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CONTINUED FRACTION SOLUTIONS OF THE RICCATI EQUATION

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It is shown that for a solution of a Riccati equation with polynomial coefficients an expansion can be constructed as a Stieltjes continued fraction, with coefficients given by a recurrence relation, which is in general non-linear. Particular expansions associated with hypergeometric and confluent hypergeometric equations are given, and are shown to have a uniquely simple form.

Introduction

Observations about continued fraction solutions of equations of Riccati type go back to Euler [2]. Some developments in more recent times are due to Fair [3], Khovanskii [4], and Merkes and Scott [5]. This paper treats Riccati equations of the form

$$xA(x)y' = xB(x) + C(x)y + D(x)y^2 \tag{1}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are polynomials. It is shown that solutions of (1) have continued fraction expansions about zero of the form

$$y = \cfrac{\alpha_0}{1+\cfrac{\alpha_1 x}{1+\cfrac{\alpha_2 x}{1+\dots}}}$$

or possibly

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$$y = \cfrac{\alpha_0 x}{1+\cfrac{\alpha_1 x}{1+\cfrac{\alpha_2 x}{1+\dots}}}$$

where the coefficients $\alpha_0, \alpha_1, \dots$ are generated by a recurrence relation, which is of finite order, but in general non-linear.

Of course, it is possible to form similar expansions about points other than zero. The theory given here can be used simply by transforming (1) so that the expansion is about zero.

Invariance under bilinear transformation

Consider the standard form

$$(2) \qquad R_i(Z_i) = xA_iZ_i' + B_i + C_iZ_i + xD_iZ_i^2 = 0$$

where A_i, B_i, C_i, D_i are polynomials in x. Let a_i, b_i, c_i, d_i be their respective orders.

Let a new function $Z_{i+1}(x)$ be given by

$$(3) \qquad Z_i = \frac{\alpha_i}{1+xZ_{i+1}}$$

where $\alpha_i = -B_i(0)/C_i(0)$. Then by substitution,

$$\alpha_iA_ix(xZ_{i+1}') = \left(B_i+\alpha_iC_i+\alpha_i^2xD_i\right) + Z_{i+1}x(\alpha_iC_i+2B_i-\alpha_iA_i) + Z_{i+1}^2x^2B_i .$$

This equation can be divided by $x\alpha_i$ and then Z_{i+1} is seen to satisfy the new Riccati equation

$$R_{i+1}(Z_{i+1}) = 0$$

where

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$$(4)\qquad \begin{cases} A_{i+1} = -A_i \ , \\ B_{i+1} = \left(\left(B_i/\alpha_i\right)+C_i+\alpha_i xD_i\right)/x \ , \\ C_{i+1} = \left(C_i+\left(2B_i/\alpha_i\right)-A_i\right) \ , \\ D_{i+1} = B_i/\alpha_i \ . \end{cases}$$

The coefficient α_i has been chosen to ensure that B_{i+1} is a polynomial.

Let M_i be defined for each i as the maximum of the orders of the polynomials A_i, B_i, C_i, D_i. Then it is clear from (4) that

$$M_{i+1} \le M_i \ .$$

Suppose that a Riccati equation

$$(5)\qquad R_0\left(Z_0\right) = 0$$

has been given, in the standard form (2) with $i = 0$, and the transformation (3) is applied repeatedly with $i = 0, 1, 2, \ldots$. Then a sequence $\{\alpha_i;\ i = 0, 1, 2, \ldots\}$ is generated which, from the form of (3), is clearly the set of coefficients for a continued fraction expansion of a solution $Z_0(x)$ of (5), which is regular at $x = 0$. The equation (5) is singular, and $Z_0(x)$ is unique.

The relations (4) give recursively the coefficients of the four polynomials A_{i+1}, B_{i+1}, C_{i+1} and D_{i+1} from those of A_i, B_i, C_i and D_i. Since $M_i \le M_{i-1} \le \ldots \le M_0$ for each $i = 1, 2, \ldots$, there are not more than $4\left(M_0+1\right)$ such coefficients for each i. Then (4), with $i = 0, 1, 2, \ldots$, constitutes a true recurrence relation of order not more than $4\left(M_0+1\right)$. It is nonlinear because of the occurrence of the ratio $\alpha_i = -B_i(0)/C_i(0)$.

