\phi_+,s) = \begin{cases} 0 & \textit{if } v(\alpha)\le 0\\ |\alpha|^{-s}\frac{X^{v(\alpha)}-1}{X-1} & \textit{if } v(\alpha)>0.\end{cases} \tag{6.20}$$

Proof. The orbital integral only depends on the orbit of γ', so we may choose $\gamma' = \left(\begin{smallmatrix}1&\alpha\\1&1\end{smallmatrix}\right)$ for the computation. Let $h_1=\mathrm{diag}(a,1)$ and $h_2=\mathrm{diag}(c,d)$. Then

$$h_1^{-1}\gamma' h_2\in I_+ \quad\iff\quad v(d)=0,\ v(c)\ge 0,\ v(a)=v(c),\ v(\alpha)>v(a).$$

Assuming these equivalent conditions are met, the transfer factor is given by $\Omega(\gamma',s)=q^{v(\alpha)s}$, and the character that occurs in the integrand is

$$|h_1h_2|^s\eta(h_2) = (-1)^{v(c)}q^{-2v(c)s}.$$

A direct evaluation of the definition in (3.4) now gives

$$\mathrm{Orb}(\gamma',\phi_+,s) = q^{v(\alpha)s}\sum_{i=0}^{v(\alpha)-1}(-q^{-2s})^i$$

as was to be shown. □

Proposition 6.6. *Let $\gamma\in G'_{\mathrm{rs}}$ be regular semi-simple and hyperbolic with* $\mathrm{Inv}(\gamma;T)=(T-\alpha)(T-\beta)$. *Set $X=-q^{-2s}$.*

(1) *The parahoric orbital integral* $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s)$ *vanishes if $v(\alpha)\le 0$ or $v(\beta)\le 0$. Otherwise, it is given by*

$$\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s) \;=\; q^{-1}|\alpha-\beta|^{-1}\,|\alpha\beta|^{-s}\,\frac{(X^{v(\alpha)}-1)(X^{v(\beta)}-1)}{(X-1)^2}. \tag{6.21}$$

(2) *The Iwahori orbital integral* $\mathrm{Orb}(\gamma,f'_{\mathrm{Iw}},s)$ *vanishes if $v(\alpha)\le 0$ or $v(\beta)\le 0$. Otherwise, it is given by*

$$\mathrm{Orb}(\gamma,f'_{\mathrm{Iw}},s) \;=\; 2q\,(X+1)\,\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s). \tag{6.22}$$

Proof. If $v(\alpha)\le 0$ or $v(\beta)\le 0$, then $z=z_\gamma$ is not topologically nilpotent. It follows that $\mathcal{L}(\gamma)=\emptyset$ and $\mathcal{P}(\gamma)=\emptyset$, so both orbital integrals vanish by (3.25).

Now assume that $v(\alpha),v(\beta)>0$. We first deal with the function f'_{Par}. Let $\Gamma\subset L_\gamma^\times$ be the subgroup $(\pi,1)^{\mathbb{Z}}\times(1,\pi)^{\mathbb{Z}}$. It has the property that $\mathrm{vol}(L_\gamma^\times/\Gamma)=1$, so we may rewrite the combinatorial description

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of the orbital integral (3.25) as

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}, s) = \sum_{\Lambda_\bullet \in \Gamma\backslash\mathcal{P}(\gamma)} \Omega(\gamma, \Lambda_0, s).$$

By (6.13) and (6.16), this equals

$$\sum_{(\Lambda^0_\bullet \times \Lambda^1_\bullet) \in \left(F^\times\backslash\mathcal{L}^0_+(\gamma^0)\right)\times\left(F^\times\backslash\mathcal{L}^1_+(\gamma^1)\right)} q^{-1}|\alpha-\beta|^{-1}\, \Omega\Big(\gamma^0, \Lambda^0_0, s\Big)\, \Omega\Big(\gamma^1, \Lambda^1_0, s\Big),$$

which coincides with $q^{-1}|\alpha-\beta|^{-1}\,\mathrm{Orb}(\gamma^0, \phi_+, s)\,\mathrm{Orb}(\gamma^1, \phi_+, s)$ by (6.19). Substituting (6.20) yields (6.21).

Now we turn to the test function f'_{Iw}. Arguing as before and using (3.25), we obtain the combinatorial formula

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s) = \sum_{\Lambda_\bullet \in \Gamma\backslash\mathcal{L}(\gamma)} \Omega(\gamma, \Lambda_0, s).$$

By (6.14) and (6.16), this expression may be rewritten as the following sum of three terms:

$$\begin{aligned}
&\sum_{(\Lambda^0_\bullet \times \Lambda^1_\bullet) \in \left(F^\times\backslash\mathcal{L}^0_+(\gamma^0)\right)\times\left(F^\times\backslash\mathcal{L}^1_+(\gamma^1)\right)} 2\,|\alpha-\beta|^{-1}\, \Omega\Big(\gamma^0, \Lambda^0_0, s\Big)\, \Omega\Big(\gamma^1, \Lambda^1_0, s\Big)\\
+&\sum_{(\Lambda^0_\bullet \times \Lambda^1_\bullet) \in \left(F^\times\backslash\mathcal{L}^0_+(\gamma^0)\right)\times\left(F^\times\backslash\mathcal{L}^1_-(\gamma^1)\right)} |\alpha-\beta|^{-1}\, \Omega\Big(\gamma^0, \Lambda^0_0, s\Big)\, \Omega\Big(\gamma^1, \Lambda^1_0, s\Big)\\
+&\sum_{(\Lambda^0_\bullet \times \Lambda^1_\bullet) \in \left(F^\times\backslash\mathcal{L}^0_-(\gamma^0)\right)\times\left(F^\times\backslash\mathcal{L}^1_+(\gamma^1)\right)} |\alpha-\beta|^{-1}\, \Omega\Big(\gamma^0, \Lambda^0_0, s\Big)\, \Omega\Big(\gamma^1, \Lambda^1_0, s\Big).
\end{aligned}$$

So we obtain from (6.18) and (6.19) that

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s) = 2\,|\alpha-\beta|^{-1}\,(X+1)\,\mathrm{Orb}(\gamma^0, \phi_+, s)\,\mathrm{Orb}(\gamma^1, \phi_+, s).$$

Substituting (6.20) again and comparing the result with the formula for $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}, s)$ yields (6.22). □

7. Germ expansion

In this section, we prove a germ expansion principle for the parahoric and Iwahori orbital integrals. Together with the results from §6, this will allow us to compute $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}, s)$ and $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ in all cases. We will first focus on the case of f'_{Iw}; the case of f'_{Par} is much simpler and will be treated in §7.4.

7.1. Simplified lattice counting

Our first aim is to give a more concrete description of the set $\mathcal{L}(\gamma)$ from Definition 6.3. (Note that its definition does not require γ to be hyperbolic.) Throughout, $w \in GL_2(F)$ denotes an element such that $\gamma(w) := \left(\begin{smallmatrix} 1 & 1 \\ w & 1 \end{smallmatrix}\right)$ lies in G'_{rs}. In other words, we assume that $L := F[w]$ is a quadratic étale extension of F and that $w, 1-w \in L^\times$. In this situation, $z^2_{\gamma(w)} = \mathrm{diag}(w, w)$, so $L_{\gamma(w)}$ can be identified with L. More precisely, $L_{\gamma(w)}$ equals the image of the diagonal embedding $L \longrightarrow M_2(F) \times M_2(F) \subset M_4(F)$.

Definition 7.1. Let $\mathcal{L}(w)$ be the set of quadruples $(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat)$ of O_F-lattices in L that have the following three properties.

(1) It holds that $\Lambda_0^\flat \subset \Lambda_0$ and $\Lambda_1^\flat \subset \Lambda_1$, and each of these two inclusions is of index 1.
(2) It holds that $O_L \cdot \Lambda_0 = O_L$.

(3) The four lattices fit into the diagram

$$\begin{array}{ccccccc} \Lambda_0 & \supset & \Lambda_0^\flat & \supseteq & \Lambda_1 & \supseteq & w\Lambda_0 \\ & & \cup & & \cup & & \cup \\ & & \pi\Lambda_0 & \supseteq & \Lambda_1^\flat & \supseteq & w\Lambda_0^\flat. \end{array} \tag{7.1}$$

For the next statement, choose a free discrete subgroup $\Gamma \subset L^\times$ such that $\mathrm{vol}(L^\times/\Gamma) = 1$. Concretely, take $\Gamma = \varpi^{\mathbb{Z}}$ with some uniformizer $\varpi \in L$ if L is a field or $\Gamma \cong (\varpi_1^{\mathbb{Z}}, \varpi_2^{\mathbb{Z}})$ for two uniformizers $\varpi_1, \varpi_2 \in F$ if $L \cong F \times F$.

Also let (L, r, d) be the numerical triple of $\gamma(w)$ from 5.1: The quadratic F-algebra $L = F[w]$ is already given while

$$r = v(N_{L/F}(w)) \quad \text{and} \quad d = \mathrm{cond}(O_F[\pi^k w]) - k - r/2, \quad k \gg 0. \tag{7.2}$$

Lemma 7.2. *Given $(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat) \in \mathcal{L}(w)$, the following lattice chain lies in $\mathcal{L}(\gamma(w))$:*

$$\Lambda_0 \oplus \Lambda_1 \ \supset\ \Lambda_0^\flat \oplus \Lambda_1 \ \supset\ \Lambda_0^\flat \oplus \Lambda_1^\flat \ \supset\ \pi\Lambda_0 \oplus \Lambda_1^\flat \ \supset\ \pi(\Lambda_0 \oplus \Lambda_1).$$

This assignment defines a bijection

$$\psi : \mathcal{L}(w) \xrightarrow{\sim} \Gamma\backslash\mathcal{L}(\gamma(w)) \tag{7.3}$$

which has the property that

$$\Omega(\gamma(w),\ \psi(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat),\ s) = (-q^s)^{-r(w)}(-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.4}$$

In particular, the Iwahori orbital integral has the expressions

$$\mathrm{Orb}(\gamma(w), f'_{\mathrm{Iw}}, s) \ = \ (-q^s)^{-r(w)} \sum_{(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat) \in \mathcal{L}(w)} (-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.5}$$

Proof. The well-definedness and bijectivity of (7.3) follows directly from definitions. For the description of the transfer factor (7.4), we first note that $\mathcal{L}(w) \neq \emptyset$ implies that w and hence z_γ are topologically nilpotent. The element $\gamma = \gamma(w)$ has the form $\gamma = 1 + z_\gamma$, so if $\Lambda \subset F^4$ is both O_K-stable and $\gamma^{-1}O_K\gamma$-stable, then already $\gamma\Lambda = \Lambda$ by Lemma 3.21. In this case, (3.30) simplifies to

$$\Omega(\gamma, \Lambda, s) = (-1)^{[\Lambda_-:z\Lambda_+]} q^{([\Lambda_+:z\Lambda_-] - [\Lambda_-:z\Lambda_+])s}.$$

Then (7.4) is obtained from evaluating this with $\Lambda_+ = \Lambda_0 \oplus 0$ and $\Lambda_- = 0 \oplus \Lambda_1$ with $z = \begin{pmatrix} & 1 \\ w & \end{pmatrix}$. Finally, (7.5) follows from the previous two statements and (3.25). □

7.2. The germ expansion

Any O_F-lattice $\Lambda \subset L$ defines an order $O_\Lambda = \{x \in L \mid x\Lambda \subseteq \Lambda\}$. We define the conductor of Λ as the conductor of its order,

$$\mathrm{cond}(\Lambda) := \mathrm{cond}(O_\Lambda).$$

Definition 7.3. We define the principal germ orbital integral of w as

$$P(w, s) := \sum_{(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat) \in \mathcal{L}(w),\ \mathrm{cond}(\Lambda_0) \ = \ \mathrm{cond}(\Lambda_0^\flat) \ = \ 0} (-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.6}$$

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Note that $\mathrm{cond}(\Lambda_0) = 0$ just says $\Lambda_0 = O_L$. Let $i(L)$ denote the index $[O_L^\times : (O_F + \pi O_L)^\times]$. Define the unipotent germ orbital integral as

$$U(w,s) := i(L)^{-1} \sum_{(\Lambda_0,\Lambda_0^\flat,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w),\ (\mathrm{cond}(\Lambda_0),\mathrm{cond}(\Lambda_0^\flat)) \neq (0,0)} (-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.7}$$

Proposition 7.4. *The Iwahori orbital integral, the principal germ, and the unipotent germ are related by the following identity. For every $w \in GL_2(F)$ such that $\gamma(w) \in G'_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma(w), f'_{\mathrm{Iw}}, s) = (-q^s)^{-r(w)}[P(w,s) + i(L)U(w,s)]. \tag{7.8}$$

Proof. This follows directly from the combinatorial description in (7.5) and the definition of the two germs. □

Proposition 7.5. *The principal germ $P(w,s)$ depends only on the triple $(L, r(w), d(w))$. The unipotent germ is independent of L and only depends on the pair $(r(w), d(w))$.*

Proof. The claim about the principal germ will be proved as part of Proposition 7.9 below. We only deal with the unipotent germ here, which relies on the following lemmas.

Lemma 7.6 (Classification of numerical invariants). *Let $r = r(w)$ and $d = d(w)$ in the following.*

(1) *Assume that $d \geq 0$ and let $\zeta \in O_L$ be an O_F-algebra generator. Then r is even, $d \in \mathbb{Z}$, and $\pi^{-r/2}w \in O_L^\times$. Moreover, w is of the form $\pi^{r/2}(a + \pi^d b\zeta)$ for suitable elements $a \in O_F$ and $b \in O_F^\times$.*

(2) *Assume that $d < 0$. Then L is either ramified or split.*

(3) *Assume that $d < 0$ and that L is ramified. Then r is odd, $d = -1/2$ and $\pi^{(r-1)/2}w$ is a uniformizer of L.*

(4) *Assume that $d < 0$ and that $L = F \times F$. Then $w = (x, y)$ for two element $x, y \in F^\times$ such that*

$$v(x) + v(y) = r \quad \textit{and} \quad |v(x) - v(y)| = -2d.$$

Proof. This is proved by an easy case-by-case analysis.

(a) Assume that L/F is an unramified field extension. Then $r = 2v_L(w)$ is even. The element $w_0 = \pi^{-r/2}w$ lies in $O_L^\times$. By definition (see (7.2)), $d = \mathrm{cond}(O_F[w_0])$ is ≥ 0. Moreover, given a generator $O_L = O_F[\zeta]$, there obviously are $a, b \in O_F^\times$ such that $w_0 = a + \pi^d b\zeta$. This proves (2), as well as (1) whenever L is an unramified field extension.

(b1) Assume that L/F is a ramified field extension. Then $r = v_L(w)$. First assume that r is even. Then $w_0 = \pi^{-r/2}w$ lies in $O_L^\times$ and, just as in (a), $d = \mathrm{cond}(O_F[w_0])$ is ≥ 0. Given a generator $O_L = O_F[\zeta]$, it is again clear that there are $a, b \in O_F^\times$ such that $w_0 = a + \pi^d b\zeta$. This shows (1) whenever L is ramified.

(b2) Now assume r is odd. Then $w = \pi^{(r-1)/2}\varpi$ for a uniformizer $\varpi \in L$. We find that

$$\begin{aligned} d &= \mathrm{cond}(O_F[\pi^{-(r-1)/2}w]) + (r-1)/2 - r/2 \\ &= \mathrm{cond}(O_L) - 1/2 = -1/2. \end{aligned}$$

This proves (3).

(c1) Now assume that $L = F \times F$ and $w = (x, y)$ with $v(x) \leq v(y)$. The identity $r = v(x) + v(y)$ is clear from definition. Set $w_0 = \pi^{-v(x)}w$. Then

$$\begin{aligned} d &= \mathrm{cond}(O_F[w_0]) + v(x) - (v(x) + v(y))/2 \\ &= \mathrm{cond}(O_F[w_0]) + (v(x) - v(y))/2. \end{aligned} \tag{7.9}$$

Assume that $v(x) = v(y)$. Then we get $d = \mathrm{cond}(O_F[w_0]) \geq 0$. Given a generator $\zeta \in O_L$, there are $a, b \in O_F^\times$ such that $w_0 = a + \pi^d b\zeta$. Since $v(x) = r/2$, this shows (1).

(c2) Finally, assume that $v(x) < v(y)$. Then $O_F[w_0] = O_L$, so (7.9) gives $-2d = v(y) - v(x)$ which shows (4). □

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Lemma 7.7. *Let L_1 and L_2 be two quadratic étale F-algebras. Let $R_{i,c} = O_F + \pi^c O_{L_i}$ denote the order of conductor c in L_i. Let furthermore $w_1 \in L_1^\times \setminus F$ and $w_2 \in L_2^\times \setminus F$ be elements with*

$$r(w_1) = r(w_2) \quad \textit{and} \quad d(w_1) = d(w_2).$$

Then there exists an F-linear map $\phi : L_1 \to L_2$ that has the following property. For every $c \geq 0$, both

$$\phi(R_{1,c}) = R_{2,c} \quad \textit{and} \quad \phi(w_1 R_{1,c}) = w_2 R_{2,c}.$$

Proof. Given $x, y \in O_F^\times$, there is a unique F-linear map $\phi_{x,y} : L_1 \to L_2$ such that

$$\phi_{x,y}(1) = x \quad \text{and} \quad \phi_{x,y}(w_1) = y w_2.$$

We will show that there are $x, y \in O_F$ such that $\phi = \phi_{x,y}$ has the properties claimed in the lemma. (In fact, ϕ is necessarily of such a form.) Note that for all x, y as above, $\phi_{x,y}(O_F) = O_F$ and $\phi_{x,y}(w_1 O_F) = w_2 O_F$. Since $R_{i,c} = O_F + \pi^c O_{L_i}$, our task is thus to find x and y such that also $\phi_{x,y}(O_{L_1}) = O_{L_2}$ and $\phi_{x,y}(w_1 O_{L_1}) = w_2 O_{L_2}$. Put $r = r(w_1)$ and $d = d(w_1)$ in the following.

First assume that $d \geq 0$. Let $\zeta_i \in O_{L_i}$ be two O_F-algebra generators. Using Lemma 7.6 (1), there are $a_i \in O_F$ and $b_i \in O_F^\times$ such that

$$w_1 = \pi^{r/2}(a_1 + \pi^d b_1 \zeta_1), \quad w_2 = \pi^{r/2}(a_2 + \pi^d b_2 \zeta_2).$$

Consider first the case $d = 0$. Then we claim that $\phi = \phi_{1,1}$ satisfies the assertion of the lemma. Indeed,

$$\phi(\zeta_1) = b_1^{-1}(\pi^{-r/2} w_2 - a_1) = b_1^{-1}(a_2 - a_1) + b_1^{-1} b_2 \zeta_2$$

is an O_F-algebra generator of O_{L_2}. Furthermore, $w_i O_{L_i} = \pi^{r/2} O_{L_i}$ because $d \geq 0$, so it also follows that $\phi(w_1 O_{L_1}) = w_2 O_{L_2}$.

Consider now the case $d > 0$. Then $a_1, a_2 \in O_F^\times$, and we claim that $\phi = \phi_{x,y}$ with $x = a_1^{-1} a_2$ and $y = 1$ satisfies the assertion of the lemma. Indeed, we obtain

$$\phi(\zeta_1) = \pi^{-d} b_1^{-1}(\pi^{-r/2} w_2 - a_2) = b_1^{-1} b_2 \zeta_2,$$

which is again an O_F-algebra generator of O_{L_2}. Hence, $\phi(O_{L_1}) = O_{L_2}$, and the desired statement $\phi(w_1 O_{L_1}) = w_2 O_{L_2}$ is obtained just as in the previous case. This proves the lemma in case $d \geq 0$.

Now assume that $d < 0$. No matter which of the two cases (3) and (4) of Lemma 7.6 occur for w_1 and w_2, it always holds that $\zeta_i = \pi^{-r/2-d} w_i$ is an O_F-algebra generator of O_{L_i}. So take $\phi = \phi_{1,1}$. Then $\phi(\zeta_1) = \zeta_2$ and hence $\phi(O_{L_1}) = O_{L_2}$. Furthermore, the element π^{-2d}/ζ_i is again an O_F-algebra generator of O_{L_i}; see Lemma 7.6 (4). We obtain

$$\phi(w_1 O_{L_1}) = \pi^{r/2+d} \phi(\zeta_1(O_F + \pi^{-2d}/\zeta_1 O_F)) = \pi^{r/2+d}(\zeta_2 O_F + \pi^{-2d} O_F) = w_2 O_{L_2},$$

as desired, and the proof of the lemma is complete. □

We now come to the main part of the proof of Proposition 7.5. Let $\mathcal{B}$ be the PGL-building for L viewed as F-vector space. Concretely, $\mathcal{B}$ is the graph (a tree in fact) with the following description. Its vertices are in bijection with O_F-lattices $\Lambda \subset O_L$ such that O_L/Λ is a cyclic O_F-module. Its edges are given by unordered pairs $\{\Lambda, \Lambda'\}$ such that one lattice is a sublattice of index 1 of the other. We write $d(\Lambda, \Lambda')$ for the distance between two vertices Λ and Λ'. Consider the subtree $\mathcal{E} \subset \mathcal{B}$ that is spanned by the vertices

$$\mathcal{E} = \{\Lambda \in \mathcal{B} \mid O_L \cdot \Lambda = O_L\}. \tag{7.10}$$

(An equivalent description is as follows: Let $\mathcal{C} = \{\Lambda \in \mathcal{B} \mid O_L \cdot \Lambda = \Lambda\}$ be the subtree spanned by all O_L-lattices. It equals $\{O_L\}$, the edge $\{O_L, \varpi O_L\}$ or the apartment $\{(\pi^k, 1)O_L,\ (1, \pi^k)O_L \mid k \geq 0\}$, depending on whether L is inert, ramified with uniformizer ϖ, or isomorphic to $F \times F$. Then $\mathcal{E}$ consists of all points of $\mathcal{B}$ whose shortest path to O_L does not contain any other point of $\mathcal{C}$.) Let $\mathcal{E}_c = \{\Lambda \in \mathcal{E} \mid d(O_L, \Lambda) = c\}$. The following statements hold in this situation:

(1) For any vertex $\Lambda \in \mathcal{E}$, we have that $\mathrm{cond}(\Lambda) = d(O_L, \Lambda)$.

(2) The group $O_L^\times$ acts transitively on $\mathcal{E}_c$ with stabilizer $R_c^\times$. Here, $R_c = O_F + \pi^c O_L$ again denotes the order of conductor c. In particular, $\#\mathcal{E}_c = [O_L^\times : R_c^\times]$ which equals $i(L)q^{c-1}$ if $c \geq 1$.

(3) Since $\mathcal{E}$ is a tree, $O_L^\times$ then also acts transitively on the set of edges of $\mathcal{E}$ with distance c from O_L. In particular, every edge of distance c from O_L is an $O_L^\times$-translate of the edge $\{R_c, R_{c+1}\}$.

Consider now a pair $(\Lambda_0, \Lambda_0^\flat)$ of lattices in L with $\Lambda_0 \in \mathcal{E}$ and such that $\Lambda_0^\flat \subset \Lambda_0$ with index 1. Assume that $\mathrm{cond}(\Lambda_0, \Lambda_0^\flat) \neq (0,0)$. This is equivalent to $\{\Lambda_0, \Lambda_0^\flat\} \not\subset \mathcal{C}$, which means that $\{\Lambda_0, \Lambda_0^\flat\}$ is an edge of $\mathcal{E}$. In particular, Λ_0 and $\Lambda_0^\flat$ have different conductors. So if $c = \min\{\mathrm{cond}(\Lambda_0), \mathrm{cond}(\Lambda_0^\flat)\}$, then

$$\mathrm{cond}(\Lambda_0, \Lambda_0^\flat) \in \{(c, c+1),\ (c+1, c)\}. \tag{7.11}$$

In the first case, $\Lambda_0^\flat$ lies in $\mathcal{B}$. In the second case, it is instead $\pi^{-1}\Lambda_0^\flat$ that lies in $\mathcal{B}$. Depending on which case occurs, $\{\Lambda_0, \Lambda_0^\flat\}$ or $\{\Lambda_0, \pi^{-1}\Lambda_0^\flat\}$ defines an edge of $\mathcal{B}$ that lies in $\mathcal{E}$ (because $\Lambda_0 \in \mathcal{E}$) and that has distance c from O_L.

For a fixed pair $(\Lambda_0, \Lambda_0^\flat)$, let $\mathcal{L}(w; \Lambda_0, \Lambda_0^\flat)$ denote the set of quadruples $(\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat) \in \mathcal{L}(w)$. Assuming that $\Lambda_0^\flat \subseteq \pi O_L$, there is the symmetry

$$\begin{aligned} \mathcal{L}(w; \Lambda_0, \Lambda_0^\flat) &\xrightarrow{\sim} \mathcal{L}(w; \pi^{-1}\Lambda_0^\flat, \Lambda_0) \\ (\Lambda_0, \Lambda_0^\flat, \Lambda_1, \Lambda_1^\flat) &\longmapsto (\pi^{-1}\Lambda_0^\flat, \Lambda_0, \pi^{-1}\Lambda_1^\flat, \Lambda_1). \end{aligned}$$

We deduce that no matter which case occurs in (7.11), $\#\mathcal{L}(w; \Lambda_0, \Lambda_0^\flat) = \#\mathcal{L}(w; R_c, R_{c+1})$. It follows that we can rewrite the definition of the unipotent germ in (7.7) as

$$U(w, s) \ = \ 2 \sum_{c \geq 0} q^c \sum_{(R_c, R_{c+1}, \Lambda_1, \Lambda_1^\flat) \in \mathcal{L}(w)} (-q^{2s})^{[R_c : \Lambda_1]}. \tag{7.12}$$

It now follows from Lemma 7.7 that the outer diagram in (7.1),

$$\begin{array}{ccccccc} R_c & \supset & R_{c+1} & \supseteq & \ldots & \supseteq & wR_c \\ & & \cup & & & & \cup \\ & & \pi R_c & \supseteq & \ldots & \supseteq & wR_{c+1}, \end{array}$$

only depends on the invariants $r(w)$ and $d(w)$ up to F-linear isomorphism. We conclude that the Expression (7.12) only depends on $(r(w), d(w))$ and not on L, as was to be shown. □

7.3. *The principal germ*

The purpose of this section is to explicitly compute the principal germ. The relative position $(M_0 : M_1) \in \mathbb{Z}^2$ of two lattices $M_0, M_1 \subset F^2$ is, by definition, the pair (a, b) with $a \leq b$ that consists of the valuations of their elementary divisors (Cartan decomposition). For example, $(M_0 : M_1) = (0, k)$ with $k \geq 0$ if and only if $M_0 \supseteq M_1$ with cyclic quotient M_0/M_1 of length k. By lattice pair in F^2, we mean a pair of lattices $(M, M^\flat)$ such that $(M : M^\flat) = (0, 1)$.

For nonnegative integers $0 \leq a \leq b$ and a third integer $0 \leq k \leq a + b$, we define the quantity

$$\Xi_k(a, b) = 1 + 2q + \ldots + 2q^{\min\{k, a, a+b-k\}}. \tag{7.13}$$

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The boundary cases here are understood as $\Xi_k(a,b) = 1$ whenever $\min\{k, a, a+b-k\} = 0$. We also need a slight modification of $\Xi_k(a,b)$ which will only be considered for $a \geq 1$:

$$\Xi'_k(a,b) = \begin{cases} 1 + 2q + \ldots + 2q^{\min\{k, a+b-k\}} & \text{if } k < a \text{ or } b < k \\ 1 + 2q + \ldots + 2q^{a-1} + q^a & \text{if } a \leq k \leq b \end{cases} \tag{7.14}$$

with boundary cases $\Xi'_k(a,b) = 1$ whenever $k \in \{0, a+b\}$.

Lemma 7.8. *Let $(M_0, M_0^\flat)$ and $(M_1, M_1^\flat)$ be two lattice pairs in F^2 such that $M_0 \supseteq M_1$ and $M_0^\flat \supseteq M_1^\flat$. Let $0 \leq a \leq b$ be such that $(M_0 : M_1) = (a : b)$. Then the number of lattice pairs $(\Lambda, \Lambda^\flat)$ that fit into the diagram*

$$\begin{array}{ccccc} M_0 & \supseteq & \Lambda & \supseteq & M_1 \\ \cup & & \cup & & \cup \\ M_0^\flat & \supseteq & \Lambda^\flat & \supseteq & M_1^\flat \end{array} \tag{7.15}$$

and furthermore satisfy $[M_0 : \Lambda] = k$ is given by

$$\begin{cases} \Xi_k(a,b) & \text{if } (M_0^\flat : M_1^\flat) = (a : b) \\ \Xi'_k(a,b) & \text{if } (M_0^\flat : M_1^\flat) = (a-1 : b+1) \\ \Xi'_k(a+1, b-1) & \text{if } a+2 \leq b \text{ and } (M_0^\flat : M_1^\flat) = (a+1 : b-1). \end{cases} \tag{7.16}$$

Part 1 of the proof: Auxiliary results. We begin with a few easier counting formulas. Let $0 \leq a \leq b$ and $0 \leq k \leq a+b$ be integers and let $M_0 \supseteq M_1$ be two lattices of relative position $(a : b)$. Put

$$\Phi_k^{\mathrm{prim}}(a,b) = \#\{M_0 \supseteq \Lambda \supseteq M_1 \mid [M_0 : \Lambda] = k,\ M_0/\Lambda \text{ cyclic}\}. \tag{7.17}$$

We have

$$\#\mathrm{Surjections}\big(O_F/(\pi)^a \oplus O_F/(\pi)^b,\ O_F/(\pi)^k\big) = \begin{cases} 1 & \text{if } k = 0 \\ q^{2k} - q^{2k-2} & \text{if } 1 \leq k \leq a \\ q^a(q^k - q^{k-1}) & \text{if } a < k \leq b \\ 0 & \text{if } b < k. \end{cases}$$

Since $\#\,\mathrm{Aut}\,\big(O_F/(\pi)^k\big) = q^k - q^{k-1}$, it follows that

$$\Phi_k^{\mathrm{prim}}(a,b) = \begin{cases} 1 & \text{if } k = 0 \\ q^{k-1} + q^k & \text{if } 1 \leq k \leq a \\ q^a & \text{if } a < k \leq b \\ 0 & \text{if } b < k. \end{cases} \tag{7.18}$$

We next consider the quantities

$$\Phi_k(a,b) = \#\{M_0 \supseteq \Lambda \supseteq M_1 \mid [M_0 : \Lambda] = k\}. \tag{7.19}$$

It holds that $\Phi_0(a,b) = 1$ and that

$$\Phi_1(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ 1 & \text{if } a = 0 \text{ and } 1 \leq b \\ 1+q & \text{if } 1 \leq a. \end{cases} \tag{7.20}$$

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A sublattice $\Lambda \subseteq M_0$ has the property that M_0/Λ is not cyclic if and only if it is contained in πM_0. So there is a recursion formula for $2 \leq k$:

$$\Phi_k(a,b) = \Phi_{k-2}(a-1,b-1) + \Phi_k^{\mathrm{prim}}(a,b). \tag{7.21}$$

It follows from this and (7.18) that

$$\Phi_k(a,b) = 1 + q + \ldots + q^{\min\{k,a,a+b-k\}}. \tag{7.22}$$

We next count lattice pairs that lie between M_0 and M_1. More precisely, we consider the quantity

$$\Psi_k(a,b) = \#\{M_0 \supseteq \Lambda \supset \Lambda^\flat \supseteq M_1 \mid [M_0:\Lambda] = k,\ [\Lambda:\Lambda^\flat] = 1\}. \tag{7.23}$$

If we are given $M_0 \supseteq \Lambda^\flat \supseteq M_1$, then there are either 1 or $1+q$ possibilities for finding a lattice Λ as in (7.23), depending on whether $M_0/\Lambda^\flat$ is cyclic or not. We obtain that

$$\Psi_k(a,b) = \Phi_{k+1}^{\mathrm{prim}}(a,b) + (1+q)\Phi_{k-1}(a-1,b-1).$$

Evaluating this expression with (7.22), it follows that

$$\Psi_k(a,b) = \begin{cases} 1 + 2q + \ldots + 2q^k + q^{k+1} & \text{if } k < a \\ 1 + 2q + \ldots + 2q^a & \text{if } a \leq k < b \\ 1 + 2q + \ldots + 2q^{a+b-k-1} + q^{a+b-k} & \text{if } b \leq k \leq a+b-1. \end{cases} \tag{7.24}$$

Part 2 of the proof: Main result. We now come back to the setting of the lemma. That is, $(M_0, M_0^\flat)$ and $(M_1, M_1^\flat)$ denote lattice pairs with $(M_0 : M_1) = (a:b)$ and $M_0^\flat \supseteq M_1^\flat$; our aim is to show (7.16). We begin by noting that diagrams of the form (7.15) have the symmetry

$$\begin{array}{ccccc} M_0 & \supseteq & \Lambda & \supseteq & M_1 \\ \cup & & \cup & & \cup \\ M_0^\flat & \supseteq & \Lambda^\flat & \supseteq & M_1^\flat \end{array} \quad \longmapsto \quad \begin{array}{ccccc} M_0^\flat & \supseteq & \Lambda^\flat & \supseteq & M_1^\flat \\ \cup & & \cup & & \cup \\ \pi M_0 & \supseteq & \pi\Lambda & \supseteq & \pi M_1. \end{array}$$

Thus, the third case in (7.16) follows directly from the second one and will not be considered anymore.

Next, we settle the case $a = 0$: Then M_0/M_1 is cyclic. The second case cannot occur (and the third case has already been dealt with), so we are in the first case meaning that the quotient $M_0^\flat/M_1^\flat$ is cyclic as well. It follows that for every $0 \leq k \leq b$, there is a unique lattice pair $(\Lambda, \Lambda^\flat)$ that fits (7.15) and satisfies $[M_0:\Lambda] = k$. This fits the special case $\Xi_k(0,b) = 1$ in (7.13).

