THE CURRENT THEORY OF ANALYTIC SETS

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1. Introduction. In this paper we describe the outlines of the theory of analytic sets from the point of view of recent work on the subject. Our aim is to present the concepts and some of the principal results in a setting useful to workers in analysis, especially those workers not familiar with the field or its current developments. No attempt has been made to include all the results concerning analytic sets—not even in a particular category. There are some excellent monographs **(12; 18)** as well as chapters in books **(2, 8, 9, 15, 17)** where the subject is treated extensively. These, however, do not contain the recent results and consider only metric spaces. Our emphasis is on a general topological setting.

Three Borel families, the family of Souslin sets, and the family of analytic sets are introduced in Section 3. Sections 4 and 5 are concerned with relations between these families. Section 6 is concerned with approximation theorems and miscellaneous results. Many of these theorems are generalizations of known ones. Although an attempt has been made to make this paper reasonably self-contained, the proofs of several theorems have been omitted, because they are too long, or highly technical, or similar to others which are given.

Analytic sets were first introduced by Souslin **(27)** in 1917, and an extensive theory was developed during the following two decades. After that, interest in the subject diminished considerably. In the early fifties, however, several isolated results appeared in different fields and brought renewed interest in the subject. In the past few years the theory has been extended considerably.

Perhaps the best known of recent results is Choquet's theorem in potential theory about the capacitability of analytic sets **(5)**. Choquet's theorem is concerned with approximation from below. So are many of the recent results. Also typical is the fact that although he was not primarily interested in analytic sets *per se*, Choquet was led to their use by the methods of proof. It is now apparent that this is no accident. In many problems involving approximation from below the natural family to consider is that of analytic sets, not some Borel family. On this point, readers may find Section 6 of particular interest.

2. Notation.

2.0. 0 is both the empty set and the number zero.

2.1. ω is the set of natural numbers (including zero).

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2.2. For n in ω, $\bar{n} = \{m: m \in \omega \text{ and } 0 \leqslant m \leqslant n\}$.

2.3. S is the set of all sequences of natural numbers; i.e., all functions on ω to ω.

2.4. S_n' is the set of all ordered $(n + 1)$-tuples of natural numbers; i.e., all functions on $\bar{n}$ to ω.

If f is a function and A is a set, then

2.5. $f[A]$ is the direct f-image of A, i.e.,

$$f[A] = \{y: y = f(x) \text{ for some } x \in A\};$$

2.6. $f^{-1}[A]$ is the counter f-image of A, i.e.

$$f^{-1}[A] = \{x: y = f(x) \text{ for some } y \in A\};$$

2.7. $f|A$ is the restriction of f to A, i.e., the function g on $A \cap \text{domain } f$ such that for each $x \in A \cap \text{domain } f$, $g(x) = f(x)$.

Let H be a family of sets. Then

2.8. $H_\sigma = \{A: A = \bigcup_{n \in \omega} B_n$ for some sequence B of sets in $H\}$,

2.9. $H_\delta = \{A: A = \bigcap_{n \in \omega} B_n$ for some sequence B of sets in $H\}$,

2.10. $H_{\sigma\delta} = (H_\sigma)_\delta$.

3. Definitions of Borel, Souslin, and analytic families. In this section, the five families of sets with which this paper is concerned are introduced. First, the formal definitions are given, and then some remarks are made on ways in which these families arise.

In 3.1 and 3.2 it is helpful to think of H as a family of subsets of some space X which covers X.

3.1. *Definitions: The Borel Families.*

3.1.1. Borelian H is the smallest family which contains H and is closed to countable, non-vacuous union and intersection.

3.1.2. Borel ring H is the smallest family which contains H and is closed to countable, non-vacuous union and set difference.

3.1.3. Borel field H is the smallest family which contains H and is closed to countable (including vacuous) union and complementation with respect to $\bigcup_{\alpha \in H} \alpha$.

3.2. *Definition.* Souslin H is the family of all sets A such that

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n})$$

for some function h on $\bigcup_{n \in \omega} S_n'$ to H.

3.3. *Definition.* For X a topological space, Analytic in X is the family of all sets A such that

$$A = f[\alpha]$$

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for some $\alpha \in K'_{\sigma\delta}$ and some continuous function f on α to X, where K' is the family of closed, compact sets in a topological space X'.

The Borel families, in general distinct, are well known and their very definitions indicate the types of set operations which give rise to them.

