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UNIT PRESERVING ISOMETRIES ARE HOMOMORPHISMS IN CERTAIN L^p

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1. Introduction and notation.

(a) σ_1 and σ_2 will always denote positive bounded measures of equal mass defined on sets X and Y respectively. $L^p(\sigma_1)$ and $L^p(\sigma_2)$ will always be *complex* L^p spaces.

(b) $M \subseteq L^\infty(\sigma_1)$ will always denote a *subalgebra* of $L^\infty(\sigma_1)$ *containing constants.*

(c) Let $T : M \rightarrow L^p(\sigma_2)$ be a linear map of M into $L^p(\sigma_2)$. We shall say that T is a linear isometry in L^p norm if

$$\int |Tf|^p d\sigma_2 = \int |f|^p d\sigma_1.$$

We shall prove the following:

THEOREM B. *If $2 < p < \infty$ and $T : M \rightarrow L^p(\sigma_2)$ is a linear isometry in the L^p norm with $T(1) = 1$ then T is a homomorphism on M; that is*

(a) $\quad T(fg) = T(f)T(g)$

for all f and g in M. Furthermore,

(b) $$\int T(f)\overline{T(g)}d\sigma_2 = \int f\bar{g}d\sigma_1$$

for all f and g in M.

This theorem extends a result of Forelli's [**2**] by eliminating his extra hypothesis that $Tf \not\equiv 0$ a.e. σ_2 if $f \not\equiv 0$. For results when $p = \infty$, see [**3**].

2. The proof of Theorem B is an extension of Proposition 1 and Proposition 2 [**1**] of Forelli. We shall use similar language where we can so that the reader familiar with Forelli's work can follow more easily.

THEOREM A. *Let $\infty > p > 2$ and assume that f_k is in $L^p(\sigma_k)$ $(k = 1, 2)$ and that for all complex numbers z*

(1) $$\int |1 + zf_1|^p d\sigma_1 = \int |1 + zf_2|^p d\sigma_2.$$

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Then

(a) $$\int |f_1|^2 d\sigma_1 = \int |f_2|^2 d\sigma_2$$

and

(b) $$\int |f_1|^4 d\sigma_1 = \int |f_2|^4 d\sigma_2.$$

Proof. Forelli's Proposition 1 gives part (a). Also note that since $p > 2$, $f_k \in L^2(\sigma_k)$. If for both $k = 1, 2$

$$\int |f_k|^4 d\sigma_k$$

is infinite we are done. Assume that

$$\int |f_1|^4 d\sigma_1$$

is finite. Consider

(2) $$\frac{1}{2\pi} \int_0^{2\pi} |1 + ze^{ix}|^p dx - \frac{p^2}{4} |z|^2 - 1.$$

When $|z| < 1$

$$(1 + ze^{ix})^{p/2} = \sum_{j \geqslant 0} \binom{p/2}{j} z^j e^{ijx}$$

and (2) is given by

(3) $$\binom{p/2}{2}^2 |z|^4 + \sum_{j \geqslant 3} \binom{p/2}{j}^2 |z|^{2j}$$

and therefore

(4) $$r^{-4} \left(\frac{1}{2\pi} \int_0^{2\pi} |1 + rf_k e^{ix}|^p dx - \frac{p^2}{4} |r|^2 |f_k|^2 - 1 \right) \rightarrow \binom{p/2}{2} |f_k|^4$$

pointwise a.e. when $r \rightarrow 0$.

We wish to show that (2) is nonnegative for all z. For $|z| < 1$ this is clear from (3). For $|z| > 1$ note that

$$\frac{1}{2\pi} \int_0^{2\pi} |1 + ze^{ix}|^p dx = \frac{1}{2\pi} |z|^p \int_0^{2\pi} |1 + \frac{1}{z} e^{ix}|^p dx$$

and therefore (2) is given by

(5) $$|z|^p + \frac{p^2}{4} |z|^{p-2} - \frac{p^2}{4} |z|^2 - 1 + \sum_{j \geqslant 2} \binom{p/2}{j}^2 |z|^{p-2j}$$

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for $|z| > 1$. To show (2) is nonnegative for $|z| > 1$ it therefore suffices to show that

$$(6) \quad g(x) = x^p + \frac{p^2}{4} x^{p-2} - \frac{p^2}{4} x^2 - 1$$

is nonnegative for $x > 1$. We shall just use some elementary calculus techniques for this. Note:

(a) $g(1) = 0$.

(b) Since $p > 2$ $g(x) \sim x^p$ as $x \to \infty$ and hence is positive for large x.

(c) $g'(x) = px^{p-1} + \frac{1}{4}p^2(p-2)x^{p-3} - \frac{1}{2}p^2 x$ and $g'(1) = \frac{1}{4}p(p-2)^2 > 0$ as $p \neq 2$ and $p > 0$.