From now on, we can and do assume that $1 \leq a$, which implies that $M_0^\flat \supset M_1$. The set of lattice pairs $(\Lambda, \Lambda^\flat)$ in question is then in bijection with the following disjoint union. (The condition $[M_0:\Lambda] = k$ is understood without explicit mentioning.)

$$\{M_0 \supseteq \Lambda \supseteq M_1 \mid M_0^\flat \not\supseteq \Lambda\} \ \sqcup\ \{M_0^\flat \supseteq \Lambda^\flat \supseteq M_1^\flat \mid \Lambda_0^\flat \not\supseteq M_1\} \ \sqcup\ \{M_0^\flat \supseteq \Lambda \supset \Lambda^\flat \supseteq M_1\}. \tag{7.25}$$

Indeed, for every lattice Λ from the first set or $\Lambda^\flat$ from the second set, there is a unique way to complete the diagram (7.15). It is given by setting $\Lambda^\flat = \Lambda \cap M_0^\flat$ or $\Lambda = \Lambda^\flat + M_1$, respectively. Furthermore, the cardinalities of all three sets in (7.25) are easily expressed in terms of Φ and Ψ:

$$\begin{aligned} \#\{M_0 \supseteq \Lambda \supseteq M_1 \mid M_0^\flat \not\supseteq \Lambda\} &= \#\{M_0 \supseteq \Lambda \supseteq M_1\} - \#\{M_0^\flat \supseteq \Lambda \supseteq M_1\} \\ &= \Phi_k(a,b) - \Phi_{k-1}(M_0^\flat : M_1) \\ \#\{M_0^\flat \supseteq \Lambda^\flat \supseteq M_1^\flat \mid \Lambda^\flat \not\supseteq M_1\} &= \#\{M_0^\flat \supseteq \Lambda^\flat \supseteq M_1^\flat\} - \#\{M_0^\flat \supseteq \Lambda^\flat \supseteq M_1\} \\ &= \Phi_k(M_0^\flat : M_1^\flat) - \Phi_k(M_0^\flat : M_1) \\ \#\{M_0^\flat \supseteq \Lambda \supset \Lambda^\flat \supseteq M_1\} &= \Psi_{k-1}(M_0^\flat : M_1). \end{aligned} \tag{7.26}$$

More precisely, we obtain from (7.26) that the number of $(\Lambda, \Lambda^\flat)$ in question is

$$\Phi_k(a,b) + \Phi_k(M_0^\flat : M_1^\flat) - \Phi_{k-1}(M_0^\flat : M_1) - \Phi_k(M_0^\flat : M_1) + \Psi_{k-1}(M_0^\flat : M_1). \tag{7.27}$$

We have already reduced to the first and second case in (7.16), so there are the following three possibilities left, none of which poses any difficulties:

$$\begin{cases} (M_0^\flat : M_1^\flat) = (a,b), & (M_0^\flat : M_1) = (a, b-1), \quad a \leq b-1 & \text{Case (1)} \\ (M_0^\flat : M_1^\flat) = (a,b), & (M_0^\flat : M_1) = (a-1, b) & \text{Case (2)} \\ (M_0^\flat : M_1^\flat) = (a-1,b+1), & (M_0^\flat : M_1) = (a-1, b) & \text{Case (3).} \end{cases}$$

Let Φ_k^{total} denote the sum of the four Φ-terms in (7.27). Using (7.22), one sees that in Case (1),

$$\Phi_k^{\text{total}} = \begin{cases} q^k & \text{if } 0 \leq k \leq a \\ 0 & \text{if } a < k < b \\ q^{a+b-k} & \text{if } b \leq k \leq a+b, \end{cases}$$

in Case (2),

$$\Phi_k^{\text{total}} = \begin{cases} q^k & \text{if } 0 \leq k < a \\ 2q^a & \text{if } a \leq k \leq b \\ q^{a+b-k} & \text{if } b < k \leq a+b, \end{cases}$$

and in Case (3),

$$\Phi_k^{\text{total}} = \begin{cases} q^k & \text{if } 0 \leq k < a \\ q^a & \text{if } a \leq k \leq b \\ q^{a+b-k} & \text{if } b < k \leq a+b. \end{cases}$$

Adding these to $\Psi_{k-1}(a, b-1)$ in Case (1) resp. to $\Psi_{k-1}(a-1, b)$ in Cases (2) and (3), and using (7.24), proves (7.16).

Proposition 7.9. *Let $w \in M_2(F)$ and $P(w,s)$ be as in Definition 7.3. Let (L, r, d) be the numerical invariants of w; see (7.2). For integers $0 \leq a \leq b$ and $0 \leq k \leq a+b$, let $\Xi_k(a,b)$ and $\Xi'_k(a,b)$ denote the quantities from (7.13) and (7.14); set $X = -q^{-2s}$.*

(1) *If L is an unramified field extension or if $r \leq 0$, then $P(w,s) = 0$.*

(2) *Assume that L is ramified and $r \geq 1$. Then either $d \geq 0$ and r is even or $d = -1/2$ and r is odd. The principal germ $P(w,s)$ only depends on (r,d) and equals*

$$P(L,r,d,s) := \begin{cases} \sum_{k=0}^{r-1} \Xi_k(r/2-1, r/2) X^{-k-1} & \textit{if } d \geq 0 \\ \sum_{k=0}^{r-1} \Xi_k(r/2+d, r/2-d-1) X^{-k-1} & \textit{if } d = -1/2. \end{cases} \tag{7.28}$$

(3) *Assume that $L \cong F \times F$ and $r \geq 1$. If $d \geq 0$, then r is even. The principal germ $P(w,s)$ only depends on (r,d) and equals*

$$P(F \times F, r, d, s) := 2 \cdot \begin{cases} \sum_{k=0}^{r-1} \Xi_k(r/2-1, r/2) X^{-k-1} & \textit{if } d \geq 0 \\ \sum_{k=0}^{r-1} \Xi'_k(r/2+d, r/2-d-1) X^{-k-1} & \textit{if } d < 0. \end{cases} \tag{7.29}$$

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Proof. Recall first the definition of the principal germ from (7.6):

$$P(w,s) = \sum_{(O_L,\Lambda_0^\flat,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w),\ \mathrm{cond}(\Lambda_0^\flat)=0} X^{-[O_L:\Lambda_1]}. \tag{7.30}$$

The task is thus to count the set of lattice diagrams of the form $(O_L,\Lambda_0^\flat,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w)$ with $\mathrm{cond}(\Lambda_0^\flat)=0$ and $[O_L:\Lambda_1]=k$. This counting problem was the content of Lemma 7.8, and it is only left to evaluate this lemma in dependence on (L,r,d).

If L is an unramified field extension, then $P(w,s)=0$ for the trivial reason that there are no lattices $\Lambda_0^\flat\subset O_L$ of index 1 and conductor 0. If $r\le 0$, then w is not topologically nilpotent, which implies $\mathcal{L}(w)=\emptyset$ and hence $P(w,s)=0$. This proves Part (1). We also note that the case distinctions for (r,d) in Parts (2) and (3) were already stated in Lemma 7.6, so it only left to prove (7.28) and (7.29).

Consider first the case of a ramified extension L and of $r\ge 1$. Let $\varpi\in L$ denote a uniformizer. Then $\Lambda_0^\flat=\varpi O_L$ is the unique sublattice of O_L of index 1 and conductor 0. Define $0\le a\le b$ by

$$(a,b)=(\varpi O_L:wO_L)=(\pi O_L:w\varpi O_L).$$

If $d\ge 0$, then r is even and $(a,b)=(r/2-1,r/2)$. Lemma 7.8 states that there are $\Xi_{k-1}(r/2-1,r/2)$ many choices $(\Lambda_1,\Lambda_1^\flat)$ such that $(O_L,\varpi O_L,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w)$ and such that $[O_L:\Lambda_1]=k$. Specializing (7.30) to this case, we precisely obtain the first identity in (7.28).

We use the same arguments for $d=-1/2$. In this case, r is odd and $a=b=(r-1)/2$. Lemma 7.8 states that there are $\Xi_{k-1}((r-1)/2,(r-1)/2)$ many tuples $(O_L,\varpi O_L,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w)$ with $[O_L:\Lambda_1]=k$, and we obtain the second identity in (7.28). This completes the proof of Part (2).

Consider now the case of a split extension $L\cong F\times F$ and of $r\ge 1$. Write $w=(w_1,w_2)$ with $v(w_1)\le v(w_2)$ for a fixed choice of such an isomorphism. There are two sublattices of O_L of index 1 and conductor 0 – namely, $M_1=(\pi,1)O_L$ and $M_2=(1,\pi)O_L$.

Assume first that $d\ge 0$. Then $v(w_1)=v(w_2)=r/2$ and

$$(M_i:wO_L)=(\pi O_L:wM_i)=(r/2-1,r/2)$$

for both possibilities $i\in\{1,2\}$. Lemma 7.8 states that there are $\Xi_{k-1}(r/2-1,r/2)$ many tuples $(O_L,\varpi O_L,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w)$ with $[O_L:\Lambda_1]=k$, and one obtains the first identity in (7.29) in the same way as before.

Assume now that $d<0$ and put $a:=v(w_1)$ as well as $b:=v(w_2)$. Then $(a,b)=(r/2+d,r/2-d)$, and one easily checks the identities

$$\begin{aligned}(M_1:wO_L)&=(a-1,b), & (\pi O_L:wM_1)&=(a,b-1)\\ (M_2:wO_L)&=(a,b-1), & (\pi O_L:wM_2)&=(a-1,b).\end{aligned}$$

It always holds that $a<b$. Applying the second and third identity in (7.16), we find that the number of tuples $(O_L,M_i,\Lambda_1,\Lambda_1^\flat)\in\mathcal{L}(w)$ with $[O_L:\Lambda_1]=k$ is given by $\Xi'_{k-1}(a,b-1)$ for both $i=1$ and 2. The second identity in (7.29) follows directly from (7.30) which finishes the proof of Part (3) and of the proposition. □

7.4. The parahoric case

The exact same ideas can be used to define a germ expansion for the parahoric orbital integral $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s)$ and to give a formula for the principal germ. Throughout, $w\in GL_2(F)$ is an element such that $\gamma(w)=\left(\begin{smallmatrix}1&1\\w&1\end{smallmatrix}\right)$ is regular semi-simple. Let (L,r,d) be its numerical triple, see (7.2). Define $\mathcal{P}(w)$ as the set of pairs (Λ_0,Λ_1) of O_F-lattices in L such that $O_L\cdot\Lambda_0=O_L$ and such that

$$\Lambda_0\supset\pi\Lambda_0\supseteq\Lambda_1\supseteq w\Lambda_0. \tag{7.31}$$

The set $\mathcal{P}(w)$ takes the role of $\mathcal{L}(w)$, but for the parahoric test function f'_{Par}:

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Lemma 7.10. *Given* $(\Lambda_0, \Lambda_1) \in \mathcal{P}(w)$*, the following lattice chain lies in* $\mathcal{P}(\gamma(w))$*:*

$$\Lambda_0 \oplus \Lambda_1 \supset \pi\Lambda_0 \oplus \Lambda_1 \supset \pi(\Lambda_0 \oplus \Lambda_1).$$

This assignment defines a bijection

$$\psi : \mathcal{P}(w) \xrightarrow{\sim} \Gamma\backslash\mathcal{P}(\gamma(w)) \tag{7.32}$$

with the property

$$\Omega(\gamma(w),\ \psi(\Lambda_0, \Lambda_1),\ s) = (-q^s)^{-r(w)}(-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.33}$$

In particular, the parahoric orbital integral has the expression

$$\mathrm{Orb}(\gamma(w), f'_{\mathrm{Par}}, s) \ = \ (-q^s)^{-r(w)} \sum_{(\Lambda_0,\Lambda_1)\in\mathcal{P}(w)} (-q^{2s})^{[\Lambda_0:\Lambda_1]}. \tag{7.34}$$

We again decompose the sum in (7.34) into principal and unipotent germ:

$$\begin{aligned} P_{\mathrm{Par}}(w,s) \ &= \sum_{(O_L,\Lambda_1)\in\mathcal{P}(w)} (-q^{2s})^{[\Lambda_0:\Lambda_1]}, \\ U_{\mathrm{Par}}(w,s) \ &= i(L)^{-1} \sum_{(\Lambda_0,\Lambda_1)\in\mathcal{P}(w),\ \Lambda_0\neq O_L} (-q^{2s})^{[\Lambda_0:\Lambda_1]}. \end{aligned} \tag{7.35}$$

The relation of the two germs with the orbital integral is again given by

$$O(\gamma(w), f'_{\mathrm{Par}}, s) = (-q^s)^{-r(w)}[P_{\mathrm{Par}}(w,s) + i(L)U_{\mathrm{Par}}(w,s)]. \tag{7.36}$$

Proposition 7.11. *Both the principal germ* $P_{\mathrm{Par}}(w,s)$ *and the unipotent germ* $U_{\mathrm{Par}}(w,s)$ *depend only on* (r,d) *and not on* L*.*

Proof. Let $R_c = O_F + \pi^c O_L$ again denote the order of conductor c in L. By Lemma 7.7, the relative position $(R_c : wR_c)$ only depends on (r,d). Moreover, for every $c \geq 1$, the number

$$i(L)^{-1}\#\{\Lambda_0 \subseteq O_L \mid \mathrm{cond}(\Lambda_0) = c,\ O_L \cdot \Lambda_0 = O_L\}$$

equals q^{c-1} and is hence independent of L. Combining these facts with the definition of $\mathcal{P}(w)$ and (7.35) proves the proposition. □

Proposition 7.12. *For integers* $0 \leq a \leq b$ *and* $0 \leq k \leq a+b$*, let* $\Phi_k(a,b)$ *denote the quantity from (7.22); set* $X = -q^{-2s}$*.*

(1) *If* $r \leq 0$*, then* $\mathrm{Orb}(\gamma(w), f'_{\mathrm{Par}}, s) = 0$*.*
(2) *If* $r > 0$*, then* $\mathrm{Orb}(\gamma(w), f'_{\mathrm{Par}}, s)$ *only depends on* (r,d) *and equals*

$$P_{\mathrm{Par}}(r,d,s) := \begin{cases} \sum_{k=0}^{r-2} \Phi_k(r/2-1, r/2-1)X^{-k-2} & \text{if } d \geq 0 \\ \sum_{k=0}^{r-2} \Phi_k(r/2+d-1, r/2-d-1)X^{-k-2} & \text{if } d < 0. \end{cases} \tag{7.37}$$

Proof. The vanishing statement in (1) holds because $\mathcal{P}(w) = \emptyset$ if $r \leq 0$. For (2), we note that no matter what L is, the relative position of O_L and wO_L is given by

$$(O_L : wO_L) = \begin{cases} (r/2, r/2) & \text{if } d \geq 0 \\ (r/2+d, r/2-d) & \text{if } d < 0. \end{cases}$$

It follows from the definition of $\Phi_k(a,b)$ in (7.19) and that of $\mathcal{P}(w)$ in (7.31) that the number of pairs $(O_L,\Lambda_1)\in\mathcal{P}(w)$ such that $[O_L:\Lambda_1]=k$ equals

$$\begin{cases}\Phi_{k-2}(r/2-1,r/2-1) & \text{if } d\ge 0\\ \Phi_{k-2}(r/2+d-1,r/2-d-1) & \text{if } d<0.\end{cases}$$

Substituting these quantities in (7.35) proves the proposition. □

8. $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})$, $\mathrm{Orb}(\gamma,f'_{\mathrm{Iw}})$ and $\partial\mathrm{Orb}(\gamma,f'_{\mathrm{Iw}})$

We have shown in Propositions 7.5 and 7.11 that the principal and unipotent germs for both f'_{Par} and f'_{Iw} only depend on the triple (L,r,d) resp. (r,d). Accordingly, we will write $P(L,r,d,s)$ for the Iwahori principal germ for such a numerical triple. We will similarly write $U(r,d,s)$ for the Iwahori unipotent germ as well as $P_{\mathrm{Par}}(r,d,s)$ and $U_{\mathrm{Par}}(r,d,s)$ for the parahoric germs.

8.1. The central values

Proposition 8.1. *Let $\gamma\in G'_{\mathrm{rs}}$ be regular semi-simple with numerical invariants (L,r,d); see (5.5).*

(1) *If r is odd, or if $r\le 0$, or if $r/2+d\le 0$, then* $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})=0$.

(2) *In all other cases, the parahoric orbital integral* $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})$ *is given by*

$$\begin{cases} 1+q^2+\ldots+q^{r/2-2} & \textit{if } L \textit{ ramified and } r\in 4\mathbb{Z}\\ (1+q^2+\ldots+q^{r/2-3})+\ (q^{r/2-1}+q^{r/2}+\ldots+q^{r/2+d-1}) & \textit{if } L \textit{ ramified and } r\in 2+4\mathbb{Z}\\ 2(1+q^2+\ldots+q^{r/2-2}) & \textit{if } L \textit{ inert and } r\in 4\mathbb{Z}\\ 2(1+q^2+\ldots+q^{r/2-3})+2(q^{r/2-1}+q^{r/2}+\ldots+q^{r/2+d-2})+q^{r/2+d-1} & \textit{if } L \textit{ inert and } r\in 2+4\mathbb{Z}\\ 0 & \textit{if } L \textit{ split and } r\in 4\mathbb{Z}\\ q^{r/2+d-1} & \textit{if } L \textit{ split and } r\in 2+4\mathbb{Z}.\end{cases} \tag{8.1}$$

Proof. The sign of the functional equation of $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s)$ is $(-1)^r$; see Lemma 3.16 and Proposition 3.19. So $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})=0$ whenever r is odd. Assume that $r\le 0$ or $r/2+d\le 0$. Then z_γ is not topologically nilpotent, and hence, $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}},s)=0$ by Lemma 3.21 (3). (If $r/2+d\le 0$ and $r>0$, then necessarily $d<0$, which implies that one eigenvalue of z_γ has valuation $r/2+d$; see Lemma 7.6 cases (3) and (4).) So we can henceforth assume that r is even, that $r>0$ and that $r/2+d>0$.

First, we consider the case of a split extension L. Let $\alpha,\beta\in F$ be the two eigenvalues of z_γ. Note that $v(\alpha)$ and $v(\beta)$ are both positive and of the same parity under our assumptions on (r,d). If their parity is even, which is equivalent to $r\in 4\mathbb{Z}$, then $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})=0$ by Identity (6.21). If the parity of $v(\alpha)$ and $v(\beta)$ is odd instead, then $v(\alpha-\beta)=r/2+d$ and (6.21) shows $\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})=q^{r/2+d-1}$. This proves (8.1) in case L is split.

Assume from now on that L is a field. With the standing assumption that r is even, it will necessarily hold that $d\ge 0$. Moreover, the germ expansion identity (7.36) specializes to

$$\mathrm{Orb}(\gamma,f'_{\mathrm{Par}})=P_{\mathrm{Par}}(r,d,0)+i(L)U_{\mathrm{Par}}(r,d,0). \tag{8.2}$$

We can now use our knowledge of the hyperbolic orbital integrals to compute the unipotent germ. Let $\widetilde{\gamma}\in G'_{\mathrm{rs}}$ be an auxiliary element with numerical invariants $(F\times F,r,d)$. The value at $s=0$, equivalently at $X=-1$, of the parahoric principal germ (7.37) for $d\ge 0$ is given by the geometric series

$$P_{\mathrm{Par}}(r,d,0)=1-q+q^2-\ldots+(-q)^{r/2-1}. \tag{8.3}$$

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Substituting (8.3) in the germ expansion (8.2) for $\widetilde{\gamma}$ shows that the unipotent germ is either a geometric series or a sum of two such series:

$$U_{\mathrm{Par}}(r,d,0) = \begin{cases} 1+q^2+\ldots+q^{r/2-2} & \text{if } r\in 4\mathbb{Z} \\ (1+q^2+\ldots+q^{r/2-3})+(q^{r/2-1}+q^{r/2}+\ldots+q^{r/2+d-2}) & \text{if } r\in 2+4\mathbb{Z}. \end{cases} \tag{8.4}$$

It is left to substitute (8.3) and (8.4) in (8.2) with $i(L)=q$ for L ramified and $i(L)=q+1$ for L inert. This proves the proposition in the remaining four cases. □

Theorem 8.2. *The central value of the Iwahori orbital integral is given by*

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \begin{cases} 1 & \textit{if } L \textit{ ramified and } r\geq 1 \textit{ odd} \\ 0 & \textit{otherwise.} \end{cases} \tag{8.5}$$

In particular, the fundamental lemma (Conjecture 3.10) holds in case D is a division algebra of degree 4.

Proof. We first compute the right-hand side of the FL Identity (3.16). Let $g\in G_{\mathrm{rs}}$ be regular semi-simple. Proposition 2.6 states that D contains the F-algebra $E\otimes_F L_g$ which hence has to be a field. It follows that L_g is a ramified field extension of F. Then Proposition 3.13 states that

$$\mathrm{Orb}(g, f_D) = \begin{cases} 1 & \text{if } v_D(g)\in 2\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $v_D: D^\times \to \mathbb{Z}$ denotes the normalized valuation of D. The condition $v_D(g)\in 2\mathbb{Z}$ holds if and only if $v_D(z_g)\geq 1$. Moreover, $v_D(z_g)$ is always odd, and thus, $r=v_F(N_{L_g/F}(z_g^2))\in 2\mathbb{Z}+1$. In this way, the FL for f'_{Iw} becomes Identity (8.5).

We turn to the computation of $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. The sign of the functional equation of $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ is $(-1)^{r+1}$; see (5.6) and Proposition 3.19. It follows that $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})=0$ whenever r is even. Moreover, Proposition 6.6 in particular implies that $(X+1)$ divides $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ if L is split, and hence that $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})=0$ in all such cases. Since r is always even when L is an unramified field extension, the only remaining possibility for $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$ to be nonzero is when L is ramified and r odd. In this case $d=-1/2$ by Lemma 7.6 (3). If $r\leq 0$, then (8.5) holds for the trivial reason that $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)=0$ by Lemma 3.21 (3). Thus, it is left to consider the case L ramified, $r>0$ odd, $d=-1/2$.

Our first aim is to determine $U(r,-1/2,0)$. To this end, we evaluate the principal germ for $F\times F$ from (7.29) at $s=0$, equivalently at $X=-1$:

$$P(F\times F, r, -1/2, 0) = -2[1-2q+2q^2+\ldots+2(-q)^{(r-3)/2}+(-q)^{(r-1)/2}]. \tag{8.6}$$

Using the vanishing of $\mathrm{Orb}(\,\cdot\,, f'_{\mathrm{Iw}})$ in all hyperbolic cases (see above), we obtain from the germ expansion (7.8) that

$$\begin{aligned} U(r,-1/2,0) &= (1-q)^{-1}P(F\times F, r, -1/2, 0) \\ &= -2[1-q+q^2-\ldots+(-q)^{(r-3)/2}]. \end{aligned} \tag{8.7}$$

Let L be a ramified field extension. The principal germ for the case $(L,r,-1/2)$, given by (7.28), specializes to

$$P(L,r,-1/2,0) = -[1-2q+2q^2-\ldots+2(-q)^{(r-1)/2}]. \tag{8.8}$$

Let $\gamma\in G'_{\mathrm{rs}}$ have numerical invariants (L,r,d). We substitute (8.7) and (8.8), with $i(L)=q$ and r odd in the germ expansion (7.8) for $(L,r,1/2)$ and obtain

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = -[P(L,r,-1/2,0)+q\,U(r,-1/2,0)] = 1.$$

This is precisely Identity (8.5), and the proof of the theorem is complete. □

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8.2. The central derivative

Proposition 8.3. *Let $\gamma \in G'_{\mathrm{rs}}$ be regular semi-simple with numerical invariants (L, r, d). Assume first that r is odd, meaning that the sign $\varepsilon_D(\gamma)$ of the functional equation of $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s)$ is positive. Then*

$$\partial\mathrm{Orb}(\gamma, f'_D) = 0 \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) \log(q). \tag{8.9}$$

Assume now that r is even, which implies $\partial\mathrm{Orb}(\gamma, f'_D) = \partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$. If $r \leq 0$, then $\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = 0$. If $r > 0$, it is given by

$$\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = 4q\log(q)\ \mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) + \log(q)\begin{cases} r & \text{if } L \text{ ramified} \\ 2r & \text{if } L \text{ inert} \\ 0 & \text{if } L \text{ split.}\end{cases} \tag{8.10}$$

Proof. Identity (8.9) follows immediately from the functional equation (Proposition 3.19): If its sign $\varepsilon_D(\gamma)$ is positive, then $\mathrm{Orb}(\gamma, f'_D, s)$ has an even functional equation, so $\partial\mathrm{Orb}(\gamma, f'_D) = 0$. Applying (5.3), we also have the functional equation

$$\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, -s) = q^{-2s}\varepsilon_D(\gamma)\,\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s).$$

Taking the derivative of both sides at $s = 0$ and assuming $\varepsilon_D(\gamma) = 1$ gives the other identity in (8.9).

Moreover, if $r \leq 0$, then z_γ is not topologically nilpotent, so $\mathcal{L}(\gamma) = \emptyset$, and hence, $\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}, s) = 0$ by Lemma 3.21. From now on we assume that r is even and that $r > 0$. In particular, $\varepsilon_D(\gamma) = -1$ and hence $\mathrm{Orb}(\gamma, f'_D) = 0$. Then (5.3) shows that $\partial\mathrm{Orb}(\gamma, f'_D) = \partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}})$, as claimed in the proposition. We now come to the main part of the proof.

Consider first the case that $L \cong F \times F$ is split. The factor $(X + 1)$ in (6.22) has the property that $(X + 1)|_{s=0} = 0$ and $(d/ds)_{s=0}(X + 1) = 2\log(q)$. Thus, the derivative of (6.22) at $s = 0$ is given by

$$\partial\mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = 4q\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) \log(q), \tag{8.11}$$

which proves (8.10) when L is split.

Our next aim is to compute the central derivatives

$$\partial P(L, r, d) = \frac{d}{ds}\bigg|_{s=0} P(L, r, d, s) \quad \text{and} \quad \partial U(r, d) = \frac{d}{ds}\bigg|_{s=0} U(r, d, s).$$

We are ultimately interested in the case of a field extension L, and here, r even implies $d \geq 0$. So we only compute ∂P and ∂U with this restriction. Directly from (7.29), we find

$$\partial P(F \times F, r, d) = 4[r/2 - (r-2)q + (r-4)q^2 - \ldots + 2(-q)^{r/2-1}]\log(q). \tag{8.12}$$

Let $\widetilde{\gamma}$ be an auxiliary hyperbolic element with numerical invariants (r, d). We obtain from the germ expansion (7.8) and our previous result (8.11) that

$$\partial U(r, d) = (q-1)^{-1}[4q\,\mathrm{Orb}(\widetilde{\gamma}, f'_{\mathrm{Par}})\log(q) - \partial P(F \times F, r, d)]. \tag{8.13}$$

The orbital integral $\mathrm{Orb}(\widetilde{\gamma}, f'_{\mathrm{Par}})$ here is either 0 or $q^{r/2+d-1}$ which depends on whether $r \in 4\mathbb{Z}$ or $r \in 2 + 4\mathbb{Z}$; see Proposition 8.1. Substituting this in (8.13) yields

$$\partial U(r, d) = \begin{cases} 4\left[\frac{r}{2} - (\frac{r}{2} - 2)q + (\frac{r}{2} - 2)q^2 - \ldots + 2(-q)^{r/2-3} + 2(-q)^{r/2-2}\right]\log(q) \\ 4\left[\frac{r}{2} - (\frac{r}{2} - 2)q + (\frac{r}{2} - 2)q^2 - \ldots + 3(-q)^{r/2-4} + 3(-q)^{r/2-3} \right. \\ \qquad\qquad\qquad\qquad \left. - q^{r/2-2} + q^{r/2-1} + \ldots + q^{r/2+d-1}\right]\log(q). \end{cases} \tag{8.14}$$

Here, the first line occurs if $r \in 4\mathbb{Z}$ and the second if $r \in 2+4\mathbb{Z}$. It is left to evaluate the expression

$$\partial \mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \partial P(L, r, d) + i(L)\partial U(r, d).$$

If L is ramified, then $i(L) = q$ and

$$P(L, r, d, s) = P(F \times F, r, d, s)/2$$

by Proposition 7.9. Thus, we may reuse (8.12), and we obtain

$$\partial \mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = \log(q)\begin{cases} r + 4q + 4q^3 + \ldots + 4q^{r/2-1} \\ r + 4q + 4q^3 + \ldots + 4q^{r/2-1} + 4q^{r/2} + \ldots + 4q^{r/2+d}, \end{cases} \tag{8.15}$$

where the first case is for $r \in 4\mathbb{Z}$ and the second for $r \in 2+4\mathbb{Z}$.

Consider now the case where L is an unramified field extension of F. Then $i(L) = q+1$ and $P(L, r, d, s) = 0$ by Proposition 7.9, so simply $\partial \mathrm{Orb}(\gamma, f'_{\mathrm{Iw}}) = (q+1)\partial U(r, d)$. This equals

$$\log(q)\begin{cases} 2r + 8q + 8q^3 + \ldots + 8q^{r/2-1} \\ 2r + 8q + 8q^3 + \ldots + 8q^{r/2} + 8q^{r/2+1} + \ldots + 8q^{r/2+d-1} + 4q^{r/2+d-1}. \end{cases} \tag{8.16}$$

Here, again, the first case is for $r \in 4\mathbb{Z}$ and the second for $r \in 2+4\mathbb{Z}$.

Comparing (8.15) and (8.16) with (8.1) shows (8.10), and the proof of the proposition is complete. □

Part III
Intersection numbers on $\mathcal{M}_{1/4}$ and $\mathcal{M}_{3/4}$

In this third part, we establish AT for the two division algebras of Hasse invariant $1/4$ and $3/4$. This is the main result of our paper, and we formulate it upfront; cf. Theorem 9.1. The proof will be completed in §12.4 for invariant $1/4$ and in §13.7 for invariant $3/4$.

The layout is as follows. After formulating the result in §9, there will be two short sections that equally concern both Hasse invariants. The first (§10) provides a formula for intersection numbers of regular surfaces in regular 4-space. The second (§11) provides a description of certain multiplicity functions on the Bruhat–Tits tree of $PGL_{2,E}$. These will later describe the multiplicities of the 1-dimensional components in the intersection loci $\mathcal{I}(g)$.

Subsequently, we will first complete the proof of AT for invariant $1/4$ in §12. The key point here is that Drinfeld's theorem [10] provides an explicit linear algebra description of $\mathcal{M}_D$ for $D = D_{1/4}$. So the proofs in §12 will, in fact, not involve any π-divisible groups.

In §13, we will use deformation-theoretic arguments to extend the results from Hasse invariant $1/4$ to invariant $3/4$.

9. Main results

The notation will be the same as in §4. We assume, however, that $n = 2$ and that D is a division algebra of Hasse invariant $\lambda \in \{1/4, 3/4\}$. The centralizer $C = \mathrm{Cent}_D(E)$ is then a quaternion division algebra over E. We also denote by $B = D_{1/2}$ a quaternion division algebra over F. Recall that $O_D \subset D$ denotes a maximal order such that $O_C = C \cap O_D$ is again maximal.

The set $B(H, \mu_H)$ has a single element $[b]$ in this situation (Example 4.5 (3)). The corresponding C-isocrystal $\mathbf{N}_{b,+}$ is of height 8, dimension 2 and isoclinic of slope $1/4$. We choose framing objects: Let $(\mathbb{Y}, \iota)$ denote a special O_C-module over $\mathrm{Spec}\,\mathbb{F}$ and put $(\mathbb{X}, \kappa) = O_D \otimes_{O_C} (\mathbb{Y}, \iota)$. We identify the isocrystal of $(\mathbb{Y}, \iota)$ with $\mathbf{N}_{b,+}$. In particular, we view C_b and D_b as the groups of quasi-automorphisms of $(\mathbb{Y}, \iota)$ and $(\mathbb{X}, \kappa)$. Then

$$C_b \cong M_2(E) \quad \text{and} \quad D_b \cong \begin{cases} M_4(F) & \text{if } \lambda = 1/4 \\ M_2(B) & \text{if } \lambda = 3/4. \end{cases} \tag{9.1}$$

As before, we put $H_b = C_b^\times$ and $G_b = D_b^\times$. These act from the right on the moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_D$ whose definitions we briefly recall. First, $\mathcal{M}_C$ is the formal scheme over Spf $O_{\breve F}$ with functor of points

$$\mathcal{M}_C(S) = \left\{ (Y, \iota, \rho) \middle| \begin{array}{c} (Y,\iota)/S \text{ a special } O_C\text{-module} \\ \rho : \bar S \times_{\operatorname{Spec} \mathbb{F}} \mathbb{Y} \to \bar S \times_S Y \text{ an } O_C\text{-linear quasi-isogeny} \end{array} \right\}. \tag{9.2}$$

This is the (base chang to $O_{\breve F}$) of the Drinfeld half-plane for O_E. It is a two-dimensional, regular, π-adic formal scheme whose description will be recalled in §12.1 below. Second, $\mathcal{M}_D$ is the formal scheme over Spf $O_{\breve F}$ with functor of points

$$\mathcal{M}_D(S) = \left\{ (X, \kappa, \rho) \middle| \begin{array}{c} (X,\kappa)/S \text{ a strict } O_D\text{-module} \\ \rho : \bar S \times_{\operatorname{Spec} \mathbb{F}} \mathbb{X} \to \bar S \times_S X \text{ an } O_D\text{-linear quasi-isogeny} \end{array} \right\}. \tag{9.3}$$

If $\lambda = 1/4$, then $\mathcal{M}_D$ is Drinfeld's 4-space and, in particular, a π-adic formal scheme. Its description will also be given in §12.1. If $\lambda = 3/4$, however, then there is no known explicit description of $\mathcal{M}_D$.

For every regular semi-simple $g \in G_{b,\mathrm{rs}}$, we have defined the intersection locus $\mathcal{I}(g) = \mathcal{M}_C \cap g\mathcal{M}_C$ and an intersection number $\operatorname{Int}(g) \in \mathbb{Z}$ in §4.4. We formulate our main result:

Theorem 9.1. *The AT conjecture holds for D. More precisely, let f'_{corr} be given by*

$$f'_{\mathrm{corr}} = \begin{cases} -4q\log(q) \cdot f'_{\mathrm{Par}} & \text{if } \lambda = 1/4 \\ 0 & \text{if } \lambda = 3/4. \end{cases} \tag{9.4}$$

Then, for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,

$$\partial \operatorname{Orb}(\gamma, f'_D) + \operatorname{Orb}(\gamma, f'_{\mathrm{corr}}) = \begin{cases} 2\operatorname{Int}(g)\log(q) & \text{if there exists a } g \in G_b \text{ that matches } \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{9.5}$$

This theorem will be proved as Theorems 12.13 and 13.23 at the end of sections §12 and §13. We mention here that our proof of Theorem 9.1 is not fully complete when F is of equal characteristic and $\lambda = 3/4$. The reason is that §13.6 relies on the O_F-display theory of [1], which was only developed for p-adic local fields. Completing the proof requires a duplication of §13.6 but for local shtuka. We will not carry out these arguments in order to keep the article at a reasonable length.

10. Surface intersections

The aim of this section is to derive a general formula for intersection numbers of regular surfaces in a regular 4-dimensional space. Let first Y be a regular formal scheme, pure of dimension 2, and let $Z = V(\mathcal{I}) \subseteq Y$ be a closed formal subscheme of dimension ≤ 1.

Definition 10.1. (1) Let $Z^{\mathrm{pure}} \subseteq Z$ be the maximal effective Cartier divisor on Y that is contained in Z. More precisely, we define $Z^{\mathrm{pure}} = V(\mathcal{I}^{\mathrm{pure}})$, where for every affine open $U = \operatorname{Spf} A$ of Y, say $U \cap Z = \operatorname{Spf} A/I$,

$$\mathcal{I}^{\mathrm{pure}}(U) = \left\{ f \in A \middle| \begin{array}{c} f = 0 \text{ in } (A/I)_\eta \text{ for all generic points } \eta \text{ of} \\ \text{1-dimensional irreducible components of } \operatorname{Spec} A/I \end{array} \right\}.$$

(2) Set $Z^{\mathrm{art}} = V(\mathcal{I}^{\mathrm{art}})$ with $\mathcal{I}^{\mathrm{art}} = \{f \in \mathcal{O}_X \mid f\mathcal{I}^{\mathrm{pure}} \subseteq \mathcal{I}\}$. Note that $\mathcal{O}_{Z^{\mathrm{art}}}$ is isomorphic to the quotient $\mathcal{I}^{\mathrm{pure}}/\mathcal{I}$.