The set operation involved in the definition of the Souslin family is the operation $(\mathscr{A})$. To see how it arises, consider the following situation which occurs frequently in analysis. Let A be an arbitrary set and let f be a function on A. In order to study properties of $f[A]$, the set A is in some way decomposed, thereby introducing a decomposition of $f[A]$. Now suppose $\mathfrak{P}$ is a family of subsets of A which covers A. If, for each $x \in A$, $F_x = \{\alpha \in \mathfrak{P}: x \in \alpha\}$, then

$$f[A] = \bigcup_{x\in A} \bigcap_{\alpha\in F_x} f[\alpha].$$

One is therefore led to consider sets of the form

$$\bigcup_{x\in A} \bigcap_{\alpha\in F_x} h(\alpha), \tag{3.4}$$

where h is a set function with domain $\mathfrak{P}$.

It has been suggested by some that this be taken as the definition of a Souslin set. Such a definition, however, seems to be too broad. For most of the present development of the theory, it seems necessary to require that $\mathfrak{P}$ be countable. It then turns out that this is equivalent to assuming that A is the space S of all sequences to ω and that $\mathfrak{P}$ is the family of all sets α of the form

$$\alpha = \{s \in S: s_i = x_i \text{ for } i = 0, \ldots, n\}$$

for some $n \in \omega$ and $x \in S_n'$. If in addition it be required that the $h(\alpha)$ belong to a given family H, then (3.4) yields the sets of Definition 3.2.

Analytic sets come up naturally when one is interested in continuous images of Borel sets. In fact, it was in order to show that in Euclidean space a continuous image of a Borel set is not necessarily Borel that Souslin introduced the operation $(\mathscr{A})$. His fundamental result was the following: *If H is the family of closed linear sets then* Souslin H *contains sets which are not* Borel *and* Souslin H *coincides with the family of continuous images of $\mathfrak{G}_\delta$ sets.*

For this reason, Lusin **(11, 12)**, working in metric spaces, defined an analytic set as a continuous image of a $\mathfrak{G}_\delta$, where $\mathfrak{G}$ is the family of open sets in a complete, separable, metric space. Then, Choquet **(4, 6)**, working in topological spaces, defined an analytic set as a continuous image of a $K_{\sigma\delta}$ set, where K is the family of compact sets in a compact Hausdorff space. Recently, Sion **(22)**, working in abstract spaces, defined an (f, H)-analytic set as the image under f of an $H_{\sigma\delta}$ set satisfying certain conditions. The definition given in 3.3 is a slight modification of Choquet's definition.

4. Non-topological properties of Borel and Souslin families. This section is concerned with quite general relations between the Borel and Souslin families when no topological structure is assumed on the underlying space.

In 4.1 through 4.4 are listed the main relations between the various families. The inclusions in 4.1 and 4.2 are, in general, proper. Lemmas 4.5 through 4.8 are concerned with interchanging the order of the operations $\cup$ and $\cap$. They are needed in the proofs of many theorems. Amongst other things, 4.5 and 4.6 indicate situations where a Souslin set is Borelian. Proofs and some remarks follow the list.

4.1. THEOREM. $H \subset$ Borelian $H \subset$ Borel ring $H \subset$ Borel field H.

4.2. THEOREM. Borelian $H \subset$ Souslin H.

4.3. THEOREM. Souslin Souslin H = Borelian Souslin H = Souslin H.

4.4. THEOREM. *If* $A \in$ Souslin H *and* Ω *is the first non-countable ordinal, then there exist transfinite sequences* B *and* C *such that for each* $\alpha \in \Omega$

$$B_\alpha \in \text{Borelian } H, \qquad C_\alpha \in \text{Borel ring } H,$$

and

$$\bigcap_{\alpha\in\Omega} B_\alpha = A = \bigcup_{\alpha\in\Omega} C_\alpha.$$

4.5. LEMMA. *If* h *is such a function on* $\bigcup_{n\in\omega} S_n'$ *that for each* $n \in \omega$

(i) *if* $x \in S_n'$ *and* $y \in S_n'$ *and* $x \neq y$, *then* $h(x) \cap h(y) = 0$, *and*