Some special relations

Let $A_{i,m}$ denote the mth coefficient of the polynomial A_i. Then

$$\alpha_i = -B_{i,0}/C_{i,0} \ .$$

From (4), $A_i = (-1)^i A_0$, so $A_{i,0} = (-1)^i A_{0,0}$, and

$$C_{i+1,0} = C_{i,0} - A_{i,0} - 2B_{i,0}\left(C_{i,0}/B_{i,0}\right)$$
$$= -C_{i,0} - (-1)^i A_{0,0} \ .$$

Therefore $(-1)^{i+1}C_{i+1,0} = (-1)^i C_{i,0} + A_{0,0}$; that is

(6) $$(-1)^i C_{i,0} = C_{0,0} + iA_{0,0} \ .$$

In the cases where $b_i = d_i = c_i - 1 = M - 1 \geq a_i - 1$, $i = 0, 1, \ldots$, then

$$C_{i+1,M} = C_{i,M} - \left(A_{0,M} \sum_{j=1}^{i} (-1)^j\right)$$
$$= C_{0,M} - (-1)^i A_{0,M} \ .$$

So

(7) $$C_{i,M} = C_{0,M} + \sigma_i A_{0,M}$$

where $\sigma_i = \left(1-(-1)^i\right)/2$. Further,

(8) $$D_{i+1,0} = -B_{i,0}\left(C_{i,0}/B_{i,0}\right) = -C_{i,0} \ .$$

Particular cases

I. The simplest case of interest has $c_i = 1$, $a_i = b_i = d_i = 0$. This is the Riccati equation

(9) $$x\left(Z_0'+Z_0^2\right) + (b-x)Z_0 - a = 0$$

derived from the confluent hypergeometric equation

$$xy'' + (b-x)y' - ay = 0$$

by letting

$$Z_0 = y'/y \ .$$

Then, from (7), $C_{i,1} = C_{0,1} = -1$, $i = 1, 2, \ldots$ and, from (6),

$$\begin{aligned}(-1)^i C_{i,0} &= C_{0,0} + iA_{0,0} \\ &= b + i \ , \quad i = 1, 2, \ldots \ .\end{aligned}$$

Then

$$\begin{aligned}B_{i+1,0} &= C_{i,1} + \alpha_i C_{i,0} \ , \quad i = 0, 1, 2, \ldots \ , \\ &= C_{i,1} - \left(B_{i,0}/C_{i,0}\right)C_{i-1,0} \quad \text{from (8).}\end{aligned}$$

So

$$\begin{aligned}B_{i+1,0}C_{i,0} &= C_{i,0}C_{i,1} + B_{i,0}C_{i-1,0} \\ &= C_{i,0}C_{i,1} + C_{i-1,0}C_{i-1,1} + \ldots + C_{1,0}C_{1,1} + B_{1,0}C_{0,0} \\ &= \sum_{j=0}^{i} C_{i,0}C_{i,1} + \alpha_0 D_{0,0}C_{0,0} \\ &= -\sum_{j=0}^{i} (-1)^i(b+i) + (a/b)\cdot b \\ &= \left(a-\tfrac{1}{4}-(b/2)+(-1)^i\left(\left((2i-1)/4\right)+(b/2)\right)\right)/(b+i)(-1)^i \ .\end{aligned}$$

Then

$$\alpha_i = -\left(B_{i,0}/C_{i,0}\right) = \left(a+\tfrac{1}{4}-(b/2)+(-1)^i\left(\left((2i-1)/4\right)+(b/2)\right)\right)/(b+i)(b+i-1) \ .$$

This gives the continued fraction expansion for $M'(a, b, x)/M(a, b, x)$ about zero. Replacing x by $1/x$ in (9) gives

$$-xZ_0' + Z_0^2/x + \left(b-(1/x)\right)Z_0 - a = 0 \ ,$$

and, setting $x^2 Z_0 = w - ax$,

$$-x^2 w' + w^2 + \left((b-2a)x-1\right)w - a(b-a-1) = 0 \ .$$

In this case $A_{0,0} = 0$, so $(-1)^i C_{i,0} = C_{0,0} = 1$ while $C_{i,1} = C_{0,1} - \sigma_i A_{0,1}$ where $\sigma_i = \left(1-(-1)^i\right)/2$. Then