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Let X be a regular formal scheme, pure of dimension 4, and let $Y_1, Y_2 \subseteq X$ be regular closed formal subschemes, both pure of dimension 2. Then Y_1 and Y_2 are locally defined by a regular sequence of length 2 in $\mathcal{O}_X$. Put $Z = Y_1 \cap Y_2$ and assume $\dim Z \leq 1$. The conormal bundle of Y_i in X is $\mathcal{C}_i = (\mathcal{I}_i/\mathcal{I}_i^2)|_{Y_i}$, where $Y_i = V(\mathcal{I}_i)$. Let $D = Z^{\mathrm{pure}}$ be the purely 1-dimensional locus of Z as in Definition 10.1; it is independent of whether it is defined with respect to $Z \subseteq Y_1$ or $Z \subseteq Y_2$.

Proposition 10.2. *The following identities hold for the Tor-terms* $T_i := \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})$.

(1) $T_0 = \mathcal{O}_Z$.
(2) $T_1 = (\det \mathcal{C}_1)|_D \otimes_{\mathcal{O}_D} \mathcal{O}_{Y_2}(D)|_D$.
(3) $T_2 = 0$.

Proof. The claim on T_0 is immediate. To prove the statements about the higher Tor-terms, we first assume that $Y_1 = V(f_1, f_2)$ is the vanishing locus of two elements and that $D = V(g)$ for some $g \in \mathcal{O}_{Y_2}$. Then the Koszul complex

$$K_{(f_1,f_2)} := \left[\mathcal{O}_X \xrightarrow{\binom{f_2}{-f_1}} \mathcal{O}_X^{\oplus 2} \xrightarrow{(f_1,f_2)} \mathcal{O}_X \right]$$

is quasi-isomorphic to $\mathcal{O}_{Y_1}$ and

$$T_i = H^{-i}(\mathcal{O}_{Y_2} \otimes_{\mathcal{O}_X} K_{(f_1,f_2)}).$$

Let $\overline{f}_i$ denote the image of f_i in $\mathcal{O}_{Y_2}$. Then

$$\mathcal{O}_{Y_2} \otimes_{\mathcal{O}_X} K_{(f_1,f_2)} = \left[\mathcal{O}_{Y_2} \xrightarrow{\binom{\overline{f}_2}{-\overline{f}_1}} \mathcal{O}_{Y_2}^{\oplus 2} \xrightarrow{(\overline{f}_1,\overline{f}_2)} \mathcal{O}_{Y_2} \right].$$

Set $f_i' = g^{-1}\overline{f}_i \in \mathcal{O}_{Y_2}$. Then (f_1', f_2') forms a regular sequence in $\mathcal{O}_{Y_2}$ because $V(f_1', f_2')$ is artinian by definition of D. In particular,

$$T_2 = \ker\binom{\overline{f}_2}{-\overline{f}_1} = \ker\binom{f_2'}{-f_1'} = 0.$$

Moreover,

$$T_1 = H^{-1}(\mathcal{O}_{Y_2} \otimes_{\mathcal{O}_X} K_{(f_1,f_2)}) = \mathcal{O}_{Y_2} \cdot \binom{f_2'}{-f_1'} \Big/ \mathcal{O}_{Y_2} \cdot \binom{\overline{f}_2}{-\overline{f}_1}.$$

is a line bundle on D, and our choices provide the specific generator

$$c_{f_1,f_2,g} := \binom{f_2'}{-f_1'} \bmod \mathcal{O}_{Y_2} \cdot \binom{\overline{f}_2}{-\overline{f}_1}.$$

Now we turn to the general situation. The given local argument already implies that $T_2 = 0$. We claim that the above construction glues to a map (and hence isomorphism)

$$\begin{aligned} (\det \mathcal{C}_1)|_D \otimes_{\mathcal{O}_D} \mathcal{O}_{Y_2}(D)|_D &\longrightarrow \quad T_1 \\ \left(\overline{f}_1 \wedge \overline{f}_2\right) \otimes g^{-1} &\longmapsto \quad c_{f_1,f_2,g}. \end{aligned} \tag{10.1}$$

(Here, f_1, f_2 and g denote any local generators as before.) It is clear that if g is replaced by ug with $u \in \mathcal{O}_{Y_2}^{\times}$, then f_i' gets replaced by $u^{-1}f_i'$. We find that $c_{f_1,f_2,ug} = u^{-1}c_{f_1,f_2,g}$ which shows the independence of (10.1) from the chosen trivialization $g^{-1} \in \mathcal{O}_{Y_2}(D)$.

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Now assume $(h_1, h_2) = (f_1, f_2)A$ for some $A \in GL_2(\mathcal{O}_X)$. Then A defines an isomorphism of complexes

$$\begin{tikzcd}
\mathcal{O}_{Y_2} \arrow[r, "\begin{pmatrix} \bar{f}_2 \\ -\bar{f}_1 \end{pmatrix}"] & \mathcal{O}_{Y_2}^{\oplus 2} \arrow[r, "(\bar{f}_1, \bar{f}_2)"] & \mathcal{O}_{Y_2} \\
\mathcal{O}_{Y_2} \arrow[r, "\begin{pmatrix} \bar{h}_2 \\ -\bar{h}_1 \end{pmatrix}"] \arrow[u, "\det A"] & \mathcal{O}_{Y_2}^{\oplus 2} \arrow[r, "(\bar{h}_1, \bar{h}_2)"] \arrow[u, "A"] & \mathcal{O}_{Y_2} \arrow[u, equal]
\end{tikzcd} \tag{10.2}$$

and one easily checks the relation

$$A \cdot c_{h_1, h_2, g} = \det A \cdot c_{f_1, f_2, g}.$$

Since $h_1 \wedge h_2 = \det A \cdot f_1 \wedge f_2$, this precisely says that (10.1) is also independent of the choice of trivialization of $\mathcal{C}_1$. □

Assume now that X is a Spf W-scheme of locally formally finite type where W is a complete DVR and that $Z \longrightarrow \operatorname{Spec} W$ is a proper scheme with empty generic fiber. Then we define the intersection number of Y_1 and Y_2 as

$$\langle Y_1, Y_2 \rangle_X = \chi(\mathcal{O}_{Y_1} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{Y_2}) \in \mathbb{Z}.$$

As a corollary to Proposition 10.2, this has the following more concrete description.

Corollary 10.3. *With all notation as before,*

$$\begin{aligned}
\langle Y_1, Y_2 \rangle_X &= \chi(\mathcal{O}_{Z^{\mathrm{art}}}) + \chi(\mathcal{O}_D) - \chi\big(\det \mathcal{C}_1|_D \otimes_{\mathcal{O}_D} \mathcal{O}_{Y_2}(D)|_D\big) \\
&= \operatorname{len}(\mathcal{O}_{Z^{\mathrm{art}}}) - \deg(\det \mathcal{C}_1|_D) - \langle D, D \rangle_{Y_2}.
\end{aligned}$$

Proof. The first equality is Proposition 10.2. To obtain the second, we applied the Riemann–Roch identity

$$\chi(\mathcal{O}_D) - \chi(\mathcal{L}) = -\deg(\mathcal{L}), \quad \mathcal{L} \in \operatorname{Pic}(D),$$

and rewrote $\deg \mathcal{O}_{Y_2}(D)|_D$ as the self-intersection number of D on Y_2. We refer to [40, Tag 0AYQ] for the notion of degree in this possibly non-reduced context. □

11. Multiplicity functions

Let W be a 2-dimensional E-vector space and let $\mathcal{B}$ denote the Bruhat–Tits building of the projective linear group $PGL_E(W)$. Recall that $\mathcal{B}$ is a (q^2+1)-regular tree whose vertices are the homothety classes of O_E-lattices in W. Two vertices are connected by an edge if and only if the two homothety classes have representatives Λ_0 and Λ_1 with $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$.

Let $z \in GL_F(W)$ be an E-conjugate linear endomorphism. The aim of this section is to give a precise description of the shape of the function

$$n(z, -) : \operatorname{Vert}(\mathcal{B}) \longrightarrow \mathbb{Z}, \quad \Lambda \longmapsto \max\{k \in \mathbb{Z} \mid z\Lambda \subseteq \pi^k \Lambda\}. \tag{11.1}$$

More precisely, we give a description for all z such that $1 + z$ lies in $GL_F(W)$ and is regular semi-simple with respect to $E \subseteq \operatorname{End}_F(W)$. We begin with a simple classification lemma over the residue field whose proof we omit.

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Lemma 11.1. *Denote by σ the Galois conjugation of $\mathbb{F}_{q^2}/\mathbb{F}_q$. Let $\bar{\Lambda}$ be a 2-dimensional $\mathbb{F}_{q^2}$-vector space and let $0 \neq \bar{z} \in \mathrm{End}_{\mathbb{F}_q}(\bar{\Lambda})$ be a σ-linear endomorphism. Precisely one of the following six statements applies to $\bar{z}$:*

(1) *It is nilpotent (i.e. $\bar{z}^2 = 0$). In this case, there is a unique line $\ell \subset \bar{\Lambda}$ such that $\bar{z}\ell \subseteq \ell$ – namely, $\ell = \bar{z}\bar{\Lambda}$. In a suitable basis, we have*

$$\bar{z} = \begin{pmatrix} & 1 \\ & \end{pmatrix} \sigma.$$

(2) *It is neither invertible nor nilpotent. Then there are precisely two lines $\ell_1, \ell_2 \subseteq \bar{\Lambda}$ such that $\bar{z}\ell_i \subset \ell_i$, namely $\bar{z}\bar{\Lambda}$ and* $\ker(\bar{z})$. *In a suitable basis, we have*

$$\bar{z} = \begin{pmatrix} \lambda & \\ & \end{pmatrix} \sigma$$

for some scalar $0 \neq \lambda \in \mathbb{F}_{q^2}$.

(3) *It is invertible, and there is precisely one line $\ell \subset \bar{\Lambda}$ with $\bar{z}\ell = \ell$. Then there are $\lambda, \mu \in \mathbb{F}_{q^2}^\times$ and a basis of $\bar{\Lambda}$ such that $\lambda\mu^q + \lambda^q\mu \neq 0$ and such that $\bar{z}$ is given by*

$$\bar{z} = \begin{pmatrix} \lambda & \mu \\ & \lambda \end{pmatrix} \sigma.$$

(4) *It is invertible, and there are precisely two lines $\ell_1, \ell_2 \subset W$ with $\bar{z}\ell_i = \ell_i$. Let $0 \neq v_i \in \ell_i$ be any two vectors and define λ_i by $\bar{z}v_i = \lambda_i v$. Then $\lambda_1^{q+1} \neq \lambda_2^{q+1}$ and $\bar{z}$ is given in that basis by*

$$\bar{z} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \sigma.$$

(5) *It is invertible, and there are precisely $q + 1$ lines $\ell \subset V$ with $\bar{z}\ell = \ell$. In a suitable basis and for a suitable scalar $0 \neq \lambda \in \mathbb{F}_{q^2}$, we have*

$$\bar{z} = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \sigma.$$

(6) *It is invertible and there is no $\bar{z}$-stable line.*

We return to O_E-lattices in W and the function $n(-,-)$.

Lemma 11.2. *The function $n(-,-)$ enjoys the following properties.*

(1) $n(\pi z, \Lambda) = n(z, \Lambda) + 1$
(2) $|n(z, \Lambda) - n(z, \Lambda')| \leq 1$ *whenever Λ and Λ' are neighbours in $\mathcal{B}$.*
(3) *The function $n(z, -)$ is bounded above by $v(\det_E(z^2))/4$ and, in particular, takes a maximum.*
(4) *Let $\Lambda'' \in [\Lambda, \Lambda']$ be a lattice on the unique shortest path connecting Λ and Λ' in $\mathcal{B}$. Then*

$$n(z, \Lambda'') \geq \min\{n(z, \Lambda),\ n(z, \Lambda')\}.$$

Proof. The first three claims follow directly from the definition. For the last one, choose Λ and Λ' in their homothety classes such that $\Lambda' \subseteq \Lambda$ with Λ/Λ' cyclic. Then Λ'' is the homothety class of one of the lattices $\Lambda' + \pi^i\Lambda$, $i \geq 0$, and the claim follows from the definition of $n(-,-)$. □

We next analyze the local properties of $n(z, -)$ in conjunction with Lemma 11.1. Given a lattice Λ, we obtain a nonzero σ-linear endomorphism $\bar{z}_\Lambda := (\pi^{-n(z,\Lambda)}z \bmod \pi\Lambda)$ in $\mathrm{End}_{\mathbb{F}_q}(\bar{\Lambda})$, where $\bar{\Lambda} = \Lambda/\pi\Lambda$. Let $\ell = \Lambda/\Lambda' \subseteq \bar{\Lambda}$ be the line corresponding to the neighbor lattice $\pi\Lambda \subset \Lambda' \subset \Lambda$.

Lemma 11.3. *The following cases occur:*

$$n(z,\Lambda') = \begin{cases} n(z,\Lambda)-1 & \text{if } \bar{z}_\Lambda(\ell) \not\subseteq \ell \\ n(z,\Lambda)+1 & \text{if } \bar{z}_\Lambda(\ell) \subseteq \ell, \text{ if } \bar{z}_\Lambda \text{ falls into case (1) of Lemma 11.1,} \\ & \qquad\text{and if } v(\det_E(\pi^{-2n(z,\Lambda)}z^2)) \geq 4 \\ n(z,\Lambda) & \text{if } \bar{z}_\Lambda(\ell) \subseteq \ell, \text{ but } \bar{z}_\Lambda \text{ does not satisfy the previous further two conditions.} \end{cases}$$

Proof. By the scaling invariance from Lemma 11.2 (1), it suffices to consider the case $n(z,\Lambda) = 0$. We also put $\bar{z} = \bar{z}_\Lambda$. It is clear that $n(z,\Lambda') \geq 0$ if and only if $\bar{z}\ell \subseteq \ell$. If $v_F(\det_F(z)) = 0$, then necessarily also $n(z,\Lambda') = 0$. This happens if and only if $\bar{z}$ does not fall into cases (1) and (2) of Lemma 11.1. In the remaining two cases, we may find an $\mathbb{F}_{q^2}$-basis (e_1, e_2) of $\bar{\Lambda}$ such that

$$\bar{z} = \begin{pmatrix} & 1 \\ & \end{pmatrix}\sigma \quad \text{or} \quad \bar{z} = \begin{pmatrix} \lambda & \\ & \end{pmatrix}\sigma, \ \ \lambda \neq 0, \tag{11.2}$$

and $\ell = \mathbb{F}_{q^2}e_1$ in the first case or $\ell \in \{\mathbb{F}_{q^2}e_1, \ \mathbb{F}_{q^2}e_2\}$ in the second. Lifting (e_1, e_2) to an O_E-basis of Λ, we obtain a matrix presentation

$$z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma$$

that reduces modulo $\pi\Lambda$ to (11.2). Depending on whether $\ell = \mathbb{F}_{q^2}e_1$ or $\ell = \mathbb{F}_{q^2}e_2$, we obtain that $z|_{\Lambda'}$ has a matrix presentation as

$$z = \begin{pmatrix} a & \pi b \\ \pi^{-1}c & d \end{pmatrix}\sigma \quad \text{or} \quad y = \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix}\sigma. \tag{11.3}$$

Then $n(z,\Lambda') = n(z,\Lambda)+1$ occurs if and only if all four entries in (11.3) have valuation ≥ 1. This never happens in the second case of (11.2) because here $v(a) = 0$. In the first case of (11.2), we find that $n(z,\Lambda') = n(z,\Lambda)+1$ if and only if $v(\pi^{-1}c) \geq 1$, which is equivalent to the stated condition $v(\det_E(z^2)) \geq 4$. □

Definition 11.4. Let $\mathcal{T}(z)$ denote the set of homothety classes of O_E-lattices in which $n(z,-)$ takes its maximum.

Property (4) of Lemma 11.2 shows that $\mathcal{T}(z)$ is connected. Lemmas 11.1 and 11.3 imply that each of its vertices has valency 0, 1, 2 or $q+1$.

Proposition 11.5. *Denote by $d(\Lambda, \mathcal{T}(z))$ the distance of Λ from $\mathcal{T}(z)$. Then*

$$n(z,\Lambda) = \max n(z,-) - d(\Lambda, \mathcal{T}(z)).$$

Proof. The claim is tautologically true for $\Lambda \in \mathcal{T}(z)$. For Λ with $d(\Lambda, \mathcal{T}(z)) = 1$, it follows from Statement (2) of Lemma 11.2.

Now assume $d(\Lambda, \mathcal{T}(z)) \geq 2$. Let Λ' denote the unique neighbor of Λ on the shortest path toward $\mathcal{T}(z)$. By induction on $d(-, \mathcal{T}(z))$, we find that Λ' has a neighbor Λ'' with $n(z,\Lambda'') = n(z,\Lambda')+1$ – namely, the subsequent lattice on the shortest path towards $\mathcal{T}(z)$. This implies by Lemma 11.3 that $\bar{z}_{\Lambda'}$ falls into Case (1) of Lemma 11.1, and hence that $n(z,\Lambda) = n(z,\Lambda')-1$, as claimed. □

This reduces us to describe $\mathcal{T}(z)$. We assume from now on that $1+z$ is regular semi-simple in the sense of §2. By definition, this means that $\mathrm{Inv}(1+z;T) = \mathrm{char}_E(z^2;T)$ is a separable polynomial with

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$\mathrm{Inv}(1+z;0)\mathrm{Inv}(1+z;1) \neq 0$. The description of $\mathcal{T}(z)$ will be in terms of the numerical invariant (L, r, d) of $1+z$ from Definition 5.1:

$$L = F[z^2], \quad r = v(N_{L/F}(z^2)), \quad d = \mathrm{cond}(O_F[\pi^k z^2]) - k - r/2, \ \ k \gg 0.$$

Note that L is an étale quadratic extension of F.

Lemma 11.6. *The maximum of $n(z, -)$ is given by*

$$\max\{k \in \mathbb{Z} \mid (\pi^{-k}z)^2 \in O_L\} = \begin{cases} \lfloor r/4 \rfloor & \text{if } d \geq 0 \\ \lfloor r/4 + d/2 \rfloor & \text{if } d < 0. \end{cases}$$

Proof. Considering all multiples $\pi^{\mathbb{Z}}z$, the claim is equivalent to the following statement: There exists a lattice Λ with $z\Lambda \subseteq \Lambda$ if and only if $z^2 \in O_L$.

The 'only if' direction is clear because $z\Lambda \subseteq \Lambda$ implies that z^2 has an integral characteristic polynomial. To prove the 'if' direction, observe that W is a free module of rank 1 over $E \otimes_F L$ by Proposition 2.6 (2). Pick any lattice $\Lambda' \subseteq W$ that is stable under $O_E \otimes_{O_F} O_L$. If $z^2 \in O_L$, then $\Lambda = \Lambda' + z\Lambda'$ is preserved by z. □

Since $\mathcal{T}(\pi^k z) = \mathcal{T}(z)$ for all $k \in \mathbb{Z}$, it suffices to describe $\mathcal{T}(z)$ whenever $z^2 \in O_L \setminus \pi^2 O_L$, and in this case, $\mathcal{T}(z) = \{\Lambda \mid z\Lambda \subseteq \Lambda\}$. We first treat the case of units.

Proposition 11.7. *Assume that $z^2 \in O_L^\times$. Then there exists an L-linear, E-conjugate linear involution τ on W that commutes with z such that*

$$O_E[z] = O_E[\tau, z^2].$$

In particular, $\mathcal{T}(z)$ is the set of O_E-scalar extensions of z^2-stable O_F-lattices in $W^{\tau=\mathrm{id}}$.

Proof. Let $R = O_F[z^2]$ and denote by $\mathfrak{m}$ its Jacobson radical. The norm map

$$N_{E/F} : \big(O_E \otimes_{O_F} (R/\mathfrak{m})\big)^\times \longrightarrow (R/\mathfrak{m})^\times$$

is surjective because O_E/O_F is étale. It equals the map on $(R/\mathfrak{m})$-points of the smooth morphism $N_{E/F} : \mathrm{Res}_{O_E/O_F}\mathbb{G}_m \to \mathbb{G}_m$ of smooth O_F-group schemes. (The norm morphism is smooth because O_E/O_F is étale.) Using completeness and a deformation argument, it follows that the map

$$N_{E/F} : \big(O_E \otimes_{O_F} R\big)^\times \longrightarrow R^\times$$

is surjective. Hence, there exists an element $t \in (O_E \otimes_{O_F} R)^\times$ with $N_{E/F}(t) = z^2$. Then $\tau := t^{-1}z$ lies in $O_E[z]$ and satisfies $\tau^2 = \mathrm{id}$. The identity $O_E[z] = O_E[\tau, z^2]$ follows directly. □

Proposition 11.8. *Assume that $z^2 \in O_L \setminus \pi^2 O_L$ is not a unit. Then $\mathcal{T}(z)$ takes the following shape.*

(1) *If L is a field extension, then $\mathcal{T}(z)$ consists of a single edge.*
(2) *If $L \cong F \times F$, then $\mathcal{T}(z)$ consists of an apartment.*

Proof. Let Λ be any lattice with $z\Lambda \subseteq \Lambda$, existence being ensured by Proposition 11.5. Then $\bar{z}_\Lambda$ has to fall into Case (1) or (2) of the local classification Lemma 11.1. Case (1) occurs precisely if z is topologically nilpotent, which under the assumption $z^2 \in O_L \setminus (\pi^2 O_L \cup O_L^\times)$ is equivalent to L being a field. (If L is a field and $z^2 \in O_L \setminus O_L^\times$, then z^2 is topologically nilpotent. Conversely, assume $L = F \times F$ and write $z^2 = (x, y)$. The quaternion algebra $(E \otimes L)[z]$ embeds into $\mathrm{End}_F(W)$ because this is the current setting, and so is isomorphic to $M_2(L)$. Thus, z^2 lies in the image of the norm map $E \otimes_F L \to L$, which means $v(x), v(y) \in 2\mathbb{Z}$. Hence, $z^2 \notin \pi^2 O_L$ implies that z^2 is not topologically nilpotent.)

Consider first Case (1). Then Λ has precisely one neighbor in $\mathcal{T}(z)$, say Λ'. Then $\bar{z}_{\Lambda'}$ is again of Case (1) because the property of L being a field (or z being topologically nilpotent) is independent of the lattice. Thus, Λ' also has a unique neighbor in $\mathcal{T}(z)$ and hence $\mathcal{T}(z) = \{\Lambda, \Lambda'\}$, as claimed.

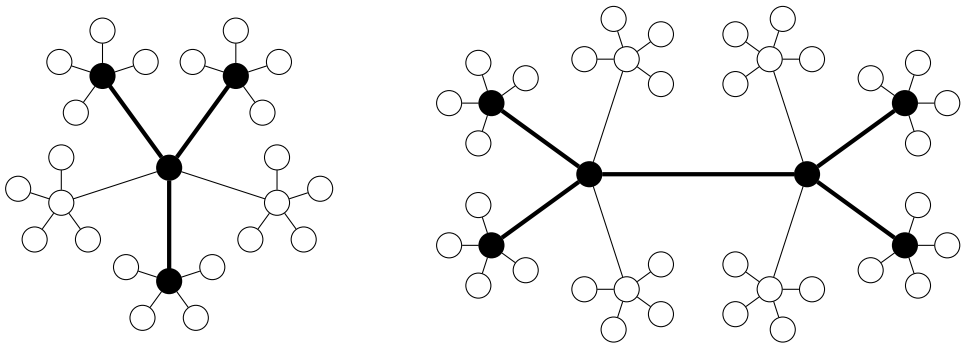

Figure 1. *Left: Case (1) of Theorem 11.10 for $d = 1$ and $q = 2$. The set $\mathcal{T}(z)$ consists of a single vertex of valency $q + 1$ and $q + 1$ vertices of valency 1 (black vertices). The ambient $(q^2 + 1)$-regular tree $\mathcal{B}$ is sketched (white vertices). Right: Similar sketch for case (3) of Theorem 11.10 for $d = 1$ and $q = 2$.*

Consider now Case (2). Then $\ell_1 = \bigcap_{i\geq 0} z^i\Lambda$ and

$$\ell_2 = \{\lambda \in \Lambda | z^i\lambda \to 0 \text{ as } i \to \infty\}$$

are complementary z-stable direct summand O_E-modules of Λ of rank 1. Picking nonzero $e_i \in \ell_i$, we see that every lattice $\pi^a O_E e_1 \oplus \pi^b O_E e_2$ is stable under z. These provide all elements of $\mathcal{T}(z)$ because any lattice in $\mathcal{T}(z)$ has exactly two neighbors in $\mathcal{T}(z)$ by the local classification Lemma 11.1 (2). □

Remark 11.9. We observe that not all triples (L, r, d) may occur. Namely, the cyclic L-algebra $L[E, z]$ has center L and embeds into $\mathrm{End}_F(W)$, and so has to be isomorphic to $M_2(L)$. It follows that $z^2 \in L$ is always a norm from $E \otimes_F L$:

(1) If L is ramified, then this means that $r = v_L(z^2) \in 2\mathbb{Z}$. In particular, it will always be the case that $d \geq 0$ by Lemma 7.6 (3).

(2) If $L = F \times F$ is split with, say, $z^2 = (z_1, z_2)$, then $v(z_1), v(z_2) \in 2\mathbb{Z}$. In particular, $r = v(z_1) + v(z_2) \in 4\mathbb{Z}$.

(3) If L is inert, then there is no such restriction on z^2.

These possibilities are the ones that lead to rows 1 and 3 in Table 1 (take $\delta = \mathrm{Inv}(1 + z; T)$, which equals $\mathrm{char}_{L/F}(z; T)$ and gives $B_\delta \cong L[E, z]$).

Theorem 11.10. *The set $\mathcal{T}(z)$ takes the following shape, depending on the numercial invariant (L, r, d) of z:*

(1) *If L is inert and $r \equiv 0$ mod 4, then $\mathcal{T}(z)$ is a $(q + 1)$-regular ball of radius d around a vertex.*

(2) *If L is inert and $r \equiv 2$ mod 4, then $\mathcal{T}(z)$ is an edge.*

(3) *If L is ramified and $r \equiv 0$ mod 4, then $\mathcal{T}(z)$ is a $(q + 1)$-regular ball of radius d around an edge.*

(4) *If L is ramified and $r \equiv 2$ mod 4, then $\mathcal{T}(z)$ is an edge.*

(5) *If $L \cong F \times F$, and if $z^2 = (z_1, z_2)$ has the property $v_F(z_1) = v_F(z_2)$, then $\mathcal{T}(z)$ is a $(q+1)$-regular ball of radius d around an apartment.*

(6) *If $L \cong F \times F$, and if $z^2 = (z_1, z_2)$ has the property $v_F(z_1) \neq v_F(z_2)$, then $\mathcal{T}(z)$ is an apartment.*

Proof. First consider the two cases when L is a field and $r \equiv 0$ mod 4. Then $(\pi^{-r/4}z)^2 \in O_L^\times$ and Proposition 11.7 states that $\mathcal{T}(z)$ is the set of homothety classes of $\pi^{-r/2}z^2$-stable O_F-lattices in a 1-dimensional L-vector space. This set is well known to be a $(q + 1)$-regular ball around a vertex (resp. a ball around an edge) of radius equal to the conductor of $O_F[\pi^{-r/2}z^2]$. This conductor equals d so (1) and (3) are proven.

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Next, stick with the case that L is a field but assume $r \equiv 2 \bmod 4$. Then $(\pi^{(2-r)/4}z)^2$ is not a unit but lies in $O_L \setminus \pi^2 O_L$. In this situation, $\mathcal{T}(z)$ is edge by Proposition 11.8 (1). This proves (2) and (4).

Finally, assume $L \cong F \times F$ and define $k \in \mathbb{Z}$ through $(\pi^k z)^2 \in O_L \setminus \pi^2 O_L$. Write $z^2 = (z_1, z_2)$ as in the proposition. If $v_F(z_1) = v_F(z_2)$, then $(\pi^k z)^2 \in O_L^\times$ and Proposition 11.7 states that $\mathcal{T}(z)$ is the set of homothety classes of $(\pi^k z)^2$-stable O_F-lattices in a free L-module of rank 1. This set is well known to be a $(q+1)$-regular ball around an apartment of radius equal to the conductor d of $O_F[\pi^{-r/2}z^2]$, as claimed. This settles (5).

If, however, $v_F(z_1) \neq v_F(z_2)$, then $(\pi^k z)^2 \notin O_L^\times$ and Proposition 11.8 (2) states that $\mathcal{T}(z)$ is an apartment, which proves (6). □

12. Invariant 1/4

The aim of this section is to prove Theorem 9.1 for $\lambda = 1/4$. To this end, we first recall Drinfeld's linear algebra description of the map $\mathcal{M}_C \to \mathcal{M}_D$ in §12.1 and §12.2. We will subsequently use this to compute all intersection numbers in question, our final result being the simple formulas in Theorem 12.13.

12.1. Drinfeld's theorem

Let, for a moment, D be a CDA over F of Hasse invariant $1/n$. The main result of Drinfeld's paper [10] states that each connected component of the RZ space $\mathcal{M}_D$ from Definition 4.12 is isomorphic to Deligne's formal scheme $\breve{\Omega}_F^{n-1}$. We will now formulate this result in more detail. Our main reference is [38, §3.54], to which we also refer for more background.

Let W be an n-dimensional F-vector space. By lattice chain in W, we mean a nonempty set $\mathcal{L}$ of O_F-lattices in W that satisfies the following two conditions:

(1) $\Lambda, \Lambda' \in \mathcal{L}$ implies $\Lambda \subseteq \Lambda'$ or $\Lambda' \subseteq \Lambda$.

(2) $\Lambda \in \mathcal{L}$ implies $\pi^{\mathbb{Z}}\Lambda \subseteq \mathcal{L}$.

Denote the set of lattice chains in W by $\mathcal{W}$. Given $\mathcal{L} \in \mathcal{W}$ and any lattice $\Lambda_0 \in \mathcal{L}$, we may consider all lattices of $\mathcal{L}$ that are contained in Λ_0 and contain $\pi\Lambda_0$ – say these are

$$\pi\Lambda_0 \subset \Lambda_k \subset \ldots \subset \Lambda_1 \subset \Lambda_0. \tag{12.1}$$

Then $\mathcal{L} = \{\pi^{\mathbb{Z}}\Lambda_i,\ i = 0, \ldots, k\}$, so we call $\Lambda_k \subset \ldots \subset \Lambda_0$ a representing chain for $\mathcal{L}$. We next define a π-adic affine formal scheme $U_{\mathcal{L}}$ over $\operatorname{Spf} O_F$. Choosing a practical approach, we give the less canonical definition in terms of a representing chain (12.1). With this convention, the points $U_{\mathcal{L}}(S)$ for a $\operatorname{Spf} O_F$-scheme S are the commutative diagrams of line bundle quotients

$$\begin{tikzcd} \mathcal{O}_S \otimes_{O_F} \Lambda_0 \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, two heads, "\varphi_0"] & \mathcal{O}_S \otimes_{O_F} \Lambda_k \arrow[r] \arrow[d, two heads, "\varphi_k"] & \ldots \arrow[r] & \mathcal{O}_S \otimes_{O_F} \Lambda_1 \arrow[r] \arrow[d, two heads, "\varphi_1"] & \mathcal{O}_S \otimes_{O_F} \Lambda_0 \arrow[d, two heads, "\varphi_0"] \\ \mathcal{L}_0 \arrow[r, "\alpha_k"] & \mathcal{L}_k \arrow[r, "\alpha_{k-1}"] & \ldots \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0 \end{tikzcd} \tag{12.2}$$

up to isomorphism in the pairs $(\mathcal{L}_i, \alpha_i)$, and such that the following condition holds: The section $\varphi_i(\lambda_i)$ is invertible whenever $\lambda_i \in \Lambda_i \setminus \Lambda_{i+1}$ (for $i = 0, 1, \ldots, k-1$) resp. $\lambda_k \in \Lambda_k \setminus \pi\Lambda_0$ (for $i = k$). A diagram of the form (12.2) may be extended in a natural way to the full chain $\mathcal{L}$, and in this way, the definition becomes independent of the chosen representing chain. The resulting $U_{\mathcal{L}}$ is isomorphic to a principal open subset of $\operatorname{Spf} O_F\langle T_0, \ldots, T_{n-1}\rangle/(T_0 \cdots T_{n-1} - \pi)$. In particular, it is π-adic, n-dimensional, and regular with semi-stable reduction over $\operatorname{Spf} O_F$.

There are open immersions $U_{\mathcal{L}'} \subseteq U_{\mathcal{L}}$ for all inclusions of lattice chains $\mathcal{L}' \subseteq \mathcal{L}$. Their definition is based on the following simple observation. Assume that $\Lambda_2 \subset \Lambda_1 \subset \Lambda_0$ are lattices and that we are

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given $(\mathcal{L}_2, \varphi_2)$, $(\mathcal{L}_0, \varphi_0)$ and $\alpha_0 \circ \alpha_1$ in the following diagram:

$$\begin{tikzcd} \mathcal{O}_S \otimes_{O_F} \Lambda_2 \arrow[r] \arrow[d, "\varphi_2"] & \mathcal{O}_S \otimes_{O_F} \Lambda_1 \arrow[r] \arrow[d, dashed, "\varphi_1"] & \mathcal{O}_S \otimes_{O_F} \Lambda_0 \arrow[d, "\varphi_0"] \\ \mathcal{L}_2 \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0. \end{tikzcd} \tag{12.3}$$

Assume further that the outer square commutes and that $\varphi_0(\lambda)$ is invertible for all $\lambda \in \Lambda_0 \setminus \Lambda_2$. Then there is a unique way (up to isomorphism) to fill in $(\mathcal{L}_1, \varphi_1)$ and to factor $\alpha_0 \circ \alpha_1$ as depicted. Namely, let $\lambda \in \Lambda_1 \setminus \Lambda_2$. Then $\alpha_0(\varphi_1(\lambda)) = \varphi_0(\lambda)$ has to be invertible, so α_0 has to be an isomorphism. Thus, we may put $\mathcal{L}_1 = \mathcal{L}_0$, $\alpha_0 = \mathrm{id}$ and $\varphi_1 = \varphi_0|_{\mathcal{O}_S \otimes \Lambda_1}$. We leave it to the reader to extend this construction to lattice chains and diagrams as in (12.2).

Assume that $\Lambda_k \subset \ldots \subset \Lambda_0$ represents $\mathcal{L}$ as above and that $\mathcal{L}' \subseteq \mathcal{L}$ is a subchain. Let $I \subseteq \{0, \ldots, k\}$ be such that $\Lambda_i \in \mathcal{L}'$ if and only if $i \in I$. The above-constructed map $U_{\mathcal{L}'} \to U_{\mathcal{L}}$ identifies $U_{\mathcal{L}'}$ with the subfunctor of all those diagrams (12.2) that have the property that α_{i-1} is an isomorphism if $i \notin I$. The maps $U_{\mathcal{L}'} \to U_{\mathcal{L}}$ are hence open immersions. Uniqueness of the construction in (12.3) ensures that the family $(U_{\mathcal{L}'} \to U_{\mathcal{L}})_{\mathcal{L}' \subseteq \mathcal{L}}$ satisfies the cocycle condition.