From (c) we see that $g(x) > 0$ for $1 < x < 1 + \epsilon$. If $g(x) < 0$ for some $x > 1$ we can see from (a) and (b) and the intermediate value theorem that there would be $1 < x_1 < x_2$ for which $g(1) = g(x_1) = g(x_2) = 0$. By Rolle's Theorem, $g'(x)$ would then have at least two zeros in $x > 1$. We shall show that this is impossible and conclude that $g(x) \geqq 0$ for $x \geqq 1$. It suffices to show that the function

$$h(x) = x^{p-2} + \frac{p(p-2)}{4} x^{p-4} - \frac{p}{2}$$

does not have two zeros in $x > 1$. But, by Rolle's Theorem if $h(x)$ has two zeros in $x > 1$ then $h'(\lambda) = 0$ for some $\lambda > 1$. But

$$h'(x) = (p-2)x^{p-3} + \frac{p(p-2)(p-4)}{4} x^{p-5}$$

and $h'(\lambda) = 0$ means that

$$1 + \frac{p(p-4)}{4} \lambda^{-2} = 0$$

for some $\lambda > 1$ (note $p \neq 2$) or

$$\lambda^2 = \frac{(4-p)p}{4} = 1 - \frac{(p-2)^2}{4}$$

which is a contradiction. Therefore (2) is nonnegative for all z.

Since (2) is nonnegative for all z the left hand side of (4) is nonnegative, Using Fatou's Lemma we see that

$$\binom{p/2}{2}^2 \int |f_2|^4 d\sigma_2$$

is less than or equal to the lower limit as $r \to 0$ of

$$(7) \quad r^{-4}\left(\int \left[\frac{1}{2\pi}\int_0^{2\pi} |1 + rf_k e^{ix}|^p dx - \frac{p^2}{4}|r|^2|f_k|^2 - 1\right] d\sigma_k\right)$$

with $k = 2$. From (3) we see that if $|z| < 1/2$, (2) is bounded by $A|z|^4$, and we see from (2) that if $|z| \geqq 1/2$ (2) is bounded by $A|z|^p$ where A depends only on p. Therefore (2) is bounded by $A|z|^4 + A|z|^p$, and by $A|z|^4$ if $2 < p \leqq 4$. Thus the left hand side of (4) is bounded by

(8) $\quad A|f_k|^4 + Ar^{p-4}|f_k|^p$

if $4 \leqq p$ and by

(9) $\quad A|f_k|^4$

if $2 < p \leqq 4$. Apply the dominated convergence theorem to (4) with $k = 1$ and we see that

$$\binom{p/2}{2}^2 \int |f_1|^4 d\sigma_1$$

is the limit when $r \to 0$ of (7) with $k = 1$. But (7) does not depend on k by our assumption (1), Fubini's Theorem, and fact (a) being established previously. Thus

$$\int |f_2|^4 d\sigma_2 \leqq \int |f_1|^4 d\sigma_1. \tag{10}$$

Since this implies $\int |f_2|^4 d\sigma_2 < \infty$, the same reasoning shows the reverse inequality of (10) is also true and (b) is established.

One should note that if $f_k \in L^\infty(\sigma_k)$ we can establish (b) for *any* $0 < p < \infty$ and $p \neq 2$. This results from the dominated convergence theorem applied to (4) using (8) or (9).

If p is not an even integer and $f_k \in L^\infty(\sigma_k)$ we can establish

$$\int |f_1|^{2l} d\sigma_1 = \int |f_2|^{2l} d\sigma_2$$

for all positive integers l and hence that

$$||f_1||_\infty = ||f_2||_\infty.$$

For this we use an induction on l and subtract appropriate multiples of $|z|^{2l}$ from (2), modify (4) accordingly, and use dominated convergence.

Theorem B is now an immediate consequence of our Theorem A and the proof in Forelli [**1**].

Proof of Theorem B. Let $f \in M$. Since $T(1 + zf) = 1 + zT(f)$ and T is an L^p isometry,

$$\int |1 + zf|^p d\sigma_1 = \int |1 + zTf|^p d\sigma_2.$$

By Theorem A,

$$\int |1 + zf|^4 d\sigma_1 = \int |1 + zTf|^4 d\sigma_2$$

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and since $f \in L^\infty(\sigma_1)$ both of these are finite. From here one need only copy the proof of Proposition 2 in [**1**], with $p = 4$, noting that the infinite series are finite binomial expansions valid for all z, to obtain that T is a homomorphism.

We must also show that

$$(11) \quad \int T(f)\overline{T(g)}d\sigma_2 = \int f\bar{g}d\sigma_1$$

for f and g in M. But part (a) of Theorem A shows that (11) follows from well known facts about isometries of complex inner product spaces.

COROLLARY. *Under the hypothesis of Theorem* B, *if* $f \in M$ *then*

$$||Tf||_\infty = ||f||_\infty.$$

Proof. The proof is the same as in Forelli [**2**]. For any l,

$$\int |Tf|^{2l}d\sigma_2 = \int (Tf)^l\overline{(Tf)^l}$$

and using the homomorphism property the above equals

$$\int T(f^l)\overline{T(f^l)}.$$

Hence, by part (b) of Theorem B

$$\int |Tf|^{2l}d\sigma_2 = \int |f|^{2l}d\sigma_1$$

for all l and the corollary follows since $f \in L^\infty(\sigma_1)$.

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