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$$B_{i,0}C_{i-1,0} = \sum_{j=0}^{i-1} C_{j,0}C_{j,1} - D_{0,0}B_{0,0}$$
$$= \sigma_{i-1}C_{0,1} - \left((i+\sigma_i)/2\right)A_{0,1} + a(b-a-1) \ ,$$

$$B_{i,0} = (-1)^i\left[\left((i+\sigma_i)/2\right)-\sigma_{i-1}(2a-b)-a(b-a-1)\right]$$

and

$$\alpha_i = -\left(B_{i,0}/C_{i,0}\right) = -\left[\left((i+\sigma_i)/2\right)-\sigma_{i-1}(2a-b)+a(b-a-1)\right] \ .$$

This gives coefficients for the continued fraction expansion about ∞ for

$$x^{-a}\left(x^aU(a,\ b,\ x)\right)'/U(a,\ b,\ x) = a/x + U'(a,\ b,\ x)/U(a,\ b,\ x)$$
$$= a\left((1/x)-\left(U(a+1,\ b+1,\ x)/U(a,\ b,\ x)\right)\right) \ .$$

It appears to be new in the literature.

II. The next simple case has $a_i = c_i = 1$, $b_i = d_i = 0$. The Riccati equation can be represented as

$$(10) \qquad x(1-x)Z_0' + axZ_0^2 + \ c+(a-b)x\ Z_0 + c - b = 0$$

where the notation has been chosen to be consistent with that for hypergeometric functions. The solution of (10) regular at $x = 0$ is

$$Z_0 = \left((b-c)/c\right)\left(F(a+1,\ b;\ c+1;\ x)/F(a,\ b;\ c;\ x)\right) \ .$$

The new relations for the coefficients of $C_i(x)$ are

$$(-1)^iC_{i,0} = C_{0,0} + iA_{0,0}$$
$$= c + i \quad \text{from (6)}$$

and

$$C_{i,1} = a - b - \sigma_i \quad \text{from (7).}$$

Then once again

$$B_{i,0}C_{i-1,0}$$

$$= \sum_{j=0}^{i-1} C_{j,0}C_{j,1} - D_{0,0}B_{0,0}$$

$$= (b-a)\left[\left((i/2)-\tfrac{1}{4}\right)(-1)^i+\tfrac{1}{4}\right]-c(a-b)\sigma_{i-1}+\left((i-\sigma_i)/2\right)c+\left((i-\sigma_i)/2\right)^2+a(c-b)$$

$$= \left((i/2)+b+\sigma_i(a-b-\tfrac{1}{2})\right)\left((i/2)+c-a+\sigma_i(a-b-\tfrac{1}{2})\right)\,.$$

So

$$\alpha_i = -B_{i,0}/C_{i,0} = \frac{\left((i/2)+c+\sigma_i(a-b-\tfrac{1}{2})\right)\left((i/2)+c-a+\sigma_i(a-b-\tfrac{1}{2})\right)}{(c+i)(c+i-1)}\,.$$

This is the Gauss continued fraction of the hypergeometric function.

Conclusion

The special cases studied here include almost all known continued fractions with coefficients given by explicit expressions. They include, for example, all those given in [1].

The above algebra shows that there is a natural association between Riccati equations and continued fractions, corresponding to that between linear differential equations and power series. In each case the equation determines a recurrence relation for the coefficients.

References

[1] Milton A. Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (National Bureau of Standards Applied Mathematics Series, 55. United States Department of Commerce, Washington, D.C., 1964).

[2] L. Euler, "Analysis facilis aequationem Riccatianam per fractionem continuam resolvendi", *Mem. Acad. Imper. Sci. Petersb.* (1813-1814).

[3] Wyman Fair, "Padé approximation to the solution of the Riccati equation", *Math. Comp.* 18 (1964), 627-634.

[4] Alexey Nikolaevitch Khovanskii, *The application of continued fractions and their generalizations to problems in approximation theory* (translated by Peter Wynn. Noordhoff, Groningen, 1963).

[5] E.P. Merkes and W.T. Scott, "Continued fraction solutions of the Riccati equation", *J. Math. Anal. Appl.* 4 (1962), 309-327.

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