Furthermore, every isomorphism $\varphi : W \to W'$ of F-vector spaces provides a compatible family of isomorphisms $\varphi : U_{\mathcal{L}} \xrightarrow{\sim} U_{\varphi(\mathcal{L})}$. In particular, an element $g \in GL_F(W)$ defines a compatible family of isomorphisms

$$g : U_{\mathcal{L}} \xrightarrow{\sim} U_{g\mathcal{L}}. \tag{12.4}$$

In the following, we write $GL_F(W)^0 = \{g \in GL_F(W) \mid v(\det(g)) = 0\}$. If, for example, $g \in GL_F(W)^0$ and $g\mathcal{L} = \mathcal{L}$, then this means that every lattice of $\mathcal{L}$ is g-stable. In this case, the g-action on $U_{\mathcal{L}}$ is the natural action of g on diagrams of the form (12.2).

Definition 12.1. Let $\Omega_F(W)$ denote the formal scheme that is obtained from the gluing datum $(U_{\mathcal{L}'} \to U_{\mathcal{L}})_{\mathcal{L}' \subseteq \mathcal{L}}$. Let $GL_F(W)$ act on $\Omega_F(W)$ by the action that is chart-wise given by (12.4). We also write $\Omega_F^{n-1} := \Omega_F(F^n)$ in the case $W = F^n$.

Let $\mathcal{M}_D^i \subset \mathcal{M}_D$ denote the open and closed formal subscheme of triples (X, κ, ρ) such that the height of ρ is i. Note that $(\mathbb{X}, \kappa, \mathrm{id}) \in \mathcal{M}_D(\mathbb{F})$, so $\mathcal{M}_D^0 \neq \emptyset$. Furthermore, an element $g \in G_b \cong GL_n(F)$ provides an isomorphism

$$g : \mathcal{M}_D^i \xrightarrow{\sim} \mathcal{M}_D^{i+4v_F(\det(g))}. \tag{12.5}$$

Finally, a simple Dieudonné module argument shows that $\mathcal{M}_D^i = \emptyset$ if $i \notin 4\mathbb{Z}$. In this way, the following result provides a complete description of $\mathcal{M}_D$.

Theorem 12.2 (Drinfeld [10]). *There is a $GL_n(F)^0$-equivariant isomorphism*

$$O_{\breve{F}} \widehat{\otimes}_{O_F} \Omega_F^{n-1} \xrightarrow{\cong} \mathcal{M}_D^0. \tag{12.6}$$

Here, we let G_b act from the left of $\mathcal{M}_D$ (instead of as from the right) by $g \mapsto g^{-1}$. In particular, M_D^0 is connected.

The special fiber $\mathbb{F}_q \otimes_{O_F} \Omega_F(W)$ is a reduced scheme. Its set of irreducible components is in bijection with the homothety classes of lattices $\Lambda \subset W$. The irreducible component associated to Λ is a blow-up of the projective spaces $\mathbb{P}(\bar{\Lambda})$ centered in the union of all $\mathbb{F}_q$-rational hyperplanes of $\mathbb{P}(\bar{\Lambda})$. In the case $n = 2$, since a blow-up does not affect smooth curves, the irreducible components of $\mathbb{F}_q \otimes_{O_F} \Omega_F^1$ are of the form $\mathbb{P}(\bar{\Lambda}) \cong \mathbb{P}^1_{\mathbb{F}_q}$.

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In light of (12.6), we will mostly be interested in the base change of $\Omega_F(W)$ to $O_{\breve{F}}$. For this reason, we introduce the notation

$$\breve{\Omega}_F(W) := O_{\breve{F}}\widehat{\otimes}_{O_F}\Omega_F(W), \quad \breve{U}_{\mathcal{L}} := O_{\breve{F}}\widehat{\otimes}_{O_F}U_{\mathcal{L}}. \tag{12.7}$$

12.2. The basic construction

We now specialize to the situation of a 2-dimensional E-vector space W. It is simultaneously a 4-dimensional F-vector space. If $\mathcal{L}$ is a chain of O_E-lattices in W, then we write $U_{E,\mathcal{L}} \subseteq \Omega_E(W)$ for the corresponding chart. We also put

$$\breve{\Omega}_E(W) := O_{\breve{F}}\widehat{\otimes}_{O_E}\Omega_E(W), \quad \breve{U}_{E,\mathcal{L}} := O_{\breve{F}}\widehat{\otimes}_{O_E}U_{E,\mathcal{L}}.$$

Let $\zeta \in O_E^\times$ be some fixed generator. It may be viewed as an element of $GL_F(W)^0$ and hence defines an automorphism of $\Omega_F(W)$. The isomorphism in Theorem 12.2 is $GL_F(W)^0$-equivariant, and so restricts to an isomorphism of ζ-fixed points

$$\mathcal{M}_D^{0,\zeta} \cong \breve{\Omega}_F(W)^\zeta.$$

By Proposition 4.15, $\mathcal{M}_C^0$ is contained in the fixed points $\mathcal{M}_D^{0,\zeta}$. Our aim is to describe its image in $\breve{\Omega}_F(W)^\zeta$.

Proposition 12.3. *Precisely two of the connected components of $\breve{\Omega}_F(W)^\zeta$ are flat over* Spf $O_{\breve{F}}$. *Each of these is isomorphic to $\breve{\Omega}_E(W)$. The image of $\mathcal{M}_C^0$ along (12.6) equals one of them.*

Proof. The fixed points $\Omega_F(W)^\zeta$ are contained in the union of the charts $U_{\mathcal{L}}$ for $\mathcal{L}$ that satisfy $\zeta\mathcal{L} = \mathcal{L}$. Since $v(\det(\zeta)) = 0$, the condition $\zeta\mathcal{L} = \mathcal{L}$ means that ζ fixes each lattices of $\mathcal{L}$ individually (i.e., that $\mathcal{L}$ is a chain of O_E-lattices). Given an O_E-lattice Λ, there is a natural decomposition

$$O_E \otimes_{O_F} \Lambda = \Lambda^+ \oplus \Lambda^- \tag{12.8}$$

because E/F is unramified. Here, the notation is such that Λ^+ (resp. Λ^-) is the set of elements on which the two O_E-actions coincide (resp. differ by Galois conjugation). For a Spf O_E-scheme S, a quotient line bundle

$$\mathcal{O}_S \otimes_{O_F} \Lambda \overset{\varphi}{\twoheadrightarrow} \mathcal{L} \tag{12.9}$$

is ζ-stable if and only if the quotient map factors over the projection to $\mathcal{O}_S \otimes_{O_E} \Lambda^+$ or over the projection to $\mathcal{O}_S \otimes_{O_E} \Lambda^-$.

Let $\mathcal{L} \in \mathcal{W}$ be a chain of O_E-lattices represented by $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$. Let S be a Spf O_E-scheme and consider a point of $U_{\mathcal{L}}^\zeta(S)$ represented by

$$\begin{tikzcd} \mathcal{O}_S \otimes_{O_E} (\Lambda_0^+ \oplus \Lambda_0^-) \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, two heads, "(\varphi_0^+,\varphi_0^-)"] & \mathcal{O}_S \otimes_{O_E} (\Lambda_1^+ \oplus \Lambda_1^-) \arrow[r] \arrow[d, two heads, "(\varphi_1^+,\varphi_1^-)"] & \mathcal{O}_S \otimes_{O_E} (\Lambda_0^+ \oplus \Lambda_0^-) \arrow[d, two heads, "(\varphi_0^+,\varphi_0^-)"] \\ \mathcal{L}_0 \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0. \end{tikzcd} \tag{12.10}$$

Then one can define a decomposition $S = S^+ \sqcup S^- \sqcup S^{\neq}$ into open and closed subschemes in the following way: S^+ is the locus where both φ_0^- and φ_1^- vanish. Similarly, S^- is the locus where both φ_0^+ and φ_1^+ vanish. Finally, $S^{\neq}$ is the complement. This decomposition is functorial and hence defines a decomposition

$$O_E\widehat{\otimes}_{O_F}U_{\mathcal{L}}^\zeta = U_{\mathcal{L}}^{\zeta,+} \sqcup U_{\mathcal{L}}^{\zeta,-} \sqcup U_{\mathcal{L}}^{\zeta,\neq}.$$

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It is furthermore compatible with gluing maps and stable under the $GL_E(W)$-action, and in particular defines a decomposition

$$O_E\widehat{\otimes}_{O_F}\Omega_F(W)^\zeta \;=\; \Omega_F(W)^{\zeta,+} \sqcup \Omega_F(W)^{\zeta,-} \sqcup \Omega_F(W)^{\zeta,\neq}.$$

The subscheme $\Omega_F(W)^{\zeta,\neq}$ lies above the special point $\operatorname{Spec}\mathbb{F}_{q^2} \subset \operatorname{Spf} O_E$ and is hence nowhere flat. Indeed, assume, for example, that $\varphi_0^- = 0$ and $\varphi_1^+ = 0$. Then φ_1^- is both a surjection onto a line bundle and $\pi\varphi_0^- = 0$ is divided by $\varphi_0^- = 0$. It follows that $\pi = 0$. The symmetric argument applies if $\varphi_0^+ = 0$ and $\varphi_1^- = 0$.

Recall from Theorem 12.2 that $\mathcal{M}_C^0$ is a flat and connected $O_{\breve{F}}$-scheme. We conclude that the proof of the proposition will be complete if we can show that the two formal schemes $\Omega_F(W)^{\zeta,\pm}$ are both isomorphic to $\Omega_E(W)$. To this end, first note that every E-conjugate linear element $\tau \in GL_F(W)$ defines an isomorphism

$$\tau : \Omega_F(W)^{\zeta,+} \xrightarrow{\cong} \Omega_F(W)^{\zeta,-}. \tag{12.11}$$

It hence suffices to describe an isomorphism of $\Omega_E(W)$ with $\Omega_F(W)^{\zeta,+}$, say. Let $\mathcal{L}$ be a chain of O_E-lattices that is represented by $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$. Let S be a $\operatorname{Spf} O_E$-scheme and consider an S-valued point of the chart $U_{E,\mathcal{L}}(S) \subset \Omega_E(W)(S)$ represented by

$$\begin{tikzcd}
\mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, "\varphi_0"] & \mathcal{O}_S \otimes_{O_E} \Lambda_1^+ \arrow[r] \arrow[d, "\varphi_1"] & \mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[d, "\varphi_0"] \\
\mathcal{L}_0 \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0.
\end{tikzcd} \tag{12.12}$$

Map this datum to the following point of $U_{\mathcal{L}}^{\zeta,+}(S)$:

$$\begin{tikzcd}
\mathcal{O}_S \otimes_{O_E} (\Lambda_0^+ \oplus \Lambda_0^-) \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, "(\varphi_0{,}0)"] & \mathcal{O}_S \otimes_{O_E} (\Lambda_1^+ \oplus \Lambda_1^-) \arrow[r] \arrow[d, "(\varphi_1{,}0)"] & \mathcal{O}_S \otimes_{O_E} (\Lambda_0^+ \oplus \Lambda_0^-) \arrow[d, "(\varphi_0{,}0)"] \\
\mathcal{L}_0 \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0.
\end{tikzcd} \tag{12.13}$$

It is not difficult to check that this definition is compatible with gluing maps and defines an $GL_E(W)$-equivariant isomorphism $\Omega_E(W) \xrightarrow{\cong} \Omega_F(W)^{\zeta,+}$; we omit these details. The proof of the proposition is now complete. □

Which of the two flat components of $\breve{\Omega}_F(W)^\zeta$ the cycle $\mathcal{M}_C^0$ gets identified with depends on the choice of the comparison isomorphism in Theorem 12.2. We do not need to be more precise about this identification, however, because the definitions of $\mathcal{I}(g)$ and $\operatorname{Int}(g)$ in §4.4 are purely in terms of spaces with group actions and because (12.11) allows to interchange the two flat components. So we will henceforth assume that the map $\mathcal{M}_C^0 \to \mathcal{M}_D^0$ is given by the morphism from (12.12) and (12.13).

Remark 12.4. In fact, this is also the result one would obtain from Drinfeld's construction during his proof of Theorem 12.2. Namely, his construction is such that the line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ in (12.13) occur as direct summands of the Lie algebra of the corresponding special O_D-module. Demanding that ζ acts strictly on the Lie algebra in the sense of Definition 4.8 precisely means to single out the component $\Omega_F(W)^{\zeta,+}$.

Remark 12.5. The map $\Omega_E(W) \to \Omega_F(W)$ from Proposition 12.3 was already considered by Drinfeld and called by him the 'basic construction'. His [10, Proposition 3.1 (1)] is similar to our Proposition 12.3. It seems, however, that the flatness condition in Proposition 12.3 cannot be omitted.

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12.3. Conormal bundle

Let $\Lambda = \Lambda_0 \subset W$ be an O_E-lattice. We write U_Λ instead of $U_{\pi^{\mathbb{Z}}\Lambda}$. A similar convention will apply to $U_{\Lambda_1\subset\Lambda_0}$, $U_{E,\Lambda}$ and $U_{E,\Lambda_0\subset\Lambda_1}$. The special fiber $\mathbb{F}\otimes_{O_{\breve{F}}}\breve{U}_{E,\Lambda}$ of $\breve{U}_{E,\Lambda}$ is $GL_{O_E}(\Lambda)$-equivariantly isomorphic to

$$\mathbb{F}\otimes_{O_E}\mathbb{P}(\Lambda)\setminus\mathbb{P}(\Lambda)(\mathbb{F}_{q^2}) \cong \mathbb{P}^1_{\mathbb{F}}\setminus\mathbb{P}^1(\mathbb{F}_{q^2}).$$

Let P_Λ denote its closure in $\breve{\Omega}_E(W)$. It is isomorphic to the projective line $\mathbb{F}\otimes_{O_E}\mathbb{P}(\Lambda)$.

Proposition 12.6. *Let $\mathcal{C}$ denote the conormal bundle of $\breve{\Omega}_E(W)\subset\breve{\Omega}_F(W)$. Then*

$$\deg(\det\mathcal{C}|_{P_\Lambda}) = q^2-1.$$

Proof. Our strategy is to choose a suitable generator of $(\det\mathcal{C})|_{\breve{U}_{E,\Lambda}}$ and to determine the divisor of its meromorphic extension to P_Λ.

(1) Fix an O_E-basis e_1, e_2 for Λ. Write $O_E\otimes_{O_F}\Lambda = \Lambda^+\oplus\Lambda^-$ as before. For an element $e\in\Lambda$, put

$$e^+ = \zeta\otimes e - 1\otimes\bar{\zeta}e \in \Lambda^+, \quad e^- = \zeta\otimes e - 1\otimes\zeta e\in\Lambda^-. \tag{12.14}$$

Then $(e_1^\pm, e_2^\pm)$ forms an O_E-basis of $\Lambda^\pm$. Let $\varphi : \mathcal{O}_{U_\Lambda}\otimes_{O_F}\Lambda\longrightarrow\mathcal{L}$ be the universal quotient. Using that we are working over $O_{\breve{F}}$ which contains O_E, write $\varphi = (\varphi^+, \varphi^-)$ as in (12.10). Comparing (12.10) with (12.13), we see that $\breve{U}_{E,\Lambda}\subset\breve{U}_\Lambda$ is defined by the condition $\varphi^- = 0$. Since e_1^-, e_2^- is a basis of Λ^-, this is the same as the two conditions $\varphi(e_1^-) = \varphi(e_2^-) = 0$.

For every $\lambda\in\Lambda\setminus\pi\Lambda$, the image $\varphi(\lambda)\in\mathcal{L}$ is a generator. In particular,

$$\varphi(e_i^+ - e_i^-) = \varphi((\zeta-\bar{\zeta})\otimes e_i)\neq 0.$$

Since $\varphi(e_i^-)$ vanishes along $\breve{U}_{E,\Lambda}$ as seen before, $\varphi(e_i^+)$ is invertible near $\breve{U}_{E,\Lambda}$. Thus, the two functions $\varphi(e_1^-)/\varphi(e_1^+)$ and $\varphi(e_2^-)/\varphi(e_2^+)$ are defined on a Zariski open neighborhood of $\breve{U}_{E,\Lambda}$ and generate the ideal defining $\breve{U}_{E,\Lambda}\subset\breve{U}_\Lambda$. Their wedge product

$$c_{(e_1,e_2)} := \frac{\varphi(e_1^-)}{\varphi(e_1^+)}\wedge\frac{\varphi(e_2^-)}{\varphi(e_2^+)}$$

is then a generator of $\det\mathcal{C}|_{\breve{U}_{E,\Lambda}}$.

(2) We next determine the behaviour of $c_{(e_1,e_2)}$ under change of basis. Let $f_1 = ae_1 + ce : 2$ and $f_2 = be_1 + de_2$ for some $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in GL_2(O_E)$. Then

$$\begin{aligned} c_{(f_1,f_2)} &= \frac{\varphi(e_1^+)\varphi(e_2^+)}{\varphi(f_1)^+\varphi(f_2)^+}\cdot\frac{\big(\bar{a}\varphi(e_1^-)+\bar{c}\varphi(e_2^-)\big)\wedge\big(\bar{b}\varphi(e_1^-)+\bar{d}\varphi(e_2^-)\big)}{\varphi(e_1^+)\varphi(e_2^+)}\\ &= \det(\bar{A})\,\frac{\varphi(e_1^+)\varphi(e_2^+)}{\varphi(f_1^+)\varphi(f_2^+)}\,c_{(e_1,e_2)}. \end{aligned}\tag{12.15}$$

Poles and zeroes of the proportionality factor (when restricted to P_Λ) are described as follows. For two elements $e, f\in\Lambda$, the ratio $\varphi(e^+)/\varphi(f^+)$ is a scalar if and only if $e^+\equiv f^+ \bmod \pi$. Otherwise, it is the rational function with simple zero at the line $\langle e^+\rangle$ and simple pole at $\langle f^+\rangle$.

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(3) Let $\Lambda_1 \subset \Lambda$ be the O_E-lattice generated by $\pi e_1, e_2$. We claim that $c_{(e_1,e_2)}$ extends to a generator of $\det \mathcal{C}|_{\breve{U}_{E,\Lambda_1\subset\Lambda}}$. Consider for this the universal point of $\breve{U}_{\Lambda_1\subset\Lambda}$, say

$$\begin{tikzcd} \mathcal{O}_{\breve{U}_{\Lambda_1\subset\Lambda}} \otimes_{O_F} \Lambda \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, "\varphi"] & \mathcal{O}_{\breve{U}_{\Lambda_1\subset\Lambda}} \otimes_{O_F} \Lambda_1 \arrow[r] \arrow[d, "\varphi_1"] & \mathcal{O}_{\breve{U}_{\Lambda_1\subset\Lambda}} \otimes_{O_F} \Lambda \arrow[d, "\varphi"] \\ \mathcal{L} \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}. \end{tikzcd} \tag{12.16}$$

Since $\varphi(\lambda)$ is a generator of $\mathcal{L}$ for every $\lambda \in \Lambda \setminus \Lambda_1$ and since similarly $\varphi_1(\lambda_1)$ is a generator of $\mathcal{L}_1$ for every $\lambda_1 \in \Lambda_1 \setminus \pi\Lambda_0$, the ideal defining $\breve{U}_{E,\Lambda_1\subset\Lambda} \subset \breve{U}_{\Lambda_1\subset\Lambda}$ is generated by

$$\frac{\varphi(e_1^-)}{\varphi(e_1^+)} \quad \text{and} \quad \frac{\varphi_1(e_2^-)}{\varphi_1(e_2^+)}$$

on a Zariski open neighborhood of $\breve{U}_{E,\Lambda_1\subset\Lambda}$. The map α_0 becomes an isomorphism when restricting the diagram (12.16) to the open subset $\breve{U}_\Lambda \subset \breve{U}_{\Lambda_1\subset\Lambda}$. Since also $\varphi = \alpha_0 \circ \varphi_1$, we see that

$$\left.\frac{\varphi(e_1^-)}{\varphi(e_1^+)} \wedge \frac{\varphi_1(e_2^-)}{\varphi_1(e_2^+)}\right|_{\breve{U}_{E,\Lambda_1\subset\Lambda}} = c_{(e_1,e_2)},$$

as claimed.

This argument applies symmetrically to the lattice $\langle e_1, \pi e_2\rangle \subset \Lambda$. So we have shown that the element $c_{(e_1,e_2)}$, which is a meromorphic section of the line bundle $\det \mathcal{C}|_{P_\Lambda}$, has neither a zero nor a pole at the points $\langle e_1\rangle, \langle e_2\rangle \in \mathbb{P}(\Lambda)(\mathbb{F}_{q^2})$.

(4) It is left to show that $c_{(e_1,e_2)}$ extends with a simple zero over all other $\mathbb{F}_{q^2}$-rational points $\langle e_1\rangle, \langle e_2\rangle \neq \langle f\rangle \in \mathbb{P}(\Lambda)(\mathbb{F}_{q^2})$. We know from Step (3) that $c_{(f,e_2)}$ is a generator of $\det \mathcal{C}|_{P_\Lambda}$ at $\langle f\rangle$. From Step (2), we have that

$$c_{(e_1,e_2)}/c_{(f,e_2)} \in O_E^\times \cdot \varphi(f^+)/\varphi(e_1^+).$$

Moreover, the function $\varphi(f^+)|_{P_\Lambda}$ vanishes with simple zero at $\langle f\rangle$ while $\varphi(e_1^+)$ is regular in $\langle f\rangle$ because $\langle f\rangle \neq \langle e_1\rangle$. Thus, we have proved that

$$\mathrm{div}(c_{(e_1,e_2)}) = P_\Lambda(\mathbb{F}_{q^2}) \setminus \{\langle e_1\rangle, \langle e_2\rangle\}$$

and obtain the claimed identity $\deg(\det \mathcal{C}|_{P_\Lambda}) = q^2 - 1$. □

12.4. Intersection numbers

Let $g = 1 + z_g \in GL_F(W)$ be a regular semi-simple element; set $z = z_g$. Recall that $\mathcal{I}(g) \neq \emptyset$ only for topologically nilpotent z (Proposition 4.23), so we also impose this condition on z. Then g lies in $GL_F(W)^0$. Let $(L = F[z^2], r, d)$ be the numerical invariant of z. Let $\breve{\Omega}_E(W) \to \breve{\Omega}_F(W)$ be the closed immersion defined by (12.12) and (12.13). Our aim is to determine the intersection locus

$$\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W).$$

Let $\Lambda \subseteq W$ be an O_E-lattice such that $z\Lambda \subseteq \Lambda$. Define $O_E \otimes_{O_F} \Lambda = \Lambda^+ \oplus \Lambda^-$ as in (12.8). Then z satisfies $z\Lambda^+ \subseteq \Lambda^-$ and $z\Lambda^- \subseteq \Lambda^+$ because it is E-conjugate linear.

Definition 12.7. Let $\mathcal{L}$ be a chain of O_E lattices in W, represented by a single lattice Λ or a pair $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$. We define $\breve{U}^z_{E,\mathcal{L}} \subseteq \breve{U}_{E,\mathcal{L}}$ as the closed formal subscheme of all those S-valued points

$$\begin{tikzcd} \mathcal{O}_S \otimes_{O_E} \Lambda^+ \arrow[d, "\varphi"] \\ \mathcal{L} \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} \mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[r, "\mathrm{id}\otimes\pi"] \arrow[d, "\varphi_0"] & \mathcal{O}_S \otimes_{O_E} \Lambda_1^+ \arrow[r] \arrow[d, "\varphi_1"] & \mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[d, "\varphi_0"] \\ \mathcal{L}_0 \arrow[r, "\alpha_1"] & \mathcal{L}_1 \arrow[r, "\alpha_0"] & \mathcal{L}_0 \end{tikzcd}$$

that satisfy $[\varphi \circ z : \Lambda^- \to \mathcal{L}] = 0$, resp. $[\varphi_i \circ z : \Lambda_i^- \to \mathcal{L}_i] = 0$ for both $i = 0, 1$.

Proposition 12.8. *Let $\mathcal{W}^g$ denote the set of g-stable chains of O_E-lattices. Then*

$$\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W) = \bigcup_{\mathcal{L} \in \mathcal{W}^g} \breve{U}^z_{E,\mathcal{L}}.$$

Proof. Consider a chart $\breve{U}_{E,\mathcal{L}} \subset \breve{\Omega}_E(W)$. Its image under g is contained in $\breve{U}_{g\mathcal{L}}$, which can only intersect $\breve{\Omega}_E(W)$ nontrivially if $g\mathcal{L}$ is again a chain of O_E-lattices. This is equivalent to $z\Lambda \subseteq \Lambda$ because z is topologically nilpotent by assumption (compare Lemma 3.21 (1)). Then we obtain that $g\mathcal{L} = \mathcal{L}$. Thus, we find

$$\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W) = \bigcup_{\mathcal{L} \in \mathcal{W}^g} \breve{U}_{E,\mathcal{L}} \cap g \cdot \breve{U}_{E,\mathcal{L}}.$$

Recall that $z\Lambda^\pm \subseteq \Lambda^\mp$. So given an S-valued point $(\mathcal{L}, (\varphi, 0))$ resp. $(\mathcal{L}_i, (\varphi_i, 0))_{i=0,1}$ of $\breve{U}_{E,\mathcal{L}}$ as in (12.13), we obtain that

$$g(\mathcal{L}, (\varphi, 0)) = (\mathcal{L}, (\varphi, \varphi \circ z)), \quad \text{resp.} \quad g \cdot (\mathcal{L}_i, (\varphi_i, 0))_{i=0,1} = (\mathcal{L}_i, (\varphi_i, \varphi_i \circ z))_{i=0,1}.$$

This point lies again in $\breve{U}_{E,\mathcal{L}}$ if and only if $\varphi \circ z$ vanishes, resp. $\varphi_i \circ z$ for both $i = 0, 1$ vanishes. □

Recall that we defined the function $n(z, \Lambda) = \max\{k \in \mathbb{Z} \mid z\Lambda \subseteq \pi^k \Lambda\}$ in §11. Denote by $m(z, \Lambda) := \max\{0, n(z, \Lambda)\}$ its nonnegative cut-off.

Proposition 12.9. *Let $g = 1 + z_g \in GL_F(W)$ be regular semi-simple with $z = z_g$ topologically nilpotent. Then $m(z, \Lambda)$ equals the multiplicity of P_Λ in $\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W)$ in the sense that*

$$(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{pure}} = \sum_{\{\Lambda \subset W\, O_E\text{-lattice}\}/E^\times} m(z, \Lambda) \cdot [P_\Lambda]$$

as 1-cycles on $\breve{\Omega}_E(W)$. Here, the pure locus is meant in the sense of Definition 10.1.

Proof. By Proposition 12.8, the multiplicity of P_Λ can only be positive if $z\Lambda \subseteq \Lambda$. In this situation, it equals the maximal integer k such that

$$\pi^k \mid [\varphi \circ z : \Lambda^- \longrightarrow \mathcal{L}], \tag{12.17}$$

where $(\mathcal{L}, \varphi)$ denotes the universal point over $\breve{U}_{E,\Lambda}$. This integer is evidently equal to $n(z, \Lambda)$. □

Definition 10.1 also provides a definition of the artinian locus $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$. Furthermore, recall that we defined $\mathcal{T}(z)$ as the set of homothety classes of O_E-lattices in which $n(z, -)$ takes its maximum (Definition 11.4). Also recall the following terminology for points on $\Omega_E(W)$:

Definition 12.10. A closed point of $\breve{\Omega}_E(W)$ is called superspecial if it is defined over $\mathbb{F}_{q^2}$. The superspecial points are hence precisely the intersection points $P_\Lambda \cap P_{\Lambda'}$ for lattice chains $\pi\Lambda \subset \Lambda' \subset \Lambda$ and in bijection with the edges of $\mathcal{B}$.

Proposition 12.11. *The artinian part $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ is nonempty if and only if $\mathcal{T}(z)$ is an edge and $r \in 4\mathbb{Z} + 2$. In this case, the artinian part is of length one and located in the superspecial point of that edge.*

Proof. We first reconsider the situation from (12.17). Write $z = \pi^{m(z,\Lambda)} z_0$. Then, by definition of the artinian part, $\varphi \circ z_0$ is a defining equation for $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ on $\breve{U}_{E,\Lambda}$. If the kernel of z_0 is nonzero, however, then it defines an $\mathbb{F}_{q^2}$-point of P_Λ which, in particular, does not lie in $\breve{U}_{E,\Lambda}$. It follows that the support of $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ is contained in the superspecial points. We next compute the local equations in such a point with Proposition 12.8.

Let $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$ be a representative of a chain of O_E-lattices. Assume that $z\Lambda_i \subseteq \Lambda_i$, otherwise $\breve{U}^z_{E,\mathcal{L}} = \emptyset$. Pick a compatible basis, say $\Lambda_0 = O_E e_1 + O_E e_2$ and $\Lambda_1 = \pi O_E e_1 + O_E e_2$. Then $\Lambda_0^\pm$ and $\Lambda_1^\pm$ have the bases $(e_1^\pm, e_2^\pm)$ and $(\pi e_1^\pm, e_2^\pm)$; see (12.14). In these coordinates, the universal point over $\breve{U}_{E,\mathcal{L}}$ may be written as

$$\begin{tikzcd}
\mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[r, "\left(\begin{smallmatrix}1 & \\ & \pi\end{smallmatrix}\right)"] \arrow[d, "\varphi_0=(1\ \mathbf{u})"] & \mathcal{O}_S \otimes_{O_E} \Lambda_1^+ \arrow[r, "\left(\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right)"] \arrow[d, "\varphi_1=(\mathbf{v}\ 1)"] & \mathcal{O}_S \otimes_{O_E} \Lambda_0^+ \arrow[d, "\varphi_0=(1\ \mathbf{u})"] \\
\mathcal{O}_{U_{E,\mathcal{L}}} \arrow[r, "\mathbf{v}"] & \mathcal{O}_{U_{E,\mathcal{L}}} \arrow[r, "\mathbf{u}"] & \mathcal{O}_{U_{E,\mathcal{L}}}
\end{tikzcd} \tag{12.18}$$

where $\breve{U}_{E,\mathcal{L}} \subset \operatorname{Spf} O_E\langle \mathbf{u}, \mathbf{v}\rangle/(\mathbf{uv}-\pi)$ is an open that contains the superspecial point $P_{\Lambda_0} \cap P_{\Lambda_1} = V(\mathbf{u}, \mathbf{v})$. We have already seen that $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ is supported in superspecial points. So we henceforth work over the formal completion $\operatorname{Spf} O_E[[\mathbf{u}, \mathbf{v}]]/(\mathbf{uv}-\pi)$. Write $z = \left(\begin{smallmatrix} a & b \\ \pi c & d\end{smallmatrix}\right)\sigma \in M_2(O_E)\sigma$with respect to the basis (e_1, e_2). Here, $\sigma \in \operatorname{Gal}(E/F)$ denotes the Galois conjugation. Note that $\sigma(e_i^+) = e_i^-$ and $\sigma(e_i^-) = e_i^+$. Thus, the map from Λ_0^- to Λ_0^+ defined by z is given by $\left(\begin{smallmatrix} a & b \\ \pi c & d\end{smallmatrix}\right)$with respect to the bases (e_1^+, e_2^+) and (e_1^-, e_2^-). The vanishing conditions defining $\breve{U}^z_{E,\mathcal{L}} \cap \operatorname{Spf} O_E[[\mathbf{u}, \mathbf{v}]]/(\mathbf{uv}-\pi)$ then become

$$(\mathbf{u}\ \ 1)\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix} = 0, \quad \text{and} \quad (1\ \ \mathbf{v})\begin{pmatrix} a & \pi b \\ c & d\end{pmatrix} = 0. \tag{12.19}$$

Note that

$$(1\ \ \mathbf{v})\begin{pmatrix} a \\ c\end{pmatrix} = 0 \implies (\mathbf{u}\ \ 1)\begin{pmatrix} a \\ \pi c\end{pmatrix} = 0.$$

and

$$(\mathbf{u}\ \ 1)\begin{pmatrix} b \\ d\end{pmatrix} = 0 \implies (1\ \ \mathbf{v})\begin{pmatrix} \pi b \\ d\end{pmatrix} = 0.$$

Therefore, (12.19) is equivalent to just

$$\begin{cases} a + c\mathbf{v} & = 0 \\ b\mathbf{u} + d & = 0. \end{cases} \tag{12.20}$$

Write

$$\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix} = \pi^m \begin{pmatrix} a' & b' \\ \pi c' & d'\end{pmatrix},$$

where $m = m(z, \Lambda_0)$ is chosen maximally. We claim that the ideal $(a + c\mathbf{v}, b\mathbf{u} + d)$ is principal unless $a', d' \in \pi O_E$ and $b', c' \in O_E^\times$. We furthermore claim that if the ideal is not principal, then it equals $\pi^m(\mathbf{u}, \mathbf{v})$.

Note that $(a + c\mathbf{v}, b\mathbf{u} + d)$ is principal if and only if $P_{\Lambda_0} \cap P_{\Lambda_1} \notin (\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$. Moreover, if it equals $\pi^m(\mathbf{u}, \mathbf{v})$, then

$$(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}} \cap \operatorname{Spf} O_E[[\mathbf{u}, \mathbf{v}]]/(\mathbf{uv}-\pi) = V(\mathbf{u}, \mathbf{v}).$$

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In order to prove the claim, observe that

$$a + c\mathbf{v} \in \pi^r R^\times \quad \text{or} \quad a + c\mathbf{v} \in \pi^r \mathbf{v} R^\times$$

and

$$b\mathbf{u} + d \in \pi^s R^\times \quad \text{or} \quad b\mathbf{u} + d \in \pi^s \mathbf{u} R^\times$$

for a uniquely determined pair of integers (r, s). The only possibility for (12.20) giving a non-principal ideal is $r = s$, in which case $r = s = m$ and

$$\begin{cases} a + c\mathbf{v} \in \pi^m \mathbf{v} R^\times \\ b\mathbf{u} + d \in \pi^m \mathbf{u} R^\times. \end{cases}$$

This is equivalent to $a', d' \in \pi O_E$ and $b', c' \in O_E^\times$, which proves our claim. The property $a', d' \in \pi O_E$ and $b', c' \in O_E^\times$ implies that $m = \max n(z, -)$ and that $r - 4m = v(\det_E(\pi^{-2m} z^2)) = 2$. Since $r \in 4\mathbb{Z}$ whenever L is split (see Remark 11.9), this shows that we are in cases (2) or (4) of Theorem 11.10, as claimed. (Being in one of these two cases is equivalent to $r \in 4\mathbb{Z} + 2$ and $\mathcal{T}(z)$ being an edge.)

Conversely, assume that $r \in 4\mathbb{Z} + 2$ and that $\mathcal{T}(z)$ is an edge. Let $\pi\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$ be the lattices representing that edge. Choose a compatible basis e_1, e_2 of Λ_0 as above. We have $m(z, \Lambda_0) = m(z, \Lambda_1) =: m$ because $\mathcal{T}(z) = \{\Lambda_0, \Lambda_1\}$ by assumption. In other words, there are $a', b', c', d' \in O_E$ such that $\pi^{-m} z$ is given by the matrices

$$\begin{pmatrix} a' & b' \\ \pi c' & d' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a' & \pi b' \\ c' & d' \end{pmatrix}$$

with respect to the bases $e_1, e_2 \in \Lambda_0$ and $\pi e_1, e_2 \in \Lambda_1$. Each of these two matrices has an invertible entry because m was chosen maximally. Furthermore, both matrices are still topologically nilpotent because $v(\det_E(\pi^{-2m} z^2)) = 2$. Thus, $a', d' \in \pi O_E$ and $b', c' \in O_E^\times$. The previous calculation now shows that $P_{\Lambda_0} \cap P_{\Lambda_1} \in (\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ with local ring of length 1, as claimed. □

We define the following auxiliary intersection number. Write $L^\times = \Gamma \times O_L^\times$ for some subgroup $\Gamma \subset L^\times$ as in Definition 4.21 and let $\Gamma_0 = \Gamma \cap GL_F(W)^0$. Then $\Gamma_0 = \{1\}$ if L is a field or $\Gamma_0 \cong \mathbb{Z}$ if $L \cong F \times F$. Note that the action of $L^\times$ preserves both $\breve{\Omega}_E(W)$ and $g \cdot \breve{\Omega}_E(W)$, so we can define

$$\mathrm{Int}_0(g) = \langle \Gamma_0 \backslash \breve{\Omega}_E(W),\ \Gamma_0 \backslash g \cdot \breve{\Omega}_E(W) \rangle_{\Gamma_0 \backslash \breve{\Omega}_F(W)}.$$

Proposition 12.12. *This intersection number is given by the following formula.*

(1) *If L is a field, then* $\mathrm{Int}_0(g) = r/2$.
(2) *If $L \cong F \times F$, then* $\mathrm{Int}_0(g) = 0$.

Proof. Proposition 12.9 states that the multiplicity of P_Λ in $\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W)$ is $m(z, \Lambda)$. Define

$$p_\Lambda := -m(z, \Lambda)\big[(q^2 - 1) + \langle P_\Lambda,\ (\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{pure}} \rangle\big].$$

Here, the term $q^2 - 1$ is the degree of the conormal bundle (Proposition 12.6), and the intersection pairing is that of divisors on $\breve{\Omega}_E(W)$. By Corollary 10.3, we have

$$\mathrm{Int}_0(g) = \mathrm{len}\Big(\mathcal{O}_{\Gamma_0 \backslash (\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}}\Big) + \sum_{\Lambda \in \Gamma_0 \backslash \{O_E\text{-lattices in } W\}/E^\times} p_\Lambda. \tag{12.21}$$

We next compute the summands p_Λ for all Λ with $m(z, \Lambda) \geq 1$. Put $m = \max m(z, -)$ and $\mathcal{T} = \mathcal{T}(z)$; assume that $m \geq 1$. By [23, Lemma 4.7], the intersection numbers of the curves P_Λ are given by

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$$\langle P_\Lambda, P_{\Lambda'}\rangle_{\breve{\Omega}_E(W)} = \begin{cases} 0 & P_\Lambda \cap P_{\Lambda'} = \emptyset \\ 1 & P_\Lambda \cap P_{\Lambda'} = \{\mathrm{pt}\} \\ -(q^2+1) & P_\Lambda = P_{\Lambda'}. \end{cases}$$

(1) First assume that $\Lambda \notin \mathcal{T}$. Then Λ has some multiplicity $i = m(z, \Lambda)$ with $1 \le i < m$. Precisely q^2 of its neighbors have multiplicity $i-1$ and a single neighbor has multiplicity $i+1$ (Proposition 11.5). Thus,

$$p_\Lambda = -i\big[(q^2-1) - i(q^2+1) + q^2(i-1) + (i+1)\big] = 0.$$

(2) Now assume that Λ has multiplicity m (i.e., lies in $\mathcal{T}$). By Theorem 11.10, the valency v_Λ of Λ in $\mathcal{T}$ is 0, 1, 2 or $q+1$. Then Λ has v_Λ many neighbors of multiplicity m and q^2+1-v_Λ many neighbors with multiplicity $m-1$. It follows that

$$p_\Lambda = -m\big[(q^2-1) - m(q^2+1) + mv_\Lambda + (m-1)(q^2+1-v_\Lambda)\big] = (2-v_\Lambda)m.$$

We now evaluate (12.21) for the six possible shapes of $\mathcal{T}$ from Theorem 11.10. An observation that applies in all cases is that $p_\Lambda \neq 0$ only for $\Lambda \in \mathcal{T}$ (see (i) above), so the discussion will only involve the set $\mathcal{T}$. Moreover, Proposition 12.11 states that the artinian part $(\breve{\Omega}_E(W) \cap g \cdot \breve{\Omega}_E(W))^{\mathrm{art}}$ is of length 1 precisely in cases (2) and (4), and 0 otherwise. We will also use Lemma 11.6 in every case to relate m with r.

(1) Assume that L/F is inert and that $r \in 4\mathbb{Z}$. Then $\mathcal{T}$ is a $(q+1)$-regular ball of radius d around a single vertex, $\Gamma_0 = \{1\}$, there are no embedded components, and $4m = r$. If $d = 0$, then (ii) above shows that

$$\mathrm{Int}_0(g) = 2m = r/2.$$

For $d > 1$, let A be the number of vertices of $\mathcal{T}$ with valency 1 and let B be the number of those with valency $q+1$. It is easy to check that $A - (q-1)B = 2$ for every $d \ge 1$. Applying (ii) again, we find

$$\mathrm{Int}_0(g) = m(A - (q-1)B) = 2m = r/2.$$

(2) Assume that L/F is inert and that $r \in 4\mathbb{Z}+2$. Then $\mathcal{T}$ is an edge, $\Gamma_0 = \{1\}$, there is a single embedded component of length 1, and $4m + 2 = r$. We obtain from (ii) that

$$\mathrm{Int}_0(g) = 1 + 2m = r/2.$$

(3) Assume that L/F is ramified and that $r \in 4\mathbb{Z}$. Then $\mathcal{T}$ is a $(q+1)$-regular ball of radius d around an edge, $\Gamma_0 = \{1\}$, there are no embedded components, and $4m = r$. If $d = 0$, then (ii) immediately shows

$$\mathrm{Int}_0(g) = 2m = r/2.$$

For $d \ge 1$, let again A be the number of vertices of $\mathcal{T}$ with valency 1 and let B be the number of those with valency $q+1$. It is again checked that $A-(q-1)B = 2$ for every $d \ge 1$. Applying (ii) again, we find

$$\mathrm{Int}_0(g) = m(A - (q-1)B) = 2m = r/2.$$

(4) Assume that L/F ramified and that $r \in 4\mathbb{Z}+2$. Just like in case (2), we obtain

$$\mathrm{Int}_0(g) = r/2.$$

(5) Assume that $L = F \times F$ and that $z^2 = (z_1, z_2)$ has the property $v(z_1) = v(z_2)$. Then $\mathcal{T}$ is a $(q+1)$-regular ball of radius d around an apartment. The action of the group $\Gamma_0 \cong \mathbb{Z}$ on this apartment is by a translation with two orbits. Moreover, there is no artinian contribution.

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Assume first that $d = 0$. Then every $\Lambda \in \mathcal{T}$ has valency 2, and hence, $p_\Lambda = 0$ by (ii) above. It follows that

$$\mathrm{Int}_0(g) = 0.$$

Assume now that $d \geq 1$. Let A be the number of vertices of $\Gamma_0\backslash\mathcal{T}$ of valency 1 and let B denote those of valency $q+1$. One checks that $A - (q-1)B = 0$ for all $d \geq 1$, so

$$\mathrm{Int}_0(g) = m[A - (q-1)B] = 0.$$

(6) Assume finally that $L \cong F \times F$ and that $z^2 = (z_1, z_2)$ has the property $v(z_1) \neq v(z_2)$. Then $\mathcal{T}$ is an apartment on which Γ_0 acts with two orbits. There is no artinian contribution, and one obtains just as before that

$$\mathrm{Int}_0(g) = 0. \qquad \square$$

We can now determine the intersection numbers $\mathrm{Int}(g)$ for $D = D_{1/4}$ and prove our arithmetic transfer conjecture (Conjecture 4.26) in this situation. Let the notation be as in §9; in particular, $G' = GL_4(F)$ and G_b denote the two groups that intervene in the formulation of the AT conjecture. Let f'_{Par} and f'_{Iw} denote the two test functions from §5.

Theorem 12.13. *Let $g \in G_{b,\mathrm{rs}}$ be a regular semi-simple element with numerical invariants (L, r, d). The intersection number* $\mathrm{Int}(g)$ *is nonzero only if $r > 0$. In this case, it is given by*

$$\mathrm{Int}(g) = \begin{cases} r & L/F \text{ unramified} \\ r/2 & L/F \text{ ramified} \\ 0 & L/F \text{ split.} \end{cases} \tag{12.22}$$

In particular, Conjecture 4.26 holds for $D = D_{1/4}$ with correction function $-4q\log(q)\cdot f'_{\mathrm{Par}}$. In other words, for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,

$$\partial O(\gamma, f'_D) - 4q\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})\log(q) = \begin{cases} 2\,\mathrm{Int}(g)\log(q) & \text{if there exists a matching } g \in G_b \\ 0 & \text{otherwise.} \end{cases} \tag{12.23}$$

Proof of Identity (12.22). The statement about the vanishing of $\mathrm{Int}(g)$ for $r \leq 0$ follows from Proposition 4.23. We henceforth assume that $r > 0$ and even that z_g is topologically nilpotent.

Recall that $\mathcal{M}_D^i \subset \mathcal{M}_D$ and $\mathcal{M}_C^i \subset \mathcal{M}_C$ denote the connected components triples (Y, ι, ρ) resp. (X, κ, ρ) where the height of ρ is i. Also recall that $\mathcal{M}_D^i$ and $\mathcal{M}_C^i$ nonempty precisely if $i \in 4\mathbb{Z}$. Moreover, an element $h \in G_b \cong GL_4(F)$, resp. $h \in H_b \cong GL_2(E)$, has the property

$$h : \mathcal{M}_D^i \xrightarrow{\cong} \mathcal{M}_D^{i+4v(\det_F(h))}, \quad \text{resp.} \quad h : \mathcal{M}_C^i \xrightarrow{\cong} \mathcal{M}_C^{i+4v(\det_E(h))}.$$

By definition, the Serre tensor construction doubles the height (i.e., is such that $\mathcal{M}_C^i = \mathcal{M}_C \cap \mathcal{M}_D^{2i}$).

Recall that we wrote $L^\times = \Gamma \times O_L^\times$ and $\Gamma_0 = \Gamma \cap GL_F(W)^0$ before. Let $\Gamma_1 \subseteq \Gamma$ be a complement to Γ_0 and let $\theta \in \Gamma_1 \cap O_L$ be a generator. Then

$$\theta\mathcal{M}_D^i = \begin{cases} \mathcal{M}_D^{i+8} & \text{if } L/F \text{ is ramified or split} \\ \mathcal{M}_D^{i+16} & \text{if } L/F \text{ is unramified.} \end{cases}$$

In other words, $\Gamma_1\backslash\pi_0(\mathcal{M}_C) = \{0\}$ or $\{0, 8\}$, depending on the case. Thus, if L is ramified or split, then

$$\mathrm{Int}(g) = \mathrm{Int}_0(g),$$

and we are done by Theorem 12.12. If L is inert, however, then we obtain

$$\mathrm{Int}(g) = \mathrm{Int}_0(g) + \langle \Gamma_0\backslash\mathcal{M}_C^4,\ \Gamma_0\backslash(g\cdot\mathcal{M}_C^4)\rangle_{\Gamma_0\backslash\mathcal{M}_D^8}.$$

Let $h \in H_b$ be any with $v_E(\det(h)) = 1$. Then $h : \mathcal{M}_D^0 \xrightarrow{\sim} \mathcal{M}_D^8$ as well as $h^{-1}(\mathcal{M}_C^4) = \mathcal{M}_C^0$ and $h^{-1}(g\mathcal{M}_C^4) = hgh^{-1}\mathcal{M}_C^0$. (Recall that the G_b-action is a right action.) We obtain that

$$\mathrm{Int}(g) = \mathrm{Int}_0(g) + \mathrm{Int}_0(hgh^{-1}).$$

But hgh^{-1} and g lie in the same H_b double coset, so have the same numerical invariant (L, r, d). Proposition 12.12 shows that $\mathrm{Int}_0(g)$ only depends on the numerical invariant, so we obtain $\mathrm{Int}_0(hgh^{-1}) = \mathrm{Int}_0(g)$ and then $\mathrm{Int}(g) = r$, as claimed. □

Proof of Identity (12.23). Let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element with numerical invariant (L, r, d). First consider the case that r is odd. Then there is no matching element $g \in G_b$ (see rows 2 and 5 of Table 1), so we need to show that the left-hand side of (12.23) vanishes.

The sign of the functional equation of f'_{Par} is $(-1)^r$ and hence negative if r is odd. This shows $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) = 0$. Proposition 8.3 for odd r moreover states that $\partial\mathrm{Orb}(\gamma, f'_D) = 0$, which is the desired vanishing.

Now we consider the case where r is even. There exists a matching element $g \in G_b$ for γ if and only if L is a field or if $L \cong F \times F$ and $r \in 4\mathbb{Z}$; see rows 1 and 3 of Table 1. No matter which case, (5.10) shows the equality of the two sides in (12.23). □

13. Invariant 3/4

The aim of this section is to prove Theorem 9.1 for Hasse invariant 3/4. It will turn out, however, that the geometry for invariant 3/4 is closely related to the one for invariant 1/4. So we will, in fact, consider the two intersection problems for $D \in \{D_{1/4}, D_{3/4}\}$ simultaneously. For this reason, we introduce the following notation: We write $\mathcal{M}_\lambda$, $\lambda \in \{1/4, 3/4\}$, for the RZ space for $D = D_\lambda$. We similarly write $G_\lambda = D_\lambda^{\mathrm{op},\times}$ for the group G and $G_{\lambda,b}$ for the group G_b.

We also choose compatible presentations of $D_{1/4}$ and $D_{3/4}$: Let F_4/F denote an unramified field extension of degree 4 and let $\sigma \in \mathrm{Gal}(F_4/F)$ be its Frobenius. For both choices of λ, we fix an embedding $F_4 \to D_\lambda$ and a uniformizer $\Pi \in D_\lambda$ that normalizes F_4 and satisfies $\Pi^4 = \pi \in F$. Then $\Pi a = \sigma(a)\Pi$ if $\lambda = 1/4$ and $\Pi a = \sigma^3(a)\Pi$ if $\lambda = 3/4$, for $a \in F_4$. We assume that the embedding $E \to D$ is such that $E \subset F_4$. Then $\varpi = \Pi^2$ is a uniformizer of O_C, and we obtain the presentation $C = F_4[\varpi]$.

13.1. Conormal bundle

Proposition 13.1. *Let $P \subseteq \mathcal{M}_C$ be any irreducible component of the special fiber. The degree of the conormal bundle $\mathcal{C}$ of $\mathcal{M}_C \to \mathcal{M}_{3/4}$ on P is the same as in the case of invariant 1/4,*

$$\deg(\det \mathcal{C}|_P) = q^2 - 1.$$

Proof. Our proof is by showing that the degrees of the conormal bundles for $\mathcal{M}_C \to \mathcal{M}_{1/4}$ and $\mathcal{M}_C \to \mathcal{M}_{3/4}$ agree. Then Proposition 12.6 yields that the degree is $q^2 - 1$ in both cases. So let $\mathcal{M} = \mathcal{M}_{1/4}$ or $\mathcal{M} = \mathcal{M}_{3/4}$ and let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{M}}$ be the ideal sheaf such that $\mathcal{M}_C = V(\mathcal{I})$. The conormal bundle is $\mathcal{I}/\mathcal{I}^2$.

Let (Y, ι, ρ) be the universal point over $\mathcal{M}_C$ and let $\mathcal{D}$ be the covariant O_F-Grothendieck–Messing crystal of Y evaluated at the thickening $V(\mathcal{I}^2)$, viewed with trivial PD-structure. It is endowed with an $O_C = O_{F_4}[\varpi]$-action ι by functoriality. This provides a $\mathbb{Z}/4$-grading $\mathcal{D} = \bigoplus \mathcal{D}_i$ where

$$\mathcal{D}_i = \{x \in \mathcal{D} \mid \iota(a)(x) = \sigma^i(a)x,\ \ a \in O_{F_4}\}.$$

Then ϖ is homogeneous of degree 2, and each graded piece is a vector bundle of rank 2.

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Write $\overline{\mathcal{D}} = \mathcal{D}/\mathcal{I}\mathcal{D}$, $\overline{\mathcal{D}_i} = \mathcal{D}_i/\mathcal{I}\mathcal{D}_i$ and denote by $\overline{\mathcal{F}_i} \subset \overline{\mathcal{D}_i}$ the Hodge filtration of Y. Recall that $\mathcal{D}/\mathcal{F} = \mathrm{Lie}(Y)$ is the Lie algebra. The special condition (see Definition 4.9) in particular requires that $O_E \subset O_{F_4}$ acts via the natural map $O_E \to \mathcal{O}_{M_C}$ on $\mathrm{Lie}(Y)$, which implies that $\overline{\mathcal{F}_i} = \overline{\mathcal{D}_i}$ for $i = 1, 3$.

Next, consider $X := O_D \otimes_{O_C} Y$ with its natural O_D-action. The evaluation of its O_F-Grothendieck–Messing crystal at $V(\mathcal{I}^2)$ is $\mathcal{P} := O_D \otimes_{O_C} \mathcal{D}$ by functoriality. The action of $O_{F_4} \subset O_D$ again provides a $\mathbb{Z}/4$-grading $\mathcal{P} = \bigoplus \mathcal{P}_i$. It may be refined as follows: Write $O_D = O_B \oplus \Pi O_B$, where Π is the previously chosen uniformizer of D. We denote by ΠY, $\Pi\mathcal{D}$ etc. the summands $\Pi \otimes Y$, $\Pi \otimes \mathcal{D}$ etc. Then $\mathcal{P}$ is a direct sum of eight terms:

$$\mathcal{P} = \begin{array}{ccccccc} \mathcal{D}_0 & \oplus & \mathcal{D}_1 & \oplus & \mathcal{D}_2 & \oplus & \mathcal{D}_3 \\ \oplus & & \oplus & & \oplus & & \oplus \\ \Pi\mathcal{D}_{4\lambda} & \oplus & \Pi\mathcal{D}_{1+4\lambda} & \oplus & \Pi\mathcal{D}_{2+4\lambda} & \oplus & \Pi\mathcal{D}_{3+4\lambda}, \end{array} \tag{13.1}$$

where $\mathcal{P}_i = \mathcal{D}_i \oplus \Pi\mathcal{D}_{i+4\lambda}$. The operator Π acts homogeneously of degree -4λ. Let $\mathcal{Q} \subset \mathcal{P}$ denote the Hodge filtration of the restriction to $V(\mathcal{I}^2)$ of the universal point of $\mathcal{M}$. It is O_D-stable, meaning it is Π-stable and graded ($\mathcal{Q} = \bigoplus \mathcal{Q}_i$ with $\mathcal{Q}_i \subset \mathcal{P}_i$).

The ideal $\mathcal{I}/\mathcal{I}^2$ tautologically defines the closed subscheme $\mathcal{M}_C \subset V(\mathcal{I}^2)$. This subscheme is also characterized by the three properties from Proposition 4.15. Consider the first one, $\mathcal{M}_C \subseteq \mathcal{Z}(\iota(O_E))$. The vertical grading $\mathcal{P} = \mathcal{D} \oplus \Pi\mathcal{D}$ in (13.1) also equals the decomposition into the two eigenspaces of $\mathcal{P}$ under the $\kappa(O_E) \otimes_{O_F} \rho\iota(O_E)\rho^{-1}$-action. (This action exists on $O_D \otimes_{O_C} Y$ and lifts to the crystal evaluated at $V(\mathcal{I}^2)$.) Thus, the intersection $V(\mathcal{I}^2) \cap \mathcal{Z}(\iota(O_E))$ as closed subscheme of $V(\mathcal{I}^2)$ is defined by the condition that $\mathcal{Q}$ is vertically graded in the sense $\mathcal{Q} = (\mathcal{Q} \cap \mathcal{D}) \oplus (\mathcal{Q} \cap \Pi\mathcal{D})$. As $\mathcal{Q}$ is already $\mathbb{Z}/4$-graded, this is equivalent to

$$\mathcal{Q}_i = (\mathcal{Q}_i \cap \mathcal{D}_i) \oplus (\mathcal{Q}_i \cap \Pi\mathcal{D}_{i+4\lambda}) \quad \forall i = 0, \ldots, 3. \tag{13.2}$$

We claim that in fact $V(\mathcal{I}) = V(\mathcal{I}^2) \cap \mathcal{Z}(\iota(O_E))$. For this, we need to check that the further conditions (1) and (2) from Proposition 4.15 are implied by (13.2).

Condition (1) just says that the rank of $\mathcal{Q}_i \cap \mathcal{D}_i$ is 1 for $i = 0, 2$ and 2 for $i = 1, 3$. This already holds on $V(\mathcal{I})$ and extends to any infinitesimal thickening. (The rank of a locally free module is locally constant.)

Condition (2) states that $\kappa(\Pi)$ defines an isomorphism $\Pi : X_+ \xrightarrow{\sim} X_-$, where $X = X_+ \oplus X_-$ is the decomposition into eigenspaces of X defined on $V(\mathcal{I}^2) \cap \mathcal{Z}(\iota(O_E))$. Just like (1) above, this condition can be checked over $V(\mathcal{I})$.

In summary, we see that $V(\mathcal{I}) \subset V(\mathcal{I}^2)$ is defined by (13.2). This condition is further equivalent to $\mathcal{D}_i \subset \mathcal{Q}_i$ and $\Pi\mathcal{D}_{i+1+4\lambda} \subset \mathcal{Q}_{i+1}$ for $i = 1, 3$, because these inclusions hold over $V(\mathcal{I})$. Since $\mathcal{Q}$ is Π-stable, it is equivalent to only require $\mathcal{D}_i \subset \mathcal{Q}_i$ for $i = 1, 3$. So we see that $\mathcal{I}/\mathcal{I}^2$ is defined by the vanishing of the two maps

$$\mathcal{D}_1 \longrightarrow \mathcal{L}_1, \quad \mathcal{D}_3 \longrightarrow \mathcal{L}_3, \qquad \mathcal{L}_i := \mathcal{P}_i/\mathcal{Q}_i.$$

These two maps are known to vanish modulo $\mathcal{I}$, so they factor over $\overline{\mathcal{D}_1}$ and $\overline{\mathcal{D}_3}$. We thus obtain an exact sequence of vector bundles on $\mathcal{M}_C$,

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\mathcal{D}_1 \otimes \mathcal{L}_1^{-1}} \oplus \overline{\mathcal{D}_3 \otimes \mathcal{L}_3^{-1}} \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow 0. \tag{13.3}$$

Denote its middle term by $\mathcal{E}$. Note that $\Pi : \overline{\mathcal{L}_{i+4\lambda}} \cong \overline{\mathcal{L}_i}$ for $i = 1, 3$ and that $\overline{\mathcal{L}_{1+4\lambda}} \oplus \overline{\mathcal{L}_{3+4\lambda}}$ is the Lie algebra of Y. It follows that the determinant of $\mathcal{E}$ is independent of whether $\lambda = 1/4$ or $3/4$. What is left to show is that the determinant of $\mathcal{K}|_{V(\pi)}$ is also independent. (Here, $V(\pi)$ denotes the special fiber of $\mathcal{M}_C$.) This relies on the commutative diagram

$$\begin{tikzcd} \overline{\mathcal{D}_1} \arrow[r, "\varpi"] \arrow[d, "\varphi_1"'] & \overline{\mathcal{D}_3} \arrow[r, "\varpi"] \arrow[d, "\varphi_3"'] & \overline{\mathcal{D}_1} \arrow[d, "\varphi_1"'] \\ \overline{\mathcal{L}_1} \arrow[r, "\varpi"] & \overline{\mathcal{L}_3} \arrow[r, "\varpi"] & \overline{\mathcal{L}_3}. \end{tikzcd} \tag{13.4}$$

Write $\varpi^\vee : \overline{\mathcal{L}_{i+2}}^{-1} \to \overline{\mathcal{L}_i}^{-1}$ for the dual map on inverse line bundles. We claim that $\mathcal{K}$ is generated by all sections of the form

$$(u_1 \otimes \varpi^\vee s_3, -\varpi u_1 \otimes s_3),\ (\varpi u_3 \otimes s_1, -u_3 \otimes \varpi^\vee s_1),\quad u_i \in \overline{\mathcal{D}_i},\ s_{i+2} \in \overline{\mathcal{L}_{i+2}}^{-1}. \tag{13.5}$$

To prove this, it is sufficient to locally exhibit elements of the form (13.5) that generate a rank 2 direct summand of $\mathcal{E}$. The top row of (13.4) has a normal form, meaning there locally exist bases e_1, f_1 of $\mathcal{D}_1$ and e_3, f_3 of $\mathcal{D}_3$ such that ϖ is given by

$$\varpi\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} e_3 \\ \pi f_3 \end{pmatrix} \quad \text{and} \quad \varpi\begin{pmatrix} e_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} \pi e_1 \\ f_1 \end{pmatrix}.$$

In particular, ϖe_1 and ϖf_3 are nowhere vanishing sections. Also assume that $s_i \in \overline{\mathcal{L}_i}^{-1}$, with $i = 1, 3$, are local generators. Then

$$(e_1 \otimes \varpi^\vee s_3, -\varpi e_1 \otimes s_3) \quad \text{and} \quad (\varpi f_3 \otimes s_1, -f_3 \otimes \varpi^\vee s_1)$$

lie in $\mathcal{K}$ and are fiberwise linearly indpendent, and hence generate $\mathcal{K}$.

From now on, we restrict to the special fiber $V(\pi) \subset \mathcal{M}_C$. The above normal form statement implies that the outer terms in the following canonical exact sequences are line bundles:

$$0 \longrightarrow \ker(\varpi|_{\mathcal{D}_i}) \longrightarrow \mathcal{D}_i \longrightarrow \operatorname{Im}(\varpi|_{\mathcal{D}_i}) \longrightarrow 0.$$

The previous computation specializes to the fact that

$$\mathcal{K}/\pi\mathcal{K} = \ker(\varpi|_{\mathcal{D}_1}) \otimes \mathcal{L}_1^{-1} \oplus \ker(\varpi|_{\mathcal{D}_3}) \otimes \mathcal{L}_3^{-1} \tag{13.6}$$

as subsheaf of $\mathcal{E}/\pi\mathcal{E}$. The determinant of $\mathcal{K}/\pi\mathcal{K}$ is then the tensor product of all four line bundles that occur on the right-hand side of (13.6). This product is independent of λ, as was to be shown. □

13.2. Intersection locus (simplified formulation)

Let $g \in G_{\lambda,b,\mathrm{rs}}$ be a regular semi-simple element. Write $z = z_g$. Our next aim is to rephrase the definition of $\mathcal{I}(g) = \mathcal{M}_C \cap g \cdot \mathcal{M}_C$ in a simpler way.

Since the framing object $(\mathbb{Y}, \iota)$ that goes into the definition of $\mathcal{M}_C$ has no étale part, Lemma 4.22 states that $\mathcal{M}_C \cap g \cdot \mathcal{M}_C = \emptyset$ unless z is toplogically nilpotent. So we assume for the following discussion that z is topologically nilpotent. Then Proposition 4.23 states that

$$\mathcal{M}_C \cap g \cdot \mathcal{M}_C = \mathcal{M}_C \cap \mathcal{Z}(z). \tag{13.7}$$

(We recall that $\mathcal{Z}(z)$ denotes all those $(X, \kappa, \rho) \in \mathcal{M}_\lambda$ such that $\rho z \rho^{-1} \in \operatorname{End}(X)$; see (4.16).) In terms of the element $\Pi \in D_\lambda$ we chose at the beginning of §13, we have $O_{D_\lambda} = O_C \oplus \Pi O_C$ and obtain a presentation of $(\mathbb{X}, \kappa)$ as

$$\mathbb{X} = \mathbb{Y} \oplus \Pi\mathbb{Y}, \quad \kappa(\Pi) = \begin{pmatrix} & \iota(\varpi) \\ 1 & \end{pmatrix}, \quad \kappa(c) = \begin{pmatrix} \iota(c) & \\ & \iota(\Pi^{-1} c \Pi) \end{pmatrix} \quad c \in C. \tag{13.8}$$

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The endomorphism ring of $(\mathbb{X},\kappa)$ is then

$$\mathrm{End}^0_D(\mathbb{X},\kappa) = \left\{ \begin{pmatrix} x & y\varpi \\ y & x \end{pmatrix} \middle| \, x \in \mathrm{End}^0_C(\mathbb{Y}),\ y \in \mathrm{End}^0_F(\mathbb{Y}) \text{ s.th. } yc = \Pi^{-1}c\Pi y \text{ for all } c \in C \right\}. \tag{13.9}$$

Description (13.8) applies to every Serre tensor construction, not just to framing objects. Thus, writing $z = \left(\begin{smallmatrix} & y\varpi \\ y & \end{smallmatrix}\right)$, we obtain that

$$\mathcal{M}_C \cap \mathcal{Z}(z) = \mathcal{Z}(y) = \{(Y,\iota,\rho) \in \mathcal{M}_C \mid \rho y \rho^{-1} \in \mathrm{End}(Y)\}. \tag{13.10}$$

The automorphism $c \mapsto \Pi^{-1}c\Pi$ of C satisfies $\Pi^{-1}\varpi\Pi = \varpi$, but its effect on F_4 depends on λ: It is given by $\Pi^{-1}a\Pi = \sigma^{-1}(a)$ if $\lambda = 1/4$ and by $\Pi^{-1}a\Pi = \sigma^{-3}(a)$ if $\lambda = 3/4$. For both choices of λ we define, with $\Pi = \Pi_\lambda \in D_\lambda$,

$$S_\lambda = \{y \in \mathrm{End}^0_F(\mathbb{Y})^\times \mid ycy^{-1} = \Pi_\lambda^{-1}c\Pi_\lambda \text{ for } c \in C\}. \tag{13.11}$$

Let S be the union $S_{1/4} \sqcup S_{3/4}$. Then $\varpi S_\lambda = S_{\lambda+1/2}$ and, for every $y \in S$, we have inclusions of closed subschemes of $\mathcal{M}_C$

$$\mathcal{Z}(\varpi^{-1}y) \subseteq \mathcal{Z}(y) \subseteq \mathcal{Z}(\varpi y).$$

This relates the intersection loci for the two different invariants. Note that for every $y \in S$, the element ϖy^2 lies in the centralizer $\mathrm{End}^0_C(\mathbb{Y})$, which is isomorphic to $M_2(E)$. Moreover, if $z = \left(\begin{smallmatrix} & y\varpi \\ y & \end{smallmatrix}\right)$ with $y \in S$ such that $1 + z \in G_\lambda$, then the following relation of invariant polynomials holds:

$$\mathrm{Inv}(1+z;T) = \mathrm{Inv}(y;T) := \mathrm{charred}_{\mathrm{End}^0_C(\mathbb{Y})/E}(\varpi y^2;T). \tag{13.12}$$

(Here, the right-hand side will always lie in $F[T]$.) We call $y \in S$ regular semi-simple if $\mathrm{Inv}(y;T)$ is regular semi-simple in the sense of §2. Let S_{rs} and $S_{\lambda,\mathrm{rs}}$ denote the regular semi-simple elements of S and S_λ. The main task in the following sections is to determine the formal scheme $\mathcal{Z}(y) \subseteq \mathcal{M}_C$ for $y \in S_{3/4,\mathrm{rs}}$.

13.3. Intersection locus (set-theoretic support)

Given $y \in S_{\mathrm{rs}}$, our first result describes the support $\mathcal{Z}(y)(\mathbb{F})$ in terms of Dieudonné theory. To this end, we first recall from [5] some more details on Drinfeld's isomorphism.

Construction 13.2. Let (M,F,V,ι) be the covariant O_F-Dieudonné module of a special O_C-module (Y,ι) over $\mathbb{F}$. Fix an embedding $F_4 \to \breve{F}$. Then the contained ring of integers $O_{F_4} \subset O_C$ induces a $\mathbb{Z}/4\mathbb{Z}$-grading $M = M_0 \oplus \ldots \oplus M_3$. Each summand is free of rank 2 as $O_{\breve{F}}$-module, and the operators F, ϖ and V are all homogeneous of degrees

$$\deg F = -1, \quad \deg V = 1, \quad \deg \varpi = 2.$$

It follows from the special condition that

$$[M_i : VM_{i-1}] = \begin{cases} 1 & \text{if } i \equiv 0 \bmod 2 \\ 0 & \text{if } i \equiv 1 \bmod 2 \end{cases} \tag{13.13}$$

and that $[M_i : \varpi M_{i-2}] = 1$ for all i. Since $\varpi^2 = \pi$ vanishes on $\mathrm{Lie}(Y) = M/VM$, there exists an index $i \in \{0,2\}$ such that $\varpi M_i = V^2 M_i$. Such indices are called critical.

The existence of critical indices implies that the operator $\tau = V^{-2}\varpi$ is homogeneous of degree 0 and σ^2-linear with all slopes 0. We put $\Lambda_i = M_i^{\tau=\mathrm{id}}$, which is a free rank 2 module over $O_E = O_{\breve{F}}^{\sigma^2=\mathrm{id}}$. There are two cases:

$$O_{\breve{F}} \otimes_{O_E} \Lambda_i = \begin{cases} M_i & \text{if } i \text{ critical} \\ \varpi M_{i-2} & \text{if } i \text{ not critical.} \end{cases} \tag{13.14}$$

Assume that i is critical. We obtain a line $\ell = \varpi M_{i-2}/\pi M_i \in \mathbb{P}(\Lambda_i/\pi\Lambda_i)(\mathbb{F})$, and the triple (i, Λ_i, ℓ) allows for a unique (up to isomorphism) reconstruction of (M, F, V, ι).

Let us now bring the framing object $(\mathbb{Y}, \iota)$ into play. Denote its isocrystal by (N, F, V, ι). We have seen that $\tau := V^{-2}\varpi$ is of degree 0 and σ^2-linear with all slopes 0. Put $W = N_0^{\tau=\mathrm{id}}$, which is a 2-dimensional E-vector space. (Recall that this is a general statement: If N is an n-dimensional $\breve{F}$-vector space and $\tau : N \to N$ a σ^t-linear automorphism with all slopes 0, then $N^{\tau=\mathrm{id}}$ is an n-dimensional F_t-vector space where $F_t = \breve{F}^{\sigma^t=\mathrm{id}}$ is the degree t unramified field extension of F. Moreover, we have $(\breve{F} \otimes_{F_t} N^{\tau=\mathrm{id}}, \sigma^t \otimes \mathrm{id}) \xrightarrow{\sim} (N, \tau)$.)

We may define a map $\mathcal{M}_C(\mathbb{F}) \to \breve{\Omega}_E(W)(\mathbb{F})$ by the following construction: An $\iota(O_C)$-stable and special Dieudonné lattice $M \subseteq N$ with $\Lambda_i = M_i^{\tau=\mathrm{id}}$ as above is sent to

$$\begin{cases} \varpi M_2/\pi M_0 \in \mathbb{P}(\Lambda_0/\pi\Lambda_0)(\mathbb{F}) & \text{if 0 is critical} \\ M_0/\varpi^{-1}(\pi M_2) \in \mathbb{P}(\varpi^{-1}\Lambda_2/\varpi^{-1}(\pi\Lambda_2))(\mathbb{F}) & \text{if 2 is critical.} \end{cases} \tag{13.15}$$

It may happen, of course, that both indices are critical. In this case, the two lines in (13.15) coincide as points of $\breve{\Omega}_E(W)$ and correspond to the diagram

$$\begin{tikzcd} \varpi\Lambda_2 \arrow[r] \arrow[d, two heads] & \Lambda_0 \arrow[r] \arrow[d, two heads] & \varpi^{-1}\Lambda_2 \arrow[d, two heads] \\ \varpi M_2/\pi M_0 \arrow[r, "0"] & M_0/\varpi M_2 \arrow[r, "0"] & \varpi^{-1}M_2/M_0, \end{tikzcd}$$

where the lower outer terms have to be identified along $\pi : \varpi^{-1}M_2/M_0 \cong \varpi M_2/\pi M_0$. The restriction of the map $\mathcal{M}_C(\mathbb{F}) \to \breve{\Omega}_E(W)(\mathbb{F})$ to the height 0 connected component $\mathcal{M}_C^0(\mathbb{F})$ agrees with the map from Drinfeld's isomorphism in Theorem 12.2.

Definition 13.3. Given $y \in S_\lambda$, consider its action on the isocrystal (N, F, V, ι) of $(\mathbb{Y}, \iota)$. Then y is homogeneous of degree

$$\deg y = \begin{cases} -1 & \text{if } \lambda = 1/4 \\ 1 & \text{if } \lambda = 3/4. \end{cases}$$

Define $w(y) = Vy$ if $\lambda = 1/4$ and $w(y) = V^{-1}y$ if $\lambda = 3/4$. Then $w(y)$ is of degree 0 and commutes with V and ϖ. It hence commutes with τ and acts as a E-conjugate linear endomorphism on $W = N_0^{\tau=\mathrm{id}}$.

In the following, we will also formulate some results for the invariant $1/4$ case. We will not use these again but hope that they clarify why and in which sense the two possibilities for λ are different.

Lemma 13.4. *Let $(Y, \iota, \rho) \in \mathcal{M}_C(\mathbb{F})$ be a point with Dieudonné lattice $M = M_0 \oplus \ldots \oplus M_3 \subset N$. Let $i \in \{0, 2\}$ be a critical index of (Y, ι) and let $\Lambda = \Lambda_0$ (if $i = 0$) or $\Lambda = \varpi^{-1}\Lambda_2$ (if $i = 2$) be the resulting lattice $\Lambda \subset W$. Let $\ell \in P_\Lambda(\mathbb{F})$ be the line defined by (13.15). Then $(Y, \iota, \rho) \in \mathcal{Z}(y)$ if and only if*

$$\begin{cases} w(y)\Lambda \subseteq \Lambda \text{ and } w(y)\ell = 0 \text{ and } \mathrm{Im}(w(y)) \subseteq \ell & \text{if } \lambda = 1/4 \\ w(y)\Lambda \subseteq \Lambda \text{ and } w(y)\ell \subseteq \ell & \text{if } \lambda = 3/4. \end{cases} \tag{13.16}$$

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Proof. Assume first that $\lambda = 1/4$ so that $\deg y = -1$. Using that $VM_0 = M_1$ and $VM_2 = M_3$ (see (13.13)), we check that

(1) $yM_{i+1} \subseteq M_i$ if and only if $yVM_i \subseteq M_i$ (i.e., $w(y)\Lambda \subseteq \Lambda$),
(2) $yM_{i+2} \subseteq M_{i+1}$ if and only if $y\varpi M_{i+2} \subseteq \varpi VM_i$, meaning $Vy\varpi M_{i+2} \subseteq \pi M_i$ (i.e., $w(y)\ell = 0$),
(3) $yM_{i+3} \subseteq M_{i+2}$ if and only if $Vy\varpi M_{i+2} \subseteq \varpi M_{i+2}$ (i.e., $w(y)\ell \subseteq \ell$), and
(4) $yM_i \subseteq M_{i+3}$ if and only if $VyV^{-2}\varpi M_i \subseteq \varpi M_{i+2}$ (i.e., $\mathrm{Im}(w(y) \subseteq \ell)$.
If $\lambda = 3/4$, however, then $\deg y = 1$ and we obtain slightly different conditions:
(1) $yM_i \subseteq M_{i+1}$ if and only if $V^{-1}yM_i \subseteq M_i$ (i.e., $w(y)\Lambda \subseteq \Lambda$),
(2) $yM_{i+1} \subseteq M_{i+2}$ if and only if $V^{-1}yM_i \subseteq V^{-2}M_{i+2}$ which is redundant after (1),
(3) $yM_{i+2} \subseteq M_{i+3}$ if and only if $V^{-1}y\varpi M_{i+2} \subseteq \varpi M_{i+2}$ (i.e., $w(y)\ell \subseteq \ell$), and
(4) $yM_{i+3} \subseteq M_i$ if and only if $V^{-1}y\varpi M_{i+2} \subseteq V^{-2}\varpi M_i = M_i$ which is redundant after (3).
These two lists of properties are precisely what was claimed in (13.16). □

For a homothety class of lattices $\Lambda \subset W$ and $y \in S$, we define

$$n(y, \Lambda) := n(w(y), \Lambda), \quad m(y, \Lambda) := \max\{0, n(y, \Lambda)\}. \tag{13.17}$$

Also let $\mathcal{Z}(y)^0 := \mathcal{Z}(y) \cap \mathcal{M}_C^0$. Lemma 13.4 shows that, under the isomorphism $\mathcal{M}_C^0 \cong \breve{\Omega}_E(W)$,

$$\mathcal{Z}(y)^0(\mathbb{F}) \subseteq \bigcup_{\Lambda \subseteq W,\ n(y,\Lambda) \geq 0} P_\Lambda(\mathbb{F}).$$

Recall that $\mathcal{T}(w(y)) \subset \mathcal{B}$ denotes the set of those homothety classes of lattices $\Lambda \subseteq W$ in which $n(y, \Lambda)$ takes its maximum and that the shape of $\mathcal{T}(w(y))$ was described in Theorem 11.10. The next corollary combines this result with Lemma 13.4.

Corollary 13.5. *Assume that $y \in S_{\mathrm{rs}}$ is regular semi-simple. The set $\mathcal{Z}(y)^0(\mathbb{F})$ has the following description, in dependence on λ and the maximum $n(y) = \max_{\Lambda \subseteq W} n(y, \Lambda)$ of the multiplicity function.*

(1) *Assume $n(y) < 0$. Then $\mathcal{Z}(y)^0 = \emptyset$.*

(2) *Assume $n(y) = 0$ and $\lambda = 1/4$. Then $\mathcal{Z}(y)^0 \neq \emptyset$ if and only if $w(y)$ is topologically nilpotent. In this case, $\mathcal{T}(w(y))$ is an edge and $\mathcal{Z}(y)^0(\mathbb{F})$ the corresponding superspecial point.*

(3) *Assume $n(y) = 0$ as well as $\lambda = 3/4$ and $\det(w(y)^2) \in O_E^\times$. Then $\mathcal{Z}(y)^0 \cap P_\Lambda(\mathbb{F}) \neq \emptyset$ if and only if $\Lambda \in \mathcal{T}(w(y))$. In the nonempty case,*

$$|\mathcal{Z}(y)^0 \cap P_\Lambda(\mathbb{F})| = q + 1. \tag{13.18}$$

Moreover, for every edge of $\mathcal{T}(w(y))$, the corresponding superspecial point lies in $\mathcal{Z}(y)^0(\mathbb{F})$.

(4) *Assume $n(y) = 0$ as well as $\lambda = 3/4$ and $\det(w(y)^2) \in O_E \setminus O_E^\times$. Then $\mathcal{Z}(y)^0(\mathbb{F})$ consists of the superspecial points that correspond to edges of $\mathcal{T}(w(y))$.*

(5) *Assume $n(y) \geq 1$. Then*

$$\mathcal{Z}(y)^0(\mathbb{F}) = \bigcup_{\Lambda \in \{\Lambda | n(y,\Lambda) \geq 1\}} P_\Lambda(\mathbb{F}).$$

Proof. Cases (1) and (5) follow immediately from Lemma 13.4. For case (2), observe that (13.16) implies that $\mathcal{Z}(y)^0 \neq \emptyset$ can only hold if $w(y)$ is topologically nilpotent. Then Proposition 11.8 (1) ensures that $\mathcal{T}(w(y))$ is an edge, and Lemma 11.1 shows that the unique $w(y)$-stable line in $P_\Lambda(\mathbb{F})$ is the corresponding superspecial point. By Lemma 13.4, this point lies in $\mathcal{Z}(y)^0(\mathbb{F})$.

We turn to cases (3) and (4). Lemma 13.4 states that $\mathcal{Z}(y)^0 \cap P_\Lambda(\mathbb{F})$ is nonempty only if $w(y)\Lambda \subseteq \Lambda$. Under the assumption $n(y) = 0$, this is equivalent to $\Lambda \in \mathcal{T}(w(y))$. Moreover, if $w(y)\Lambda \subseteq \Lambda$, then it states that $\mathcal{Z}(y)^0 \cap P_\Lambda(\mathbb{F})$ equals the set of $w(y)$-stable lines $\ell \in P_\Lambda(\mathbb{F})$.

In case (3), if $\Lambda \in \mathcal{T}(w(y))$, then $w(y)$ defines a σ-linear automorphism of $\mathbb{F} \otimes_{\mathbb{F}_{q^2}} \Lambda$. It is a simple fact that every σ-linear automorphism of a two-dimensional $\mathbb{F}$-vector space has precisely $q + 1$ fixed lines,

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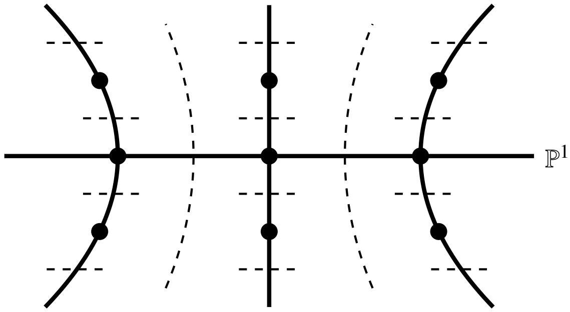

Figure 2. *Illustration of case (3) of Corollary 13.5 for L/F inert and $q = 2$. Each line represents a curve of the special fiber of $\mathcal{M}_C^0$. The four thick lines correspond to the homothety classes Λ with $n(y, \Lambda) = 0$. Their dual graph is depicted on the left in Figure 1. The scheme $\mathcal{Z}(y)^0$ consists of $q + 1$ points on each thick line. For the central curve, these points are all superspecial. For the remaining $q + 1$ curves, one point is superspecial and the other q are non-superspecial.*

so we obtain Identity (13.18). Moreover, edges emanating from Λ in $\mathcal{T}(w(y))$ are in bijection with the $\mathbb{F}_{q^2}$-rational $w(y)$-fixed points in $P_\Lambda(\mathbb{F})$. In particular, each such edge defines a point of $\mathcal{Z}(y)^0 \cap P_\Lambda(\mathbb{F})$.

The arguments for case (4) are the same. The only difference is that $w(y)|_\Lambda$, for $\Lambda \in \mathcal{T}(w(y))$, is not invertible anymore. This implies that every $w(y)$-stable line $\ell \in P_\Lambda(\mathbb{F})$ is defined over $\mathbb{F}_{q^2}$ and hence comes from an edge of Λ in $\mathcal{T}(w(y))$ as claimed. □

13.4. Cartier theory

Let R be an O_F-algebra in which π is nilpotent. Then formal π-divisible groups over R are equivalent to reduced O_F-Cartier modules over R; cf. [10, §1]. We will use this equivalence to compute the scheme structure of $\mathcal{Z}(y)$ and, to this end, collect some general results in this section.

We denote by $W_{O_F}(R)$ the ring of O_F-Witt vectors of R with respect to the chosen uniformizer π as in [10, §1]. For $x \in W_{O_F}(R)$, we write ${}^F x$ and ${}^V x$ for Frobenius and Verschiebung. They satisfy the relations

$$ {}^F[r] = [r^q], \quad {}^{FV}x = \pi x, \quad \left({}^V x\right) \cdot y = {}^V\left(x \cdot {}^F y\right). $$

We denote by $E(R) = W_{O_F}(R)[F, V]$ the O_F-Cartier ring over R. It is the non-commutative ring generated over $W_{O_F}(R)$ by two elements F and V that satisfy the relations (where $x \in W_{O_F}(R)$)

$$ VxF = {}^V x, \quad Fx = {}^F xF, \quad xV = V^F x, \quad FV = \pi. $$

The third relation implies that if M is an $E(R)$-left module, then the action of $W_{O_F}(R)$ on M/VM factors through $W(R)/{}^V W(R) = R$.

Definition 13.6. A reduced O_F-Cartier module over R is an $E(R)$-left module M such that $V : M \to M$ is injective, such that $M = \lim_{i \geq 0} M/V^i M$ and such that M/VM finite locally free over R.

In the following, we will simply say 'Witt vectors' and 'Cartier module' instead of 'O_F-Witt vectors' and 'reduced O_F-Cartier module'. This shall never lead to confusion.

Let M be a Cartier module over R such that M/VM is free over R. Let $\gamma_1, \ldots, \gamma_d \in M$ be elements that reduce to an R-basis of M/VM. Such a tuple is called a V-basis for M. Then for every element $m \in M$, there are unique coefficients $c_{n,i} \in R$, $n \geq 0$, $i = 1, \ldots, d$, such that

$$ m = \sum_{n \geq 0} \sum_{i=1}^{d} V^n [c_{n,i}] \gamma_i. \tag{13.19} $$

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Conversely, by the V-completeness of M, every tuple of coefficients $c_{n,i} \in R$ defines an element of M by (13.19). In particular, there are unique elements $r_{i,j,n} \in R$ such that

$$F\gamma_i = \sum_{n\geq 0} \sum_{j=1}^{d} V^n[r_{i,j,n}]\gamma_j. \tag{13.20}$$

These $r_{i,j,n}$ are called the structure constants of $(M, \gamma_1, \dots, \gamma_d)$, and they uniquely determine the $E(R)$-module structure on M. More precisely, by [50, Theorem 4.39], the $E(R)$-linear map $E(R)^d \to M$, $e_i \mapsto \gamma_i$ is surjective with kernel generated by the elements

$$Fe_i - \sum_{n\geq 0} \sum_{j=1}^{d} V^n[r_{i,j,n}]e_j, \qquad i = 1, \dots, d. \tag{13.21}$$

By (13.19), an $E(R)$-linear endomorphism $f : M \to M$ is uniquely determined by the images $m_i = f(\gamma_i)$. These may be (non-uniquely) lifted to elements $\widetilde{m}_i \in E(R)^d$. Conversely, let $(\widetilde{m}_1, \dots, \widetilde{m}_d) \in E(R)^d$ be a tuple of elements with images $(m_1, \dots, m_d) \in M^d$. Then the $E(R)$-linear map $E(R)^d \to E(R)^d$, $e_i \mapsto \widetilde{m}_i$ descends to M if and only if it preserves the relations (13.21), meaning that for all $i = 1, \dots, d$, the following relation holds in M:

$$Fm_i = \sum_{n\geq 0} \sum_{j=1}^{d} V^n[r_{i,j,n}]m_i. \tag{13.22}$$

Recall that $\pi - [\pi] = {}^V\xi$ for a unit $\xi \in W_{O_F}(O_F)^\times$. Using the relation ${}^V\xi = V\xi F$ in $E(R)$, we can multiply (13.20) with $V\xi$ to obtain a description of multiplication by π, say

$$\pi\gamma_i = [\pi]\gamma_i + \sum_{n\geq 1} \sum_{j=1}^{d} V^n[s_{i,j,n}]\gamma_j. \tag{13.23}$$

Multiplication by V on a Cartier module is injective by definition so the coefficients $s_{i,j,n}$ determine the structure constants $r_{i,j,n}$ uniquely. Moreover, the relation (13.22) holds for a tuple $(m_1, \dots, m_d)$ if and only if the analogous relation for multiplication by π holds,

$$\pi m_i = [\pi]m_i + \sum_{n\geq 1} \sum_{j=1}^{d} V^n[s_{i,j,n}]m_i. \tag{13.24}$$

Proposition 13.7. *Let M be a Cartier module over R such that M/VM is free over R. Let $\gamma_1, \dots, \gamma_d \in M$ be a V-basis and let $(m_1, \dots, m_d) \in M^d$ be a tuple of elements. Let $f : M \to M$ be the map of sets defined by*

$$\sum_{n\geq 0} \sum_{i=1}^{d} V^n[c_{n,i}]\gamma_i \longmapsto \sum_{n\geq 0} \sum_{i=1}^{d} V^n[c_{n,i}]m_i. \tag{13.25}$$

Then f defines a Cartier module endomorphism of M if and only if $(f \circ \pi)(\gamma_i) = (\pi \circ f)(\gamma_i)$ for all $i = 1, \dots, d$.

Proof. This is a reformulation of what was said in conjunction with (13.24). □

Definition 13.8. Let $\gamma_1, \dots, \gamma_d$ be a V-basis of a Cartier module M and let $(m_1, \dots, m_d) \in M^d$ be a tuple of elements. We call the map in (13.25) the V-series substitution map defined by $\gamma_i \mapsto m_i$, $i = 1, \dots, d$.

Assume that R is equipped with the structure of an $O_{\breve{F}}$-algebra and let M be the Cartier module of a special O_C-module (Y, ι) over R. Then the O_C-action provides a $\mathbb{Z}/4\mathbb{Z}$-grading

$$M = M_0 \oplus M_1 \oplus M_2 \oplus M_3, \quad \deg V = 1,\ \deg F = -1,\ \deg \varpi = 2. \tag{13.26}$$

We assume that each component of $\mathrm{Lie}(Y) = M_0/VM_3 \oplus M_2/VM_1$ is free as R-module and fix a homogeneous V-basis $\gamma_0 \in M_0$ and $\gamma_2 \in M_2$. Then the general descriptions in (13.23) and (13.24) simplify drastically because of the grading. Namely, a homogeneous element $m \in M_i$ has a V-series expansion of the form

$$m = \begin{cases} [r_0]\gamma_i + V^2[r_2]\gamma_{i+2} + V^4[r_4]\gamma_i + \dots & \text{if } i \text{ is even} \\ V[r_1]\gamma_{i-1} + V^3[r_3]\gamma_{i+1} + V^5[r_5]\gamma_{i-1} + \dots & \text{if } i \text{ is odd} \end{cases} \tag{13.27}$$

with unique coefficients $r_i \in R$. So an endomorphism f of M that is homogeneous of some odd degree i, say, is uniquely described by two V-series

$$f : \begin{cases} \gamma_0 \longmapsto V[a_1]\gamma_{i-1} + V^3[a_3]\gamma_{i+1} + V^5[a_5]\gamma_{i-1} + \dots, \\ \gamma_2 \longmapsto V[b_1]\gamma_{i+1} + V^3[b_3]\gamma_{i-1} + V^5[b_5]\gamma_{i+1} + \dots. \end{cases} \tag{13.28}$$

We ultimately care about the endomorphisms $y \in S_{3/4}$. These are precisely those endomorphisms that are homogeneous of degree $i = 1$ and commute with ϖ. Let

$$\varpi : \begin{cases} \gamma_0 \longmapsto [x_0]\gamma_2 + V^2[x_2]\gamma_0 + V^4[x_4]\gamma_2 + \dots, \\ \gamma_2 \longmapsto [y_0]\gamma_0 + V^2[y_2]\gamma_2 + V^4[y_4]\gamma_0 + \dots \end{cases} \tag{13.29}$$

be the V-series expansion of $\iota(\varpi)$. Note that $x_0 y_0 = \pi$ holds because $\varpi^2 = \pi$ acts as π on $\mathrm{Lie}(M) = M/VM$.

Corollary 13.9. *Let $i \in \{1, 3\}$ and let $a_1, a_3, a_5, \dots, b_1, b_3, b_5, \dots \in R$ be any elements. Let $f : M \to M$ be the V-series substitution map defined by (13.28). Then f defines a ϖ-linear endomorphism of M if and only if $(\varpi \circ f)(\gamma_j) = (f \circ \varpi)(\gamma_j)$ for both $j = 0, 2$.*

Proof. The 'only if' direction is clear. Conversely, the assumption implies that $(\pi \circ f)(\gamma_j) = (f \circ \pi)(\gamma_j)$ for both $j = 0, 2$ because $\varpi^2 = \pi$. Then apply Proposition 13.7. □

13.5. Intersection locus (superspecial points)

Throughout this section, let $y \in S_{3/4,\mathrm{rs}}$ be a regular semi-simple element. Our aim is to determine the scheme structure of $\mathcal{Z}(y) \subset \mathcal{M}_C$. Let $L = F[\varpi y^2]$ be the quadratic étale extension of F defined by $\mathrm{Inv}(y; T)$.

Recall from Definition 13.3 that $w(y) = V^{-1}y$ was defined as a σ-linear endomorphism of the isocrystal of $\mathbb{Y}$. We now change this notation slightly and only consider the restriction $w(y)|_W$ which we still denote by $w(y)$. Here, $W = N_0^{V^2 = \varpi}$ as in §13.3. The same change of notation applies to $w(\varpi y)$. Then there are the identities

$$w(\varpi y) = \pi w(y), \quad w(y)^2 = \pi^{-1} \cdot \varpi y^2 \quad \text{and} \quad w(\varpi y)^2 = \pi \cdot \varpi y^2. \tag{13.30}$$

In particular, we can view both $w(y)^2$ and $w(\varpi y)^2$ as elements of L. By Lemma 13.4, $\mathcal{Z}(y) \neq \emptyset$ only if $w(y)^2 \in O_L$. We from now on impose this assumption.

The arguments in this section will exploit the inclusions

$$\mathcal{Z}(\varpi^{-1} y) \subseteq \mathcal{Z}(y) \subseteq \mathcal{Z}(\varpi y).$$

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Here, ϖy and $\varpi^{-1} y$ both lie in $S_{1/4}$, and ϖy is regular semi-simple under the assumption $w(y) \in O_L$. In terms of (9.1), ϖy defines the element $g = 1 + \left(\begin{smallmatrix} & \varpi \cdot \varpi y \\ \varpi y & \end{smallmatrix} \right) \in G_{1/4,b,\mathrm{rs}}$. By (13.30), its invariant is

$$\mathrm{Inv}(g;T) = \mathrm{charred}_{M_2(E)/E}(\pi \cdot \varpi y^2) = \mathrm{Inv}(1 + w(\varpi y);T). \tag{13.31}$$

All results of §12 apply to the elements $g \in G_{1/4,b,\mathrm{rs}}$ and $1 + w(\varpi y) \in GL_F(W)_{\mathrm{rs}}$. In particular, the equality of invariants (13.31) shows that the following three schemes are all isomorphic:

$$\mathcal{Z}(\varpi y)^i \;\cong\; \mathcal{M}_C^i \cap g \cdot \mathcal{M}_C^i \underset{\S 12}{\cong} \breve{\Omega}_E(W) \cap (1 + w(\varpi y)) \cdot \breve{\Omega}_E(W). \tag{13.32}$$

Here, $i \in 4\mathbb{Z}$ and $\mathcal{Z}(\varpi y)^i$ as well as $\mathcal{M}_C^i$ again denote the loci where the height of ρ equals i.

After these preparations, we now formulate and prove our results. The following three propositions will respectively concern the case where $w(y)^2$ lies in $O_L^\times$, in $O_L \setminus (O_L^\times \cup \pi^2 O_L)$, or in $\pi^2 O_L$. This matches the three cases (3), (4) and (5) in Corollary 13.5.

Proposition 13.10. *Let $y \in S_{3/4,\mathrm{rs}}$ be regular semi-simple with $w(y)^2 \in O_L^\times$. Then $\mathcal{Z}(y)$ is artinian and each connected component is of length* 1.

Proof. Let $z \in \mathcal{Z}(y)(\mathbb{F})$ be any point. Using the action of H_b, we may assume $z \in \mathcal{Z}(y)^0$. Let R be the complete local ring $\widehat{\mathcal{O}}_{\mathcal{M}_C,z}$, let $\mathfrak{m}$ be its maximal ideal, and let $I \subset R$ be the ideal defining $\mathcal{O}_{\mathcal{Z}(y),z}$.

By Corollary 13.5, if $z \in P_\Lambda(\mathbb{F})$, then $n(w(y),\Lambda) = 0$. It follows that $n(w(\varpi y),\Lambda) = 1$. Moreover, by (13.30), the assumption $w(y)^2 \in O_L^\times$ implies that $w(\varpi y)^2 \in \pi^2 O_L^\times$. By Propositions 12.9 and 12.11 as well as (13.32), $\mathcal{Z}(\varpi y)$ equals the vanishing locus $V(\pi)$ in a neighborhood of z. Since $\mathcal{Z}(y) \subseteq \mathcal{Z}(\varpi y)$, this implies $\pi \in I$, and it is left to show that the structure map $O_{\breve{F}} \to R/I$ is formally unramified.

Consider for this a square-zero thickening $S \twoheadrightarrow R/\mathfrak{m}$ endowed with trivial PD-structure. Denote by $\mathcal{D} = \mathcal{D}_0 \oplus \ldots \oplus \mathcal{D}_3$ the evaluation of the Grothendieck–Messing O_F-crystal of the special O_C-module (Y,ι) over $R/\mathfrak{m}$. Then y lifts to a degree 1 homomorphism $y : \mathcal{D} \to \mathcal{D}$. We claim that if $w(y)^2 \in O_L^\times$, then there is at most one possibility for a y-stable and O_C-stable Hodge filtration $\mathcal{F} \subset \mathcal{D}$. Namely, any such filtration would be graded $\mathcal{F} = \mathcal{F}_0 \oplus \ldots \oplus \mathcal{F}_3$ with $\mathcal{F}_i \subseteq \mathcal{D}_i$ and

$$\mathrm{rk}_S(\mathcal{F}_i) = \begin{cases} 1 & \text{if } i = 0,2 \\ 2 & \text{if } i = 1,3. \end{cases}$$

So we already have $\mathcal{F}_i = \mathcal{D}_i$ if $i = 1,3$. Furthermore, $w(y)^2 \in O_L^\times$ implies that the map $y : \mathcal{D}_{i-1} \to \mathcal{D}_i$ has rank 1 mod $\mathfrak{m}$ if $i = 0,2$. (This can be read off from the Dieudonné module.) Then we have at most the possibility $\mathcal{F}_i = y\mathcal{D}_{i-1}$ for $i = 0,2$, proving both the claim and the proposition. (The images $y\mathcal{D}_{i-1}$ need not be line bundles in general, which provides an additional obstruction to deforming (Y,ι,y). This does not matter for the argument, however.) □

Proposition 13.11. *Let $y \in S_{3/4,\mathrm{rs}}$ be regular semi-simple with $w(y)^2 \in O_L \setminus (O_L^\times \cup \pi^2 O_L)$. Then $\mathcal{Z}(y)$ is artinian and has the following properties:*

(1) *If L is a field, then each connected component of $\mathcal{Z}(y)$ has length $2 + 2q$.*
(2) *If $L \cong F \times F$ is split, then each connected component of $\mathcal{Z}(y)$ has length q.*

Proof. Let $z \in \mathcal{Z}(y)$ be any point. Using the action of H_b, we may assume that $z \in \mathcal{Z}(y)^0$. Corollary 13.5 (4) states that $\mathcal{Z}(y)^0(\mathbb{F})$ consists only of superspecial points, so z is superspecial and the complete local ring $\widehat{\mathcal{O}}_{\mathcal{Z}(y),z}$ artinian.

Let $R = \widehat{\mathcal{O}}_{\mathcal{M}_C,z}$ be the complete local ring of $\mathcal{M}_C$ in z. It is isomorphic to $O_{\breve{F}}[[\mathbf{u},\mathbf{v}]]/(\mathbf{uv} - \pi)$. The elements $\mathbf{u},\mathbf{v} \in R$ are uniquely determined up to interchanging them and/or scaling them in the way $(\mathbf{u},\mathbf{v}) \mapsto (t\mathbf{u}, t^{-1}\mathbf{v})$ for a unit $t \in R^\times$. The following auxiliary result allows to make a matching choice of coordinates for both R and the Cartier module M of the universal special O_C-module over R.

Lemma 13.12. *For a suitable choice of coordinates* $\mathbf{u}, \mathbf{v} \in R$ *and a suitable* V*-basis* $\gamma_0 \in M_0$, $\gamma_2 \in M_2$, *the* V*-series presentation of* ϖ *is*

$$\varpi : \begin{cases} \gamma_0 \longmapsto [\mathbf{u}]\gamma_2 + V^2\gamma_0 \\ \gamma_2 \longmapsto [\mathbf{v}]\gamma_0 + V^2\gamma_2. \end{cases} \tag{13.33}$$

Proof. Given a V-basis $\gamma_0 \in M_0$ and $\gamma_2 \in M_2$ the V-series presentation of ϖ has the form

$$\varpi : \begin{cases} \gamma_0 \longmapsto [a_0]\gamma_2 + V^2[a_2]\gamma_0 + V^4[a_4]\gamma_2 + \dots \\ \gamma_2 \longmapsto [b_0]\gamma_0 + V^2[b_2]\gamma_2 + V^4[b_4]\gamma_0 + \dots \end{cases} \tag{13.34}$$

with $a_0 b_0 = \pi$. If $a_2 = b_2 = 1$ and if all higher coefficients vanish, then we may put $\mathbf{u} = a_0$ and $\mathbf{v} = b_0$, and we are done. So our aim is to arrange this situation for all coefficients in degree ≥ 2.

Write $M^{(n)}$ for the Cartier module obtained by base change to $R/\mathfrak{m}^n$. Then $M^{(0)}$ is precisely the Dieudonné module of the π-divisible group over the closed point. The point z in question is superspecial, meaning both indices are critical, so $V^2 M^{(0)} = \varpi M^{(0)}$. It follows that there is a V-basis $\gamma_0^{(0)} \in M_0^{(0)}$, $\gamma_2^{(0)} \in M_2^{(0)}$ with $\varpi\gamma_i^{(0)} = V^2\gamma_{i+2}^{(0)}$. We prove by induction on n, using the $\mathfrak{m}$-adic completeness of R, that such a V-basis may be lifted to one as required. This is quite standard:

Assume we already found a V-basis $\gamma_0^{(n)}, \gamma_2^{(n)}$ of $M^{(n)}$ such that $\varpi\gamma_0^{(n)} = [a_0]\gamma_2^{(n)} + V^2\gamma_0^{(n)}$ and $\varpi\gamma_2^{(n)} = [b_0]\gamma_0^{(n)} + V^2\gamma_2^{(n)}$. Let $\widetilde{\gamma}_i^{(n)} \in M_i^{(n+1)}$ be any lifts and write

$$\varpi : \begin{cases} \widetilde{\gamma}_0^{(n)} \longmapsto [a_0]\widetilde{\gamma}_2^{(n)} + V^2\widetilde{\gamma}_0^{(n)} + V^2\delta_0 \\ \widetilde{\gamma}_2^{(n)} \longmapsto [b_0]\widetilde{\gamma}_0^{(n)} + V^2\widetilde{\gamma}_2^{(n)} + V^2\delta_2 \end{cases}$$

with $\delta_i \in \ker(M^{(n+1)} \to M^{(n)})$. Any element ε in this kernel satisfies $\varpi\varepsilon = 0$ and $[r]\varepsilon = 0$, for $r \in \mathfrak{m}$. So we obtain our desired lifting as $\gamma_i^{(n+1)} = \widetilde{\gamma}_i^{(n)} + \delta_i$. □

Let $\mathfrak{m} \subset R$ be the maximal ideal and let $I \subseteq R$ be the ideal with $R/I = \mathcal{O}_{\mathcal{Z}(y),z}$. Our aim is to prove that the length of R/I is $2+2q$ if L is a field and q if L is split. Since $2+2q < q^3$, it suffices to prove that $R/(I+\mathfrak{m}^{q^3})$ is of the desired length. Moreover, by Theorem 12.13, the ideal in R that defines $\mathcal{O}_{\mathcal{Z}(\varpi y),z}$ is $\pi(\mathbf{u}, \mathbf{v})$ if L is a field or (π) if L is split. So we know a priori that $\pi(\mathbf{u}, \mathbf{v}) \subseteq I$.

Let $\overline{R} := R/(\mathfrak{m}^{q^3} + \pi(\mathbf{u}, \mathbf{v}))$ and let $\overline{I} = I\overline{R}$ be the ideal that defines $V(\mathfrak{m}^{q^3}) \cap \mathcal{Z}(y)$. We have reduced to the problem to showing that the length of $\overline{R}/\overline{I}$ is $2+2q$ resp. q. Let $\overline{M} = E(\overline{R}) \otimes_{E(R)} M$ be the reduction of M to $\overline{R}$.

We assume from now on that $\mathbf{u}, \mathbf{v} \in R$ and $\gamma_0, \gamma_2 \in M$ are chosen as in Lemma 13.12. The special fiber $\mathbb{M} := E(R/\mathfrak{m}) \otimes_{E(R)} M$ of the Cartier module is the Dieudonné module of the special fiber of the special O_C-module over R. In particular, it has the property that $V^4\mathbb{M} = \pi\mathbb{M}$ because, for a superspecial point, both indices are critical. Let $a_1, b_1, a_3, b_3, \dots \in \mathbb{F}$ be the coefficients of the V-series that define $y \in \mathrm{End}(\mathbb{M})$ in the sense of (13.28) – that is,

$$y : \begin{cases} \gamma_0 \longmapsto V[a_1]\gamma_0 + V^3[a_3]\gamma_2 + V^5[a_5]\gamma_0 + \dots, \\ \gamma_2 \longmapsto V[b_1]\gamma_2 + V^3[b_3]\gamma_0 + V^5[b_5]\gamma_2 + \dots. \end{cases} \tag{13.35}$$

Lemma 13.13. *The coefficients in (13.35) have the following properties.*

(1) *If* L *is a field, then* $a_1, b_1 = 0$ *while* $a_3, b_3 \in \mathbb{F}^\times$.
(2) *If* $L = F \times F$, *then one out of* a_1, b_1 *vanishes and the other lies in* $\mathbb{F}^\times$.

Proof. Assume first that L is a field. Then $w(y)^2 \in O_L \setminus O_L^\times$ implies that $w(y) = V^{-1}y$ is topologically nilpotent. This provides $a_1 = b_1 = 0$. As $V^2\mathbb{M} = \varpi\mathbb{M}$ because z is superspecial, we have that $\gamma_i, V^2\gamma_{i-2}$ provide an $O_{\breve{F}}$-basis for $\mathbb{M}_i/\pi\mathbb{M}_i$. Since by assumption $w(y)^2 \notin \pi^2 O_L$, the maps $y : \mathbb{M}_i/\pi\mathbb{M}_i \to$

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$\mathbb{M}_{i+1}/\pi\mathbb{M}_{i+1}$, for $i = 0, 2$, are nonzero. So we find that at least one out of the pair a_1, a_3, resp. b_1, b_3 is nonzero. We have already seen that $a_1 = b_1 = 0$, so we necessarily have $a_3, b_3 \in \mathbb{F}^\times$, as claimed.

Assume now that $L \cong F \times F$ and write $w(y)^2 = (y_1, y_2)$. Then $v(y_1)$ and $v(y_2)$ both lie in $2\mathbb{Z}$ because $1 + w(y) \in GL_4(F)$; compare with row 3 of Table 1. The assumption $w(y)^2 \notin \pi^2 O_L$ thus implies that $w(y) = V^{-1}y$ is not topologically nilpotent. Equivalently, at least one out of $\{a_1, b_1\}$ is invertible. It cannot happen that both coefficients are invertible, however, because this would imply $w(y)^2 \in O_L^\times$, which is excluded by assumption. This finishes the proof the lemma. □

Consider now any sequence of elements $a_1, b_1, a_3, b_3, \ldots \in \overline{R}$ that lift the coefficients in (13.35). (It will not lead to confusion that we denote them by the same symbols.) Let $\widetilde{y} : \overline{M} \to \overline{M}$ be the map of sets that is given by the V-series substitution

$$\widetilde{y} : \begin{cases} \gamma_0 \longmapsto V[a_1]\gamma_0 + V^3[a_3]\gamma_2 + V^5[a_5]\gamma_0 + \ldots, \\ \gamma_2 \longmapsto V[b_1]\gamma_2 + V^3[b_3]\gamma_0 + V^5[b_5]\gamma_2 + \ldots. \end{cases} \tag{13.36}$$

It lifts the map $y \in \mathrm{End}(\mathbb{M})$ by definition. By Corollary 13.9, $\widetilde{y} \in \mathrm{End}(E(\overline{R}/J) \otimes_{E(R)} M)$ for the ideal $J \subseteq \overline{R}$ that is generated by all coefficients of the two V-series $(\widetilde{y} \circ \varpi)(\gamma_0) - (\varpi \circ \widetilde{y})(\gamma_0)$ and $(\widetilde{y} \circ \varpi)(\gamma_2) - (\varpi \circ \widetilde{y})(\gamma_2)$.

We next make these V-series more explicit. Here, it will pay off that we are working with $\overline{R}$ instead of R: In $E(\overline{R})$, it holds that $[\mathbf{u}]V^n = V[\mathbf{u}^{q^n}] = 0$ and $[\mathbf{v}]V^n = V^n[\mathbf{v}^{q^n}] = 0$ whenever $n \geq 3$. The V-series expansions of $\widetilde{y} \circ \varpi$ and $\varpi \circ \widetilde{y}$ then become

$$\widetilde{y} \circ \varpi : \begin{cases} \gamma_0 \longmapsto V[b_1\mathbf{u}^q]\gamma_2 + V^3[a_1]\gamma_0 + V^5[a_3]\gamma_2 + V^7[a_5]\gamma_0 + \ldots, \\ \gamma_2 \longmapsto V[a_1\mathbf{v}^q]\gamma_0 + V^3[b_1]\gamma_2 + V^5[b_3]\gamma_0 + V^7[b_5]\gamma_2 + \ldots \end{cases} \tag{13.37}$$

and

$$\varpi \circ \widetilde{y} : \begin{cases} \gamma_0 \longmapsto V[a_1\mathbf{u}]\gamma_2 + V^3[a_3\mathbf{v}]\gamma_0 + V^5[a_5\mathbf{u}]\gamma_2 + V^7[a_7\mathbf{v}]\gamma_0 + \ldots \\ \qquad\qquad + V^3[a_1^{q^2}]\gamma_0 + V^5[a_3^{q^2}]\gamma_2 + V^7[a_5^{q^2}]\gamma_0 + \ldots, \\ \gamma_2 \longmapsto V[b_1\mathbf{v}]\gamma_0 + V^3[b_3\mathbf{u}]\gamma_2 + V^5[b_5\mathbf{v}]\gamma_0 + V^7[b_7\mathbf{u}]\gamma_2 + \ldots \\ \qquad\qquad + V^3[b_1^{q^2}]\gamma_2 + V^5[b_3^{q^2}]\gamma_0 + V^7[b_5^{q^2}]\gamma_2 + \ldots. \end{cases} \tag{13.38}$$

Lemma 13.14. *Let $J \subset \overline{R}$ be an ideal such that $y \in \mathrm{End}(E(\overline{R}/J) \otimes_{E(R)} M)$. Then the length of $\overline{R}/J$ is $\leq 2 + 2q$ if L is a field and $\leq q$ if L is split.*

Proof. Note the following two properties of sums in the ring of Witt vectors and in the Cartier ring:

(a) In $W_{O_F}(R)$, a sum of Teichmüller lifts $[a] + [b]$ lies in $[a + b] + W_{O_F}(Ra + Rb)$.

(b) In the Cartier ring $E(R)$, we have

$$V^n({}^{V^k}\varepsilon) = V^{n+k}\varepsilon F^k \in V^{n+k}E(\varepsilon R).$$

These two properties imply that

$$\begin{aligned} V^3([a_3\mathbf{v}] + [a_1^{q^2}]) &\in V^3[a_3\mathbf{v} + a_1^{q^2}] + V^5E(R) \\ V^3([b_3\mathbf{u}] + [b_1^{q^2}]) &\in V^3[b_3\mathbf{u} + b_1^{q^2}] + V^5E(R). \end{aligned}$$

Thus, comparing the V-coefficients and V^3-coefficients of (13.37) and (13.38), we obtain that the following identities hold in $\overline{R}/J$:

$$\begin{aligned} a_1\mathbf{u} &= b_1\mathbf{u}^q \qquad & a_1 &= a_3\mathbf{v} + a_1^{q^2} \\ b_1\mathbf{v} &= a_1\mathbf{v}^q \qquad & b_1 &= b_3\mathbf{u} + b_1^{q^2}. \end{aligned} \tag{13.39}$$

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The next lemma simplifies these identities. If L is split, then we will only consider the case $a_1 \in \overline{R}^\times$ and $b_1 \in \mathfrak{m}$ from now on (see Lemma 13.13 above). The reverse situation is the same by symmetry.

Lemma 13.15. *Consider the two situations from Lemma 13.13.*

(1) *If a_1, $b_1 \in \mathfrak{m}$ and a_3, $b_3 \in \overline{R}^\times$, then (13.39) is equivalent to*

$$\begin{aligned} \pi &= b_3^{-1}a_3\mathbf{v}^{q+1} \qquad & a_1 &= a_3\mathbf{v} \\ \pi &= a_3^{-1}b_3\mathbf{u}^{q+1} \qquad & b_1 &= b_3\mathbf{u}. \end{aligned} \tag{13.40}$$

(2) *If $a_1 \in \overline{R}^\times$ and $b_1 \in \mathfrak{m}$, then (13.39) is equivalent to*

$$\mathbf{u} = \mathbf{v}^q = b_1 = 0, \qquad a_1 = a_3\mathbf{v} + a_1^{q^2}. \tag{13.41}$$

Proof. We begin with the case a_1, $b_1 \in \mathfrak{m}$ and a_3, $b_3 \in \overline{R}^\times$. If the two identities on the right-hand side of (13.39) hold, then $a_1 = s\mathbf{v}$ and $b_1 = t\mathbf{u}$ for units $s, t \in \overline{R}^\times$. The left-hand side identities then imply $t^{-1}s\pi = \mathbf{u}^{q+1}$ and $s^{-1}t\pi = \mathbf{v}^{q+1}$. This forces

$$\mathbf{u}^{q+2} = \mathbf{v}^{q+2} = a_1^{q+2} = b_1^{q+2} = 0$$

because we are working modulo $\pi(\mathbf{u}, \mathbf{v})$. Then we obtain $a_1 = a_3\mathbf{v}$ and $b_1 = b_3\mathbf{u}$, meaning $s = a_3$ and $t = b_3$. It follows that(13.39) implies (13.40). The converse direction is immediate.

We consider the case $a_1 \in \overline{R}^\times$ and $b_1 \in \mathfrak{m}$. First, $a_1\mathbf{u} = b_1\mathbf{u}^q$ is equivalent to $\mathbf{u} = 0$. Then $b_1 = b_3\mathbf{u} + b_1^{q^2}$ is equivalent to $b_1 = 0$. Then $b_1\mathbf{v} = a_1\mathbf{v}^q$ is equivalent to $\mathbf{v}^q = 0$, and we have arrived at (13.41). □

Let $J' \subset \overline{R}$ be the ideal generated by the relations in (13.40) and (13.41). Then $J \subseteq J'$ and $\overline{R}/J'$ has length $\leq 2 + 2q$ resp. $\leq q$, proving Lemma 13.14. □

In particular, $\overline{R}/\bar{I}$ has length $\leq 2{+}2q$ resp. $\leq q$ because Lemma 13.14 applies to $\bar{I}$ and the deformation of y to $\mathrm{End}(E(\overline{R}/\bar{I}) \otimes_{E(R)} M)$. It remains to prove the converse inequality.

Definition 13.16. Given any two $a_3, b_3 \in \overline{R}$ as in Lemma 13.15, we from now on choose $a_1, b_1 \in \overline{R}$ and the ideal $J_{a_3,b_3} \subset \overline{R}$ in the following way.

(1) If L is a field, then a_3, $b_3 \in \overline{R}^\times$ are units. We set $a_1 = a_3\mathbf{v}$ and $b_1 = b_3\mathbf{u}$, as well as

$$J_{a_3,b_3} = (\pi - b_3^{-1}a_3\mathbf{v}^{q+1},\ \pi - a_3^{-1}b_3\mathbf{u}^{q+1}).$$

(2) If $L \cong F \times F$ and if a_3, $b_3 \in \overline{R}$ are any two elements, then we set $b_1 = 0$ and we let a_1 be the unique solution to the equation $a_1 = a_3\mathbf{v} + a_1^{q^2}$ that lifts the given solution mod $\mathfrak{m}$. (Existence and uniqueness follow from the fact that this equation is étale.) We define $J_{a_3,b_3} = (\mathbf{u}, \mathbf{v}^q)$.

With choices as in Definition 13.16, the quotient $\overline{R}/J_{a_3,b_3}$ has length $2 + 2q$ if L is a field and length q if $L \cong F \times F$. Moreover, the elements a_1, b_1, a_3 and b_3 satisfy Identity (13.40) resp. Identity (13.41) modulo J_{a_3,b_3}.

Lemma 13.17. *There exist choices for the coefficients $a_3, b_3, a_5, b_5, \ldots \in \overline{R}$ in (13.36) such that, with a_1, b_1 and J_{a_3,b_3} as in Definition 13.16, $\widetilde{y} \circ \varpi = \varpi \circ \widetilde{y}$ over $\overline{R}/J_{a_3,b_3}$.*

Proof. Using the previous properties (a) and (b) of addition in $E(R)$, we see that the V^{2k+1}-coefficients of (13.38) take the form

$$\begin{cases} a_{2k+1} + a_{2k-1}^{q^2} + p_{2k+1}(a_3\mathbf{v}, a_5\mathbf{u}, a_7\mathbf{v}, \ldots, a_{2k-1}\mathbf{s}, a_1^{q^2}, a_3^{q^2}, \ldots, a_{2k-3}^{q^2}) \\ b_{2k+1} + b_{2k-1}^{q^2} + q_{2k+1}(b_3\mathbf{u}, b_5\mathbf{v}, b_7\mathbf{u}, \ldots, b_{2k-1}\mathbf{t}, b_1^{q^2}, b_3^{q^2}, \ldots, b_{2k-3}^{q^2}) \end{cases}$$

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for certain polynomials p_{2k+1}, q_{2k+1} with coefficients in R. Here, $(\mathbf{s},\mathbf{t}) = (\mathbf{u},\mathbf{v})$ or $(\mathbf{s},\mathbf{t}) = (\mathbf{v},\mathbf{u})$ in dependence on the parity of k. Thus, (13.37) equals (13.38) if and only if the following identities hold:

$$\begin{aligned} a_3 &= a_5\mathbf{u} + a_3^{q^2} + p_3(a_3\mathbf{v}, a_1^{q^2})\\ a_5 &= a_7\mathbf{v} + a_5^{q^2} + p_5(a_3\mathbf{v}, a_5\mathbf{u}, a_1^{q^2}, a_3^{q^2})\\ a_7 &= a_9\mathbf{u} + a_7^{q^2} + p_7(a_3\mathbf{v}, a_5\mathbf{u}, a_7\mathbf{v}, a_1^{q^2}, a_3^{q^2}, a_5^{q^2})\\ &\dots\\ b_3 &= b_5\mathbf{v} + b_3^{q^2} + q_3(b_3\mathbf{u}, b_1^{q^2})\\ b_5 &= b_7\mathbf{u} + b_5^{q^2} + q_5(b_3\mathbf{u}, b_5\mathbf{v}, b_1^{q^2}, b_3^{q^2})\\ b_7 &= b_9\mathbf{v} + b_7^{q^2} + q_7(b_3\mathbf{u}, b_5\mathbf{v}, b_7\mathbf{u}, b_1^{q^2}, b_3^{q^2}, b_5^{q^2})\\ &\dots \end{aligned} \qquad (13.42)$$

Assume we have shown that these two systems of equations have a unique solution in $\overline{R}/\mathfrak{m}^n\overline{R}$ that lifts the coefficients of $y \in \mathrm{End}(\mathbb{M})$. We claim that this solution lifts uniquely to $\overline{R}/\mathfrak{m}^{n+1}\overline{R}$. Consider for this the truncations of the above two systems of equations in some degree $m = 2k+1$. The summands $a_{2k+3}\mathbf{s}$ resp. $b_{2k+3}\mathbf{t}$ (with $\mathbf{s},\mathbf{t} \in \{\mathbf{u},\mathbf{v}\}$ in dependence on k) in the two last lines are already uniquely determined by the given solution over $\overline{R}/\mathfrak{m}^n\overline{R}$. In all remaining variables, the two truncated systems of equations are étale, because their Jacobian matrices are upper triangular up to nilpotent entries. The given solution thus lifts uniquely to a solution over $\overline{R}/\mathfrak{m}^{n+1}\overline{R}$, proving the lemma. □

The proof of Proposition 13.11 is now complete. Namely, Lemma 13.17 shows the existence of an ideal $J = J_{a_3,b_3} \subset \overline{R}$ such that $\overline{I} \subseteq J$ and such that the length of $\overline{R}/J$ is $2+2q$ if L is a field and q if L is split. Lemma 13.15, on the other hand, showed that the length of R/I is $\leq 2+2q$ resp. $\leq q$. These two statements together imply the proposition. □

The third type of embedded component arises in superspecial points whenever $\max_{\Lambda\subset W} n(y,\Lambda) \geq 1$.

Proposition 13.18. *Let $y \in S_{3/4,\mathrm{rs}}$ be regular semi-simple and let $\{z\} = P_1 \cap P_2$ be a superspecial point on $\mathcal{M}_C$, where $P_1, P_2 \subseteq \mathcal{M}_C$ denote two irreducible components of the special fiber. Assume that $P_1 \nsubseteq \mathcal{Z}(y)$ but $P_2 \subseteq \mathcal{Z}(y)$. Then $z \in \mathcal{Z}(y)^{\mathrm{art}}(\mathbb{F})$ and the length of $\mathcal{O}_{\mathcal{Z}(y)^{\mathrm{art}},z}$ is q.*

Proof. The proof is analogous to the one of Proposition 13.11. Let again R be the complete local ring $\widehat{\mathcal{O}}_{\mathcal{M}_C,z} = O_{\breve{F}}[[\mathbf{u},\mathbf{v}]]/(\mathbf{uv}-\pi)$ be the complete local ring in z. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $I \subset R$ be the ideal defining $\widehat{\mathcal{O}}_{\mathcal{Z}(y),z}$. Assume that $\mathbf{v}$ corresponds to P_2 (i.e., assume that $I \subseteq (\mathbf{v})$).

Claim: The ideal I is given by $I = (\pi, \mathbf{v}^{q+1})$. This claim immediately implies the proposition: The maximal Cartier divisor dividing I is $\mathbf{v}$, so we obtain $\mathcal{O}_{\mathcal{Z}(y)^{\mathrm{art}},z} = R/(\mathbf{u},\mathbf{v}^q)$ which has length q.

In order to prove the claim, we first use our results about the case of invariant $1/4$. Let $\Lambda_1, \Lambda_2 \subseteq W$ be the two lattices defined by P_1 and P_2 in (13.15). Then $n(y,\Lambda_1) = 0$ and $n(y,\Lambda_2) = 1$ by our assumptions on P_1 and P_2. It follows from $w(\varpi y) = \pi w(y)$ that $n(\varpi,\Lambda_1) = 1$ and $n(\varpi,\Lambda_2) = 2$. Thus, $\widehat{\mathcal{O}}_{\mathcal{Z}(\varpi y),z}$ is defined by the ideal $(\pi\mathbf{v})$, and we obtain

$$(\pi\mathbf{v}) \subseteq I \subseteq (\mathbf{v}).$$

Lemma 13.19. *Let $I \subseteq R$ be an ideal such that $(\pi\mathbf{v}) \subseteq I \subseteq (\mathbf{v})$ and such that $I + \mathfrak{m}^{q^2} = (\pi, \mathbf{v}^{q+1})$. Then $I = (\pi, \mathbf{v}^{q+1})$.*

Proof. Since $(\pi\mathbf{v}) \subseteq I$ by assumption, $I + \mathfrak{m}^{q^2} = I + (\mathbf{u}^{q^2}, \mathbf{v}\mathbf{u}^{q^2-1})$. Since $I \subseteq (\mathbf{v})$ and $\pi \in I + \mathfrak{m}^{q^2}$ by assumption, we may thus write

$$\pi = a\mathbf{v} + b\mathbf{u}^{q^2} + c\mathbf{v}\mathbf{u}^{q^2-1}, \quad a \in \mathfrak{m},\ a\mathbf{v} \in I,\ b,c \in R.$$

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Then b has to be divisible by $\mathbf{v}$. So after modifying c, we may assume $b = 0$. Then we obtain that $a\mathbf{v} = \pi(1 - c\mathbf{u}^{q^2-2}) = (\text{unit}) \cdot \pi \in I$. It follows that $\pi \in I$ and hence, in particular, that $\mathbf{v}\mathbf{u}^{q^2-1} \in I$.

Since $\mathbf{v}^{q+1} \in I + \mathfrak{m}^{q^2}$ by assumption, we can now write

$$\mathbf{v}^{q+1} = a\mathbf{v} + b\mathbf{u}^{q^2}, \quad a\mathbf{v} \in I,\ b \in R.$$

Then b is divisible by $\mathbf{v}$, so $b\mathbf{u}^{q^2} \in I$ by the previous results, and hence, $\mathbf{v}^{q+1} \in I$. This finishes the proof. □

Let $\overline{R} = R/(\mathfrak{m}^{q^2} + (\pi\mathbf{v}))$ and let $\overline{I} = I\overline{R}$. By Lemma 13.19, it suffices to show that $(\operatorname{Spec}\overline{R}) \cap \mathcal{Z}(y) = V(\overline{I})$. Let M be the Cartier module of the universal point over R, and assume from now on that $\mathbf{u}, \mathbf{v} \in R$ as well as $\gamma_0, \gamma_2 \in M$ are chosen as in Lemma 13.12. Let $\mathbb{M} = E(R/\mathfrak{m}) \otimes_{E(R)} M$ be the Dieudonné module of the special fiber. Let $a_1, b_1, a_3, b_3, \ldots \in \overline{R}$ be coefficients such that the V-series datum $\widetilde{y}$ from (13.36) lifts the given endomorphism $y \in \operatorname{End}(\mathbb{M})$.

Just as in the proof of Proposition 13.11, we now extract information on the coefficients from our assumptions. By way of symmetry, we assume that 0 is the index corresponding to Λ_1, meaning that $V^{-1}y\mathbb{M}_0 \not\subset \pi\mathbb{M}_0$ while $V^{-1}y\mathbb{M}_2 \subseteq \pi\mathbb{M}_2$. The second condition means that a_1, b_1 and b_3 all lie in $\mathfrak{m}\overline{R}$. The first then implies that a_3 lies in $\overline{R}^\times$.

The compositions $\varpi \circ \widetilde{y}$ and $\widetilde{y} \circ \varpi$ are again given by the identities (13.37) and (13.38). This time, since we are working over $\overline{R}$, we even obtain that $a_1^{q^2} = b_1^{q^2} = b_3^{q^2} = 0$. The system of leading term identities of $\varpi \circ \widetilde{y} = \widetilde{y} \circ \varpi$ is then given by

$$\begin{aligned} a_1\mathbf{u} &= b_1\mathbf{u}^q & a_1 &= a_3\mathbf{v} & a_3 &= a_5\mathbf{u} + a_3^{q^2} \\ b_1\mathbf{v} &= a_1\mathbf{v}^q & b_1 &= b_3\mathbf{u} & b_3 &= b_5\mathbf{v}. \end{aligned} \tag{13.43}$$

Performing the direct substitutions for a_1, b_1 and b_3 leaves the three equations

$$\begin{aligned} a_3\pi &= b_5\pi\mathbf{u}^q & a_3 &= a_5\mathbf{u} + a_3^{q^2} \\ b_5\pi\mathbf{v} &= a_3\mathbf{v}^{q+1} \end{aligned}$$

Since $a_3 \in \overline{R}^\times$, the upper left identity is equivalent to $\pi = 0$. Now for given $a_3 \in \overline{R}^\times$ and $b_3 \in \overline{R}$, we define a_1 and b_1 by (13.43) and let J be the ideal $(\pi, \mathbf{v}^{q+1})$. Then we argue precisely as in Lemma 13.17 and obtain that $\overline{I} = J$ as claimed. The proof is now complete. □

We may now extend Propositions 13.10, 13.11 and 13.18 by a general argument.

Proposition 13.20. *Let $y \in S_{3/4,\mathrm{rs}}$ be any element and let $z \in \mathcal{Z}(y)$ be a closed point. Let $R = \widehat{\mathcal{O}}_{\mathcal{M}_C, z}$ be the complete local ring in z and let $I \subset R$ be the ideal such that $R/I = \widehat{\mathcal{O}}_{\mathcal{Z}(y), z}$. Then*

$$(\operatorname{Spf} R) \cap \mathcal{Z}(\pi y) = \operatorname{Spf}(R/\pi I).$$

Proof. For $y' \in S$ such that $z \in \mathcal{Z}(y')$, we denote by $I(y') \subseteq R$ the ideal with

$$(\operatorname{Spf} R) \cap \mathcal{Z}(y') = \operatorname{Spf}(R/I(y')).$$

In this notation, our aim is to prove that $I(\pi y) = \pi I(y)$ for the given element y. Consider the Cartier module M of the universal special O_C-module over R and write

$$\pi : \begin{cases} \gamma_0 \longmapsto [\pi]\gamma_0 + V^2[u_2]\gamma_2 + V^4[u_4]\gamma_0 + \ldots, \\ \gamma_2 \longmapsto [\pi]\gamma_2 + V^2[v_2]\gamma_0 + V^4[v_4]\gamma_2 + \ldots \end{cases} \tag{13.44}$$

for the V-series expansion of multiplication by π on M. Write $y \in \mathrm{End}(E(R/I(y)) \otimes_{E(R)} M)$ as in (13.28). Choose a lifting of all its coefficients to R, say $a_1, b_1, a_3, b_3, \dots \in R$, and denote by $\widetilde{y}$ the resulting V-series substitution map $\widetilde{y} : M \longrightarrow M$. Define the obstruction $\mathrm{ob} := \mathrm{ob}(\widetilde{y}) := \varpi \circ \widetilde{y} - \widetilde{y} \circ \varpi$. (Recall that this is just an endomorphism of M as set and need not come by $E(R)$-linear extension from $\gamma_i \mapsto \mathrm{ob}(\gamma_i)$, $i = 0, 2$.) Still, by Proposition 13.7, $\widetilde{y}$ defines an endomorphism modulo some ideal $J \subset R$ if and only if $\mathrm{ob}(\gamma_i) = 0 \bmod J$. Write

$$\mathrm{ob} : \begin{cases} \gamma_0 \longmapsto V[c_1]\gamma_2 + V^3[c_3]\gamma_0 + V^5[c_5]\gamma_2 + \dots, \\ \gamma_2 \longmapsto V[d_1]\gamma_0 + V^3[d_3]\gamma_2 + V^5[d_5]\gamma_0 + \dots. \end{cases} \tag{13.45}$$

By definition of $I(y)$ and by Proposition 13.7, $I(y)$ is precisely the ideal $(c_1, c_3, \dots, d_1, d_3, \dots)$. Now $\pi \circ \widetilde{y}$ defines a lift to R of the V-series expression for $\pi \circ y$. Its obstruction, evaluated on γ_0, γ_2, is given by

$$\begin{aligned} \mathrm{ob}(\pi \circ \widetilde{y}) &= \varpi \circ (\pi \circ \widetilde{y}) - (\pi \circ \widetilde{y}) \circ \varpi \\ &= \pi \circ \mathrm{ob} \end{aligned} \tag{13.46}$$

because $\pi = \varpi^2$ and ϖ commute as (set-theoretic) endomorphisms of M. The crucial observation now is that

$$\pi \circ \mathrm{ob} : \begin{cases} \gamma_0 \longmapsto V[c_1\pi]\gamma_2 + V^3[c_3\pi]\gamma_0 + V^5[c_5\pi]\gamma_2 + \dots, \\ \gamma_2 \longmapsto V[d_1\pi]\gamma_0 + V^3[d_3\pi]\gamma_2 + V^5[d_5\pi]\gamma_0 + \dots \end{cases} \tag{13.47}$$

modulo $I(y)^{q^2}$. Namely, $[a]V^{2k} = V^{2k}[a^{q^{2k}}]$. So when substituting (13.44) into (13.45), all the higher terms

$$V^i[c_i]V^{2k}[u_{2k}]\gamma_\varepsilon, \quad V^i[d_i]V^{2k}[u_{2k}]\gamma_\varepsilon$$

vanish modulo $I(y)^{q^2}$. Thus, we obtain that

$$I(\pi y) = \pi I(y) \mod I(y)^{q^2}. \tag{13.48}$$

It is left to show that this already implies $I(\pi y) = \pi I(y)$.

The case that $I(y) \subseteq (\pi)$. Let $n \geq 1$ be such that $I(y) \subseteq (\pi)^n$ but $I(y) \nsubseteq (\pi)^{n+1}$. This integer can also be characterized by

$$n = \max_{z \in P_\Lambda} n(y, \Lambda).$$

Here, there are either one or two curves P_Λ that contain z. Since $n(\varpi y, \Lambda) = n(y, \Lambda) + 1$ for every $y \in S_{3/4}$,

$$\max_{z \in P_\Lambda} n(\varpi y, \Lambda) = n + 1.$$

By Propositions 12.9 and 12.11, the only possibilities for $I(y)$ are $(\pi)^n$, $\pi^n \mathbf{u}$, $\pi^n \mathbf{v}$ or $\pi^n(\mathbf{u}, \mathbf{v})$, where the last three are meant for a superspecial point. Moreover, these propositions also show $I(\pi^2 y) = \pi^2 I(y)$. So we find that

$$(\pi)^{n+3} \subset I(\varpi\pi y) \subseteq I(\pi y).$$

It follows that $I(y)^{q^2} \subseteq (\pi)^{nq^2} \subseteq I(\pi y)$. The inclusion $I(y)^{q^2} \subseteq \pi I(y)$ is immediately clear, so we deduce from (13.48) that $I(\pi y) = \pi I(y)$, as desired.

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Table 2. *The possible embedded components for Hasse invariant* 3/4.

	R	$I(y)$	$I(\varpi y)$
(1)	$O_{\breve F}[[t]]$	(π, t)	(π)
(2)	$O_{\breve F}[[\mathbf{u}, \mathbf{v}]]/(\mathbf{uv}-\pi)$	$(\mathbf{u}, \mathbf{v})$	(π)
(3)		$(\mathbf{u}, \mathbf{v}^q)$	(π)
(4)		$(\pi - \mathbf{u}^{q+1}, \pi - \mathbf{v}^{q+1})$	$\pi(\mathbf{u}, \mathbf{v})$
(5)		$(\pi, \mathbf{v}^{q+1})$	$(\pi\mathbf{v})$

The case that $I(y) \nsubseteq (\pi)$. In this situation, the point z is of one of the types considered in Propositions 13.10, 13.11 and 13.18. Also taking into account Propositions 12.9 and 12.11 to determine $I(\varpi y)$, the possibilities up to isomorphism are given by the following table:

We furthermore have the inclusions

$$I(\pi\varpi y) \subseteq I(\pi y) \subseteq I(\varpi y) \subseteq I(y).$$

By Propositions 12.9 and 12.11, we know that $I(\pi\varpi y) = \pi I(\varpi y)$. It can also be verified from the above table that $\pi I(y) \subseteq I(\varpi y)$ in each case, so both $I(\pi y)$ and $\pi I(y)$ are contained in $I(\varpi y)$. It then follows from (13.48) that

$$I(\pi y) = \pi I(y) \mod I(y)^{q^2} \cap I(\varpi y).$$

By Nakayama's Lemma, the identity $I(\pi y) = \pi I(y)$ follows if we can show that

$$I(y)^{q^2} \cap I(\varpi y) \subseteq \pi\mathfrak{m}I(y). \tag{13.49}$$

The reader will have no difficulty checking this relation in cases (1), (2) and (3) of Table 2 above, and we only treat the cases (4) and (5):

Case (4). First observe that

$$(\pi - \mathbf{u}^{q+1})(\pi - \mathbf{v}^{q+1}) \in \pi(\mathbf{u}, \mathbf{v}) \subseteq I(y).$$

A general element

$$\sum_{n=0}^{q^2} a_n(\pi - \mathbf{u}^{q+1})^n(\pi - \mathbf{v}^{q+1})^{q^2-n} \in I(y)^{q^2}$$

thus lies in $\pi(\mathbf{u}, \mathbf{v})$ if and only if $\mathbf{u} \mid a_0$ and $\mathbf{v} \mid a_{q^2}$. In this situation, already $a_0(\pi - \mathbf{v}^{q+1})$ and $a_{q^2}(\pi - \mathbf{u}^{q+1})$ lie in $\pi(\mathbf{u}, \mathbf{v})$. This shows that

$$I(y)^{q^2} \cap \pi(\mathbf{u}, \mathbf{v}) \subseteq \pi(\mathbf{u}, \mathbf{v})I(y)^{q^2-2} \subseteq \pi(\mathbf{u}, \mathbf{v})\mathfrak{m}I(y)^{q^2-3}$$

which verifies (13.49) since $q^2 \geq 4$.

Case (5). Here, we can directly verify (13.49) by

$$(\pi, \mathbf{v}^{q+1})^{q^2} \cap (\mathbf{v}\pi) \subseteq \pi I(y)^{q^2-2} \subseteq \pi\mathfrak{m}I(y)^{q^2-3}.$$

The proof of Proposition 13.20 is now complete. □

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13.6. Intersection locus (non-special points)

We come to our final argument. The following proposition is essentially a converse to Proposition 13.20 and shows that there are no embedded components beyond the ones found in the previous section.

Proposition 13.21. *Assume that $z \in \mathcal{Z}(y)$ is a closed point that is not superspecial. If $\pi\mathcal{O}_{\mathcal{Z}(y),z} \neq 0$, then $z \in \mathcal{Z}(\pi^{-1}y)$.*

We would have liked to prove this with Cartier theory as before, but the V-series expressions for non-superspecial points are more complicated than in Lemma 13.12. Instead, we use the O_F-display theory from [1], which requires us to restrict to the p-adic setting. The proof in the function field setting would be analogous but in terms of local O_F-shtukas. These are equivalent to strict O_F-modules by [15, Theorem 8.3].

Proof for p-adic F. By assumption, z is a smooth point of $\mathcal{M}_C$ whose complete local ring $\widehat{\mathcal{O}}_{\mathcal{M}_C,z}$ is isomorphic to $R = O_{\breve{F}}[[t]]$. Let $I(y) \subseteq R$ be the ideal defining $\widehat{\mathcal{O}}_{\mathcal{Z}(y),z}$. The assumption $I(y) \subsetneq (\pi)$ is equivalent to the statement that for all continuous rings maps $\varphi : R \to O_{\breve{F}}$, equivalently one such map φ, it holds that $\varphi(I(y)) = (\pi)^n$ with $n \geq 2$. Thus, we need to see

$$\mathrm{Hom}_{O_{\breve{F}}}(\widehat{\mathcal{O}}_{\mathcal{Z}(y),z},\ O_{\breve{F}}/(\pi)^2) \neq \emptyset \quad \Longrightarrow \quad z \in \mathcal{Z}(\pi^{-1}y). \tag{13.50}$$

Let (Y, ι, ρ) be the triple defining the point $z \in \mathcal{M}_C(\mathbb{F})$. Recall that we have the unramified quadratic extension $E \subset F_4 \subset C$, which is normalized by ϖ. It will suffice for our arguments to consider the coarser datum $(Y, j) := (Y, \iota|_{O_{F_4}})$. Our first aim is to compute the O_F-display of a universal deformation of (Y, j). Let (M, F, V) be the O_F-Dieudonné module of (Y, ι). We choose the grading on M such that 0 is the critical index. Let

$$e_0, f_0 \in M_0^{\varpi = V^{-2}}$$

be an O_E-basis. One out of V^2e_0, V^2f_0 does not lie in πM_2. We choose our ordering such that this holds for $e_2 := V^2e_0$. Pick a complementary basis vector $f_2 \in M_2$ that can be written as

$$f_2 = \pi^{-1}(\lambda e_2 + V^2 f_0) \tag{13.51}$$

for a suitable $\lambda \in O_{\breve{F}}$. Rewriting (13.51) also provides

$$V^2 f_0 = -\lambda e_2 + \pi f_2.$$

In summary, ϖ and V are now given by the following identities:

$$\begin{array}{rclrcl} \varpi e_0 &=& e_2 & \varpi e_2 &=& \pi e_0 \\ \varpi f_0 &=& -\lambda e_2 + \pi f_2 & \varpi f_2 &=& \lambda e_0 + f_0, \\ \\ V^2 e_0 &=& e_2 & V^2 e_2 &=& \pi e_0 \\ V^2 f_0 &=& -\lambda e_2 + \pi f_2 & V^2 f_2 &=& \sigma^{-2}(\lambda) e_0 + f_0. \end{array} \tag{13.52}$$

We next rewrite this in the terminology of O_F-displays. Consider the following eight elements:

$$\begin{array}{rclrclrclrcl} t_0 &=& e_0 & m_1 &=& Vt_0 & t_2 &=& f_2 & m_3 &=& Vt_2 \\ l_0 &=& \sigma^{-2}(\lambda)e_0 + f_0 & n_1 &=& Vl_0 & l_2 &=& e_2 & n_3 &=& Vl_2. \end{array} \tag{13.53}$$

Then $M = L \oplus T$, where $L = \mathrm{span}\{l_0, m_1, n_1, l_2, m_3, n_3\}$ and $T = \mathrm{span}\{t_0, t_2\}$. This is a normal decomposition of M meaning $VM = L \oplus \pi T$. Let $\dot{F} := V^{-1}|_{L \oplus \pi T}$ be the display variant of the

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Verschiebung. Then $F|_T$ and $\dot{F}|_L$ are given by the following identities:

$$\begin{aligned} \dot{F}(l_0) &= m_3 & \dot{F}(l_2) &= m_1 \\ \dot{F}(m_1) &= t_0 & \dot{F}(m_3) &= t_2 \\ \dot{F}(n_1) &= l_0 & \dot{F}(n_3) &= l_2. \end{aligned}$$

$$\begin{aligned} F(t_0) &= \pi V^{-1}(e_0) = V\varpi e_0 = V e_2 = n_3, \\ F(t_2) &= \pi V^{-1}(f_2) = V^{-1}(\lambda e_2 + V^2 f_0) \\ &= \sigma(\lambda) V t_0 + V(l_0 - \sigma^{-2}(\lambda) t_0) = (\sigma(\lambda) - \sigma^{-3}(\lambda)) m_1 + n_1. \end{aligned}$$

Write $\mu_1 = \sigma(\lambda) - \sigma^{-3}(\lambda)$. Order the chosen basis as $(t_0, l_0, m_1, n_1, t_2, l_2, m_3, n_3)$. The structure matrix of M as O_F-display is then

$$S = \begin{bmatrix} & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \mu_1 & 1 & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}. \tag{13.54}$$

It is known that a universal deformation of M as O_F-display can be defined as follows; cf. [51, Equation (87)].[4] Let $\mu \in W_{O_F}(O_{\breve{F}})$ be any lift of μ_1. Consider the ring $A = O_{\breve{F}}[[s_{01}, \ldots, s_{06}, s_{21}, \ldots, s_{26}]]$. (The strict O_F-module Y is of height 8 and dimension 2, so one knows a priori that A is isomorphic to its universal deformation ring.) Put $L = W_{O_F}(A)^{\oplus 6}, T = W_{O_F}(A)^{\oplus 2}$ and $P = L \oplus T$. Denote and order their basis vectors just as before. Then a universal deformation of M can be defined by declaring $P = L \oplus T$ to be a normal decomposition and by taking the O_F-display for the structure matrix

$$\begin{bmatrix} 1 & [s_{01}] & [s_{02}] & [s_{03}] & & [s_{04}] & [s_{05}] & [s_{06}] \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & [s_{21}] & [s_{22}] & [s_{23}] & 1 & [s_{24}] & [s_{25}] & [s_{26}] \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \cdot S = \begin{bmatrix} [s_{06}] & [s_{05}] & 1 & [s_{01}] & \mu[s_{02}] + [s_{03}] & [s_{02}] & & [s_{04}] \\ & & & 1 & & & & \\ & & & & \mu & 1 & & \\ & & & & 1 & & & \\ [s_{26}] & [s_{25}] & & [s_{21}] & \mu[s_{22}] + [s_{23}] & [s_{22}] & 1 & [s_{24}] \\ & & & & & & & 1 \\ & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix} \tag{13.55}$$

Consider now the universal deformation of (M, j) (i.e., as O_F-display with action by O_{F_4}). It is not difficult to check that this deformation space is described by the quotient

$$O_{\breve{F}}[[s_{01}, s_{24}]] \cong \overline{A} = A/(s_{ij},\ (i, j) \notin \{(0, 1), (2, 4)\}).$$

Namely, O_{F_4}-actions on a display over an $O_{\breve{F}}$-algebra are equivalent to $\mathbb{Z}/4$-gradings such that F and $\dot{F}$ are homogeneous of degree -1. At this point, one could go even further and also determine the relation between s_{01} and s_{24} that defines the deformation space of M with special O_C-action ι, but this will not be necessary for our arguments.

[4]The reference applies directly if $O_F = \mathbb{Z}_p$. The general statement can be reduced to that case because O_F-displays are equivalent to $\mathbb{Z}_p$-displays with strict O_F-action; cf. the functor $\Gamma(\mathcal{O}, \mathcal{O}')$ in [1, (1.1)].

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We explained at the beginning (see (13.50)) that we only care about $\overline{R}$-points of $\mathcal{Z}(y)$, where $\overline{R} = O_{\breve{F}}/(\pi)^2$. So we pick any specialization map $\varphi : O_{\breve{F}}[[s_{01}, s_{24}]] \to \overline{R}$. We put $s_0 = \varphi(s_{01})$ and $s_2 = \varphi(s_{24})$. Base changing the O_F-display over $\overline{A}$ given by (13.55) along φ defines an O_F-display $(\overline{P}, \overline{Q}, F, \dot{F})$ over $\overline{R}$. It is defined by the normal decomposition $\overline{P} = \overline{L} \oplus \overline{T}$ obtained by base change from the previous normal decomposition, and the structure matrix

$$\begin{bmatrix} & & 1 & [s_0] & & & & \\ & & & 1 & & & & \\ & & & & \mu & 1 & & \\ & & & & 1 & & & \\ & & & & & & 1 & [s_2] \\ & & & & & & & 1 \\ & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix}. \tag{13.56}$$

The claim is that every homogeneous degree 1 endomorphism y of $\overline{P}$ is divisible by π after base change to $\overline{R}/(\pi)$. (Homogeneity is meant with respect to the $j(O_{F_4})$-grading.) Any such endomorphism is, in particular, a $W_{O_F}(\overline{R})$-linear endomorphism of the $W_{O_F}(\overline{R})$-module $\overline{P}$ that preserves the submodule $\overline{Q} = L \oplus {}^V W_{O_F}(\overline{R}) \cdot T$. It is hence given by four matrices of the form

$$y: \quad \overline{P}_0 \xrightarrow{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \overline{P}_1 \xrightarrow{\begin{pmatrix} {}^V b_{11} & {}^V b_{12} \\ b_{21} & b_{22} \end{pmatrix}} \overline{P}_2 \xrightarrow{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} \overline{P}_3 \xrightarrow{\begin{pmatrix} {}^V d_{11} & {}^V d_{12} \\ d_{21} & b_{22} \end{pmatrix}} \overline{P}_0 \tag{13.57}$$

where the 16 coefficients lie in $W_{O_F}(\overline{R})$. (The matrix presentation is meant with respect to the above basis $(t_0, l_0, m_1, n_1, t_2, l_2, m_3, n_3)$.) Then y being an endomorphism of $(\overline{P}, \overline{Q}, F, \dot{F})$ is equivalent to $\dot{F} \circ y = y \circ \dot{F}$. We now express this condition in terms of the 16 coefficients. We write a, b, c and d for the four matrices in (13.57) and let $\xi \in W_{O_F}(\overline{R})$ be any element. The compositions $\dot{F} \circ x$ and $x \circ \dot{F}$, for $x \in \{a, b, c, d\}$, are given as follows:

$$\begin{aligned}
(0)\quad \dot{F}(a({}^V \xi t_0)) &= \pi\xi\{\sigma(a_{11})t_0 + \sigma(a_{21})[s_0]t_0 + \sigma(a_{21})l_0\} \\
d(\dot{F}({}^V \xi t_0)) &= \xi\{{}^V d_{12}t_0 + d_{22}l_0\} \\
\dot{F}(a(l_0)) &= \sigma(a_{12})t_0 + \sigma(a_{22})[s_0]t_0 + \sigma(a_{22})l_0 \\
d(\dot{F}(l_0)) &= {}^V d_{11}t_0 + d_{21}l_0 \\[1em]
(1)\quad \dot{F}(b(m_1)) &= \mu b_{11}m_1 + \sigma(b_{21})m_1 + b_{11}n_1 \\
a(\dot{F}(m_1)) &= a_{11}m_1 + a_{21}n_1 \\
\dot{F}(b(n_1)) &= \mu b_{12}m_1 + \sigma(b_{22})m_1 + b_{12}n_1 \\
a(\dot{F}(n_1)) &= [s_0]a_{11}m_1 + a_{12}m_1 + [t_0]a_{21}n_1 + a_{22}n_1 \\[1em]
(2)\quad \dot{F}(c({}^V \xi t_2)) &= \pi\xi\{\sigma(c_{11})t_2 + [s_2]\sigma(c_{21})t_2 + \sigma(c_{21})l_2\} \\
b(\dot{F}({}^V \xi t_2)) &= \xi\{\mu {}^V b_{11}t_2 + {}^V b_{12}t_2 + \mu b_{21}l_2 + b_{22}l_2\} \\
\dot{F}(c(l_2)) &= \sigma(c_{12})t_2 + [s_2]\sigma(c_{22})t_2 + \sigma(c_{22})l_2 \\
b(\dot{F}(l_2)) &= {}^V b_{11}t_2 + b_{21}l_2 \\[1em]
(3)\quad \dot{F}(d(m_3)) &= \sigma(d_{21})m_3 + d_{11}n_3 \\
c(\dot{F}(m_3)) &= c_{11}m_3 + c_{21}n_3 \\
\dot{F}(d(n_3)) &= \sigma(d_{22})m_3 + d_{12}n_3 \\
c(\dot{F}(n_3)) &= [s_2]c_{11}m_3 + c_{12}m_3 + [s_2]c_{21}n_3 + c_{22}n_3
\end{aligned} \tag{13.58}$$

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Thus, $\dot{F} \circ y = y \circ \dot{F}$ if and only if the following identities hold.

$$\begin{array}{llcllcl} (\alpha) & a_{11} &=& \mu b_{11} + \sigma(b_{21}) & a_{12} &=& \mu b_{12} + \sigma(b_{22}) - [s_0]a_{11} \\ & a_{21} &=& b_{11} & a_{22} &=& b_{12} - [s_0]a_{21} \\ (\beta) & {}^V b_{11} &=& \sigma(c_{12}) + [s_2]\sigma(c_{22}) & {}^V b_{12} &=& \pi(\sigma(c_{11}) + [s_2]\sigma(c_{21})) - \mu {}^V b_{11} \\ & b_{21} &=& \sigma(c_{22}) & b_{22} &=& \pi\sigma(c_{21}) - \mu b_{21} \\ (\gamma) & c_{11} &=& \sigma(d_{21}) & c_{12} &=& \sigma(d_{22}) - [s_2]c_{11} \\ & c_{21} &=& d_{11} & c_{22} &=& d_{12} - [s_2]c_{21} \\ (\delta) & {}^V d_{11} &=& \sigma(a_{12}) + [s_0]\sigma(a_{22}) & {}^V d_{12} &=& \pi(\sigma(a_{11}) + [s_0]\sigma(a_{21})) \\ & d_{21} &=& \sigma(a_{22}) & d_{22} &=& \sigma(a_{21}) \end{array} \tag{13.59}$$

Assuming that all these relations hold, we claim that none of the 16 variables is a unit. (This is equivalent to claiming that all 16 coefficients are divisible by π after base change to $W_{O_F}(R/\pi) = O_{\breve{F}}$, which means that $\pi^{-1}y$ defines an endomorphism of the Dieudonné module of the closed point. This precisely means that $z \in \mathcal{Z}(\pi^{-1}y)$.) The proof of the claim is as follows. We use the matrix notation (α_{ij}), (β_{ij}) etc. to refer to the individual identities in (13.59).

(1) First, (δ_{11}) and (β_{11}) imply that a_{12} and c_{12} cannot be units. Then (γ_{12}) implies that d_{22} is no unit. Then (δ_{22}) shows that a_{21} is no unit. By (α_{21}), also b_{11} is no unit.

(2) Next, specializing (δ_{12}) along the projection map $s : W_{O_F}(\bar{R}) \to \bar{R}$, we obtain that $0 = \pi s(\sigma(a_{11})) + 0$ because $\pi s_0 = 0$ in $\bar{R}$. Since $\pi \neq 0$ in $\bar{R}$, it follows that $s(\sigma(a_{11}))$ is no unit and hence that a_{11} is no unit. The same argument but for (β_{12}) shows that c_{11} is no unit.

(3) An easy chain of substitutions now shows that all remaining variables, except for possibly $c_{21} = d_{11}$, cannot be units.

(4) For the remaining two variables, we consider identity (δ_{11}). Consider the leading terms of the Witt vector expressions for a_{12} and a_{22}:

$$a_{12} = [u_0] + {}^V u_1 \quad \text{and} \quad a_{22} = [v_0] + {}^V v_1.$$

We already know that $u_0, v_0 \notin \bar{R}^\times$. In particular, $\sigma([u_0]) = [u_0]^q = 0$ and $\sigma([v_0]) = [v_0^q] = 0$. Recall that $\sigma({}^V(x)) = \pi x$ for every $x \in W_{O_F}(\bar{R})$. Thus, we obtain from (δ_{11}) that

$${}^V d_{11} = \sigma(a_{12}) + [s_0]\sigma(a_{22}) = \pi u_1 + \pi[s_0]v_1.$$

Looking at the image of this expression under $s : W_{O_F}(\bar{R}) \to \bar{R}$ and using that $\pi \neq 0$ in $\bar{R}$, it follows that $u_1 \notin W_{O_F}(\bar{R})^\times$.

Let $\bar{a}_{12}$ and $\bar{d}_{11}$ be the images of a_{12} and d_{11} under the reduction map $W_{O_F}(\bar{R}) \to W_{O_F}(\mathbb{F}) = O_{\breve{F}}$. Identity (δ_{11}) implies that ${}^V \bar{d}_{11} = \sigma(\bar{a}_{12})$. The above showed that $\bar{a}_{12}$ is divisible by π^2, so we obtain that $\bar{d}_{11}$ is divisible by π. We deduce that d_{11} is not a unit. The proof of the proposition is now complete. □

13.7. Intersection numbers

We first summarize the results of the previous sections. For a lattice $\Lambda \subset W$, we have previously defined $m(y, \Lambda) = \max\{0, n(y, \Lambda)\}$; see (13.17).

Proposition 13.22. *Let $y \in S_{3/4,\mathrm{rs}}$ be a regular semi-simple element and let $m = \max_{\Lambda \subset W} n(y, \Lambda)$. Put $\mathcal{C}(y) = \mathcal{Z}(\pi^{-m}y)$.*

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(1) *The formal scheme $\mathcal{Z}(y)$ is nonempty if and only if $m \geq 0$. In particular, $\mathcal{C}(y) \neq \emptyset$.*

(2) *For a stratum $P \subseteq \mathcal{M}_C$, let $\Lambda(P) \subset W$ be the lattice defined by it. The pure locus of $\mathcal{Z}(y)$ is given by*

$$\mathcal{Z}(y)^{\mathrm{pure}} = \sum_{P \subseteq \mathcal{M}_{C,\mathrm{red}}} m(y, \Lambda(P)) \cdot [P].$$

(3) *The formal scheme $\mathcal{C}(y)$ is artinian. Moreover,*

$$\mathcal{Z}(y)^{\mathrm{art}} = \mathcal{C}(y) \sqcup \coprod_{z \in |\mathcal{Z}(y)| \setminus |\mathcal{C}(y)|,\ z \text{ superspecial}} \mathcal{O}_{\mathcal{Z}(y)^{\mathrm{art}}, z}, \tag{13.60}$$

and each local ring in the disjoint union on the right-hand side has length q.

Proof. (1) This follows directly from Corollary 13.5.

(2) Corollary 13.5 shows that the multiplicity of P in $\mathcal{Z}(y)$ is indeed 0 if $n(y, \Lambda(P)) \leq 0$. By the same corollary, if $n(y, \Lambda(P)) = 0$, then there exists a point $z \in \mathcal{Z}(y) \cap P(\mathbb{F})$ because every σ-linear endomorphism of a 2-dimensional $\mathbb{F}$-vector preserves some point of $P(\mathbb{F})$. Proposition 13.20 applies to that point and shows that the multiplicity of P in $\mathcal{Z}(\pi^a y)$ equals $a = m(\pi^a y, \Lambda(P))$ for every $a \geq 0$. This reasoning applies to all pairs $(\pi^{\mathbb{Z}} y, \Lambda(P))$, and statement (2) follows.

(3) For all y, by Proposition 13.20, if $z \in \mathcal{Z}(y)^{\mathrm{art}}$, then $z \in \mathcal{Z}(\pi y)^{\mathrm{art}}$ and there is an equality of local rings

$$\mathcal{O}_{\mathcal{Z}(\pi y)^{\mathrm{art}}, z} = \mathcal{O}_{\mathcal{Z}(y)^{\mathrm{art}}, z}.$$

We know from Corollary 13.5 that $\mathcal{C}(y)$ is artinian, so this shows $\mathcal{C}(y) \subseteq \mathcal{Z}(y)^{\mathrm{art}}$. Moreover, by Proposition 13.18 combined with (again) Proposition 13.20, every superspecial point $z = P \cap P'$ such that $m(y, \Lambda(P)) > m(y, \Lambda(P')) \geq 0$ lies in $\mathcal{Z}(y)^{\mathrm{art}}$ and has a local ring of length q. This shows that the right-hand side in (13.60) is an open and closed subscheme of the left-hand side.

By Corollary 13.5, every superspecial point of $\mathcal{Z}(y)$ already lies in the right-hand side of (13.60). Let $z \in \mathcal{Z}(y)^{\mathrm{art}} \cap P(\mathbb{F})$ be a non-superspecial point. The multiplicity of P in $\mathcal{Z}(\pi^{-m(y,\Lambda(P))+1} y)$ is 1. By Proposition 13.21, z even lies in $\mathcal{Z}(\pi^{-m(y,\Lambda(P))} y)$. By Corollary 13.5, the only possibility is $m(y, \Lambda(P)) = m$ and $z \in \mathcal{C}(y)$, and the proof of (3) is complete. □

Let $G^0_{3/4,b} \subset G_{3/4,b}$ denote the subgroup of elements g with reduced norm $\mathrm{Nrd}(g) \in O_F^\times$. Then $g\mathcal{M}^i_{3/4} = \mathcal{M}^i_{3/4}$ for every $i \in \mathbb{Z}$, and we may define the connected component intersection number

$$\mathrm{Int}_0(g) := \langle \mathcal{M}^0_C,\ g \cdot \mathcal{M}^0_C \rangle_{\mathcal{M}_{3/4}}.$$

Theorem 13.23. *Let $g \in G^0_{3/4,b,\mathrm{rs}}$ be a regular semi-simple element with numerical invariant (L, r, d). The intersection number $\mathrm{Int}_0(g)$ only depends on the triple (L, r, d). It is related to $\mathrm{Int}(g)$ by*

$$\mathrm{Int}(g) = \begin{cases} \mathrm{Int}_0(g) & \text{if } L/F \text{ is split or ramified} \\ 2\ \mathrm{Int}_0(g) & \text{if } L/F \text{ is an unramified field extension.} \end{cases} \tag{13.61}$$

Moreover, the arithmetic transfer conjecture (Conjecture 4.26) holds with correction function $f'_{\mathrm{corr}} = 0$. That is, for every $\gamma \in G'_{\mathrm{rs}}$,

$$\partial\mathrm{Orb}(\gamma, f'_D) = \begin{cases} 2\ \mathrm{Int}(g)\log(q) & \text{if there is a matching } g \in G_{3/4,b} \\ 0 & \text{otherwise.} \end{cases} \tag{13.62}$$

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Proof. We first determine the intersection number $\mathrm{Int}(g)$ for $g \in G_{3/4,b,\mathrm{rs}}$. We may assume that $g = 1+z$ with $z = z_g$. Since $\mathcal{M}_C \cap g \cdot \mathcal{M}_C = \emptyset$ whenever z is not topologically nilpotent by Lemma 4.22, we may assume that z is topologically nilpotent.

We work in the coordinates of §13.2. Let $y \in S_{3/4,\mathrm{rs}}$ be such that $z = \left(\begin{smallmatrix} & y\varpi \\ y & \end{smallmatrix}\right)$. In particular, $L = F[\varpi y^2]$ with $\varpi y^2 \in O_L$ and

$$r = v(N_{L/F}(\varpi y^2)), \quad d = [O_L : O_F[\varpi y^2]] - r/2.$$

The element $\widetilde{y} = \varpi^{-1} y$ lies in $S_{1/4}$. Since $\varpi \widetilde{y}^2 = \pi^{-1}\varpi y^2$, its numerical invariant is given by

$$(\widetilde{L}, \widetilde{r}, \widetilde{d}) = (L, r-2, d). \tag{13.63}$$

Let $\widetilde{z} = \left(\begin{smallmatrix} & \widetilde{y}\varpi \\ \widetilde{y} & \end{smallmatrix}\right)$ and define $\widetilde{g} = 1+\widetilde{z} \in \mathrm{End}^0_F(\mathbb{Y})$. Then $\widetilde{g}$ lies in $G_{1/4,b,\mathrm{rs}}$ unless the following exceptional case occurs: $L \cong F \times F$ and one of the eigenvalues of ϖy^2 is π. All statements that follow also apply in this exceptional case if one sets $\mathcal{I}(\widetilde{g}) = \emptyset$ and $\mathrm{Int}_0(\widetilde{g}) = 0$.

Let $\Gamma \subset L^\times$ be a subgroup such that $L^\times = \Gamma \times O_L^\times$ and set $\Gamma_0 = \Gamma \cap G^0_{3/4,b}$.

Lemma 13.24. *The following identity of intersection numbers holds:*

$$\mathrm{Int}_0(g) = \mathrm{Int}_0(\widetilde{g}) + q \cdot \#\big(\Gamma_0 E^\times \backslash \{\Lambda \subseteq W \mid m(y,\Lambda) \geq 0\}\big) + \begin{cases} 1 & \text{if } L \text{ is a field} \\ 0 & \text{if } L \cong F \times F. \end{cases}$$

Proof. First consider the divisors $\mathcal{Z}(y)^{0,\mathrm{pure}}, \mathcal{Z}(\widetilde{y})^{0,\mathrm{pure}} \subseteq \mathcal{M}^0_C$. Because

$$w(\varpi^{-1}y) = V \cdot \varpi^{-1} y = (V^2\varpi^{-1}) \cdot V^{-1} y = (V^2\varpi^{-1}) \cdot w(y)$$

by Definition 13.3, the restrictions of $w(\varpi^{-1}y)$ and $w(y)$ to the $V^2\varpi^{-1}$-invariants $W = N_0^{\tau=\mathrm{id}}$ agree. The multiplicity formula for the invariant $1/4$ case (Proposition 12.9) and the analogous formula for invariant $3/4$ (Proposition 13.22) hence give that

$$\mathcal{Z}(y)^{0,\mathrm{pure}} = \mathcal{Z}(\widetilde{y})^{0,\mathrm{pure}}. \tag{13.64}$$

By Proposition 13.1, the degree $\deg(\mathcal{C}|_P)$ of the restriction of the conormal bundle for $\mathcal{M}_C \to \mathcal{M}_D$ to an irreducible component $P \subseteq \mathcal{M}_{C,\mathrm{red}}$ equals $q^2 - 1$ in all situations. The equality in (13.64) and the general intersection number formula from Corollary 10.3 then imply that

$$\mathrm{Int}_0(g) - \mathrm{Int}_0(\widetilde{g}) = \mathrm{len}(\mathcal{O}_{\Gamma_0\backslash \mathcal{Z}(y)^{0,\mathrm{art}}}) - \mathrm{len}(\mathcal{O}_{\Gamma_0\backslash \mathcal{Z}(\widetilde{y})^{0,\mathrm{art}}}). \tag{13.65}$$

By Proposition 12.11, the length of $\Gamma_0\backslash\mathcal{Z}(\widetilde{y})^{0,\mathrm{art}}$ is given by

$$\mathrm{len}(\mathcal{O}_{\Gamma_0\backslash \mathcal{Z}(\widetilde{y})^{0,\mathrm{art}}}) = \begin{cases} 1 & \text{if } L \text{ is a field and } \widetilde{r} \in 2 + 4\mathbb{Z}_{\geq 0} \\ 0 & \text{in all other cases.} \end{cases} \tag{13.66}$$

We determine the length of $\Gamma_0\backslash\mathcal{Z}(y)^{0,\mathrm{art}}$: By (13.60), it is given as

$$\mathrm{len}(\mathcal{O}_{\Gamma_0\backslash \mathcal{C}(y)^0}) + q \cdot \#\big(\Gamma_0 E^\times \backslash \{\Lambda \subset W \mid m > m(y,\Lambda) \geq 0\}\big). \tag{13.67}$$

First assume that $\widetilde{r} \in 4\mathbb{Z}_{\geq 1}$. Then we are in case (3) of Corollary 13.5 and obtain from Proposition 13.10 that

$$\mathrm{len}(\mathcal{O}_{\Gamma_0\backslash \mathcal{C}(y)^0}) = q \cdot \#\Gamma_0\backslash\mathcal{T}(w(y)) + \begin{cases} 1 & \text{if } L \text{ is a field} \\ 0 & \text{if } L \text{ is split.} \end{cases} \tag{13.68}$$

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Now assume that $\widetilde{r} \in 2 + 4\mathbb{Z}_{\geq 0}$. Then we are in case (4) of Corollary 13.5, and Proposition 13.11 shows that

$$\mathrm{len}(\mathcal{O}_{\Gamma_0\backslash\mathcal{C}(y)^0}) = q \cdot \#\Gamma_0\backslash\mathcal{T}(w(y)) + \begin{cases} 2 & \text{if } L \text{ is a field} \\ 0 & \text{if } L \text{ is split.} \end{cases} \tag{13.69}$$

In both cases, the set $\mathcal{T}(w(y))$ is precisely the set $\{\Lambda \subseteq W \mid m(y,\Lambda) = m\}/E^\times$. So the three identities (13.67), (13.68) and (13.69) together give

$$\mathrm{len}(\mathcal{O}_{\Gamma_0\backslash\mathcal{Z}(y)^{0,\mathrm{art}}}) = q \cdot \#\big(\Gamma_0 E^\times\backslash\{\Lambda \subset W \mid m(y,\Lambda) \geq 0\}\big) + \begin{cases} 1 & \text{if } L \text{ is a field and } \widetilde{r} \in 4\mathbb{Z}_{\geq 2} \\ 2 & \text{if } L \text{ is a field and } \widetilde{r} \in 2 + 4\mathbb{Z}_{\geq 0} \\ 0 & \text{if } L \text{ is split.} \end{cases}$$

Substituting this result and Identity (13.66) in (13.65) proves the lemma. □

We can now prove the first part of Theorem 13.23: We already know from Proposition 12.12 that $\mathrm{Int}_0(\widetilde{g})$ only depends on (L, r, d). (Recall that the numerical invariant of $\widetilde{g}$ is $(L, r-2, d)$.) By Theorem 11.10 and Proposition 11.5, the number of lattices $\#(\Gamma_0 E^\times\backslash\{\Lambda \subset W \mid m(y,\Lambda) \geq 0\})$ in Lemma 13.24 also only depends on (L, r, d). So we obtain that $\mathrm{Int}_0(g)$ only depends on (L, r, d), as claimed. The claimed identity

$$\mathrm{Int}(g) = \begin{cases} \mathrm{Int}_0(g) & \text{if } L/F \text{ ramified or split} \\ 2\ \mathrm{Int}_0(g) & \text{if } L/F \text{ inert} \end{cases}$$

follows by the same argument as in the case of invariant 1/4; see the first part of the proof of Theorem 12.13. By the intersection number formula in the invariant 1/4 case (12.22), which we apply to $\widetilde{r} = r - 2$, we obtain from Lemma 13.24 that

$$\mathrm{Int}(g) = \delta \cdot q \cdot \#\big(\Gamma_0 E^\times\backslash\{\Lambda \subset W \mid n(y,\Lambda) \geq 0\}\big) + \begin{cases} r & L/F \text{ inert} \\ r/2 & L/F \text{ ramified} \\ 0 & L/F \text{ split,} \end{cases} \tag{13.70}$$

where $\delta = 2$ if L/F is inert and $\delta = 1$ otherwise. It is left to verify the arithmetic transfer identity (13.62).

Let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element with numerical invariant (L, r, d). If the sign of the functional equation $\varepsilon_D(\gamma)$ is positive, equivalently r odd, then the left-hand side of (13.62) vanishes by Proposition 5.4. There is no matching element $g \in G_{3/4,b}$ by Proposition 4.6, so the right-hand side vanishes as well.

Assume from now on that the sign $\varepsilon_D(\gamma)$ is negative, equivalently that r is even. Then $\mathrm{Orb}(\gamma, f'_D) = 0$, so the left-hand side of (13.62) equals (by Proposition 5.4)

$$\partial\mathrm{Orb}(\gamma, f'_D) = 4q\log(q)\ \mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) + \log(q) \begin{cases} 2r & \text{if } L \text{ inert} \\ r & \text{if } L \text{ ramified} \\ 0 & \text{if } L \text{ split.} \end{cases} \tag{13.71}$$

By Table 1, there exists no matching element $g \in G_{3/4,b}$ if and only if $L \cong F \times F$ and $z_\gamma = (z_1, z_2)$ with $v(z_1), v(z_2)$ both even. (The components z_1 and z_2 always have the same parity because $r = v(z_1) + v(z_2)$ was assumed even.) In this case, the derivative (13.71) vanishes by Proposition 5.2 (5).

It remains to consider the case where there exists a matching element $g \in G_{3/4,b,\mathrm{rs}}$. Then, (L, r, d) is the numerical invariant of both g and γ. We assume that $r > 0$ because otherwise all involved terms vanish. The desired equality of (13.71) and the $2\log(q)$-multiple of (13.70) is then a direct consequence of the following lemma:

Lemma 13.25. *Assume that $\gamma \in G'_{\mathrm{rs}}$ matches $g \in G_{3/4,b,\mathrm{rs}}$. Then*

$$2\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) = \#\big(\Gamma_0 E^{\times}\backslash\{\Lambda \subset W \mid n(y,\Lambda) \geq 0\}\big) \cdot \begin{cases} 2 & L/F \text{ inert} \\ 1 & L/F \text{ ramified or split}. \end{cases} \tag{13.72}$$

Proof. The left-hand side only depends on the triple (L, r, d) and is given by Proposition 5.2. Let $m = \max_{\Lambda\subset W} n(y,\Lambda)$ and assume that $m \geq 0$. The right-hand side of the lemma is described by Proposition 11.5, and our proof is by going through the cases of that proposition. (Every $g \in G_{3/4,b,\mathrm{rs}}$ has a matching element γ, so this condition will not come up anymore.)

Proposition 11.5 describes the right-hand side of (13.72) in terms of a ball of radius m around the center $\mathcal{T}(w(y))$,

$$B(\mathcal{T}(w(y)), m) = \{\Lambda \in \mathcal{B} \mid d(\Lambda, \mathcal{T}(w(y))) \leq m\}.$$

The center $\mathcal{T}(w(y))$ in turn has been determined in Theorem 11.10. Note that $w(y)^2 = \pi^{-1}\varpi y^2$, so the numerical invariant of $w(y)$ is $(L, \widetilde{r} = r - 2, d)$. We now count $\Gamma_0\backslash B(m, \mathcal{T}(w(y)))$ in dependence on (L, r, d). The group Γ_0 is trivial if L is a field and will only come up for $L = F \times F$.

(1) Assume that L/F is inert or ramified, and that $r \in 4\mathbb{Z}_{\geq 1}$. Then $\widetilde{r} \in 2 + 4\mathbb{Z}_{\geq 0}$ and Theorem 11.10 states that $\mathcal{T}(w(y))$ is a single edge. Then

$$\#B(\mathcal{T}(w(y)), m) = 2(1 + q^2 + \ldots + q^{2m}). \tag{13.73}$$

Lemma 11.6 applied to $\widetilde{r}$ states that $2m = r/2 - 2$. The expression in (13.73) equals $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ if L is inert or $2\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ if L is ramified which follows from Proposition 5.2 (1) and (3). This proves (13.72) in these two cases.

(2) Assume that L/F is inert and that $r \in 2 + 4\mathbb{Z}_{\geq 0}$. Then $\widetilde{r} \in 4\mathbb{Z}_{\geq 0}$ and Theorem 11.10 states that $\mathcal{T}(w(y))$ is a $(q+1)$-regular ball of radius d around a vertex. If $d = 0$, then we obtain

$$\#B(\mathcal{T}(w(y)), m) = 1 + (1 + q^2)(1 + q^2 + \ldots + q^{2(m-1)}) = 2(1 + q^2 + \ldots + q^{2(m-1)}) + q^{2m}.$$

By Lemma 11.6 applied to $\widetilde{r}$, we find $2m = r/2 - 1$. The equality $\#B(\mathcal{T}(w(y)), m) = \mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ follows from Proposition 5.2 (4). Assume now that $d \geq 1$. Let A be the number of vertices of valency $q+1$ of $\mathcal{T}(w(y))$ and let B be the number of vertices of valency 1. Then

$$\#B(\mathcal{T}(w(y)), m) = (1 + (q^2 - q)(1 + q^2 + \ldots + q^{2m-2}))A + (1 + q^2(1 + q^2 + \ldots + q^{2m-2}))B.$$

Evaluating this with $d \geq 1$ and $2m = r/2 - 1$, one obtains the expression for $\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ in Proposition 5.2 (4).

The verification in the remaining cases works in exactly the same way. We do not spell this out in full detail but say here which further cases one has to consider precisely.

(3) Assume that L/F is ramified and that $r \in 2 + 4\mathbb{Z}_{\geq 0}$. Then $\widetilde{r} \in 4\mathbb{Z}_{\geq 0}$, and Theorem 11.10 states that $\mathcal{T}(w(y))$ is a $q+1$-regular ball of radius d around an edge. Moreover, $2m = r/2 - 1$ as in (2) above. Direct inspection shows that the cardinality of $B(\mathcal{T}(w(y)), m)$ is given by the expression for $2\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ in Proposition 5.2 (2).

(4) Assume that L/F is split, which implies that $r \in 2 + 4\mathbb{Z}_{\geq 0}$ and hence $\widetilde{r} \in 4\mathbb{Z}_{\geq 0}$. First assume that $d < 0$. Then Theorem 11.10 states that $\mathcal{T}(w(y))$ is an apartment. The quotient $\Gamma_0\backslash\mathcal{T}(w(y))$ has two elements because Γ_0 is generated by an element of the form (π_1, π_2^{-1}) with $v(\pi_1) = v(\pi_2) = 1$. It follows that

$$\#(\Gamma_0\backslash B(\mathcal{T}(w(y)), m)) = 2[1 + (q^2 - 1)(1 + q^2 + \ldots + q^{2m-2})] = 2q^{2m}. \tag{13.74}$$

Lemma 11.6 states that $2m = r/2 + d - 1$. Proposition 5.2 (6) states that $2\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}}) = 2q^{r/2+d-1}$, which equals (13.74). In the case $d \geq 0$, we have that $2m = r/2 - 1$ and that $\mathcal{T}(w(y))$ is a $(q+1)$-regular ball of radius m around an apartment. The verification of $\#\big(\Gamma_0 \backslash B(\mathcal{T}(w(y)), m)\big) = 2\,\mathrm{Orb}(\gamma, f'_{\mathrm{Par}})$ works just as before. □

The proof of Theorem 13.23 is now complete. □

Remark 13.26. We have proved Lemma 13.25 by explicitly comparing its two sides. There is, however, a more conceptual explanation for this identity: The maximal reduced subscheme of $\mathcal{M}_{3/4}$ can be shown to admit a Bruhat–Tits type stratification indexed by O_C-lattices in C^2 which, under the Serre–Tensor construction, translates the counting problem in Lemma 13.25 into an orbital integral on $GL_2(B)$. The fundamental lemma for Hasse invariant $1/2$ from [20] expresses this as $O(\gamma, 1_{\mathrm{Par}})$.

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