(ii) *if* $x \in S'_{n+1}$, *then* $h(x) \subset h(x|\bar{n})$, *then*

$$\bigcup_{s\in S} \bigcap_{n\in\omega} h(s|\bar{n}) = \bigcap_{n\in\omega} \bigcup_{s\in S} h(s|\bar{n}) = \bigcap_{n\in\omega} \bigcup_{x\in S_n'} h(x).$$

4.6. LEMMA. *Let* $k \in S$, $T = \{s \in S\colon s_i \leqslant k_i$ *for each* $i \in \omega\}$, *and for* $n \in \omega$ *let* $T_n' = \{x \in S_n'\colon x_i \leqslant k_i$ *for each* $i \in \bar{n}\}$. *If* h *is such a function on* $\bigcup_{n\in\omega} S_n'$ *that for each* $n \in \omega$ *and* $x \in T'_{n+1}$, $h(x) \subset h(x|\bar{n})$, *then*

$$\bigcup_{s\in T} \bigcap_{n\in\omega} h(s|\bar{n}) = \bigcap_{n\in\omega} \bigcup_{s\in T} h(s|\bar{n}) = \bigcap_{n\in\omega} \bigcup_{x\in T_n'} h(x).$$

4.7. COROLLARY. *If* H *is closed to finite union and intersection and* $A \in$ Souslin H, *then*

$$A = \bigcup_{s\in S} \bigcap_{n\in\omega} h(s|\bar{n}),$$

where h *is such a function on* $\bigcup_{n\in\omega} S_n'$ *to* H *that for each* $n \in \omega$, $x \in S'_{n+1}$ *and* $j \in \omega$

$$h(x) \subset h(x|\bar{n})$$

and

$$h(x_0, \ldots, x_n, j) \subset h(x_0, \ldots, x_n, j+1).$$

4.8. LEMMA. *Let* $A \times B$ *be the Cartesian product of* A *with* B. *Let* A^B *be the set of functions on* B *to* A. *If* F *is any function on* $S \times \omega$, *then*

$$\bigcap_{n\in\omega} \bigcup_{s\in S} F(s, n) = \bigcup_{\phi\in S^\omega} \bigcap_{n\in\omega} F(\phi(n), n).$$

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Proofs and remarks.

Proof of 4.8. This lemma is a special case of the following proposition which is actually equivalent to the axiom of choice:

If X and Y are any sets and F is any function on $X \times Y$, then

$$\bigcap_{x \in X} \bigcup_{y \in Y} F(x, y) = \bigcup_{f \in Y^X} \bigcap_{x \in X} F(x, f(x)).$$

For if

$$p \in \bigcap_{x \in X} \bigcup_{y \in Y} F(x, y),$$

then for each $x \in X$ there is $y \in Y$ with $p \in F(x, y)$. Using the axiom of choice, for $x \in X$, let $f(x)$ be such that $f(x) \in Y$ and $p \in F(x, f(x))$. Then

$$p \in \bigcap_{x \in X} F(x, f(x))$$

and

$$\bigcap_{x \in X} \bigcup_{y \in Y} F(x, y) \subset \bigcup_{f \in Y^X} \bigcap_{x \in X} F(x, f(x)).$$

Conversely, if for some $f \in Y^X$ and for each $x \in X$ $\quad p \in F(x, f(x))$, then

$$p \in \bigcap_{x \in X} \bigcup_{y \in Y} F(x, y).$$

Proof of 4.1. By definition $H \subset$ Borelian H. The formulas

$$\bigcap_{n \in \omega} \alpha_n = \alpha_0 - \bigcup_{n \in \omega} (\alpha_0 - \alpha_n)$$

and

$$\alpha - \beta = X - ((X - \alpha) \cup \beta),$$

where $X = \bigcup_{\gamma \in H} \gamma$, enable one to conclude that Borel ring H is closed to countable intersection and Borel field H is closed to set difference.

Proof of 4.2 *and* 4.3. It is easy to verify that H_σ and H_δ are contained in Souslin H. Thus, if Souslin Souslin $H \subset$ Souslin H, then Souslin H is closed to countable union and countable intersection. Hence, Borelian $H \subset$ Borelian Souslin $H \subset$ Souslin H.

In view of these remarks it is enough to show that Souslin Souslin $H \subset$ Souslin H.

Let $A \in$ Souslin Souslin H. Then for some f on $\bigcup_{n \in \omega} S_n'$ to Souslin H,

$$A = \bigcup_{x \in S} \bigcap_{m \in \omega} f(x|\bar{m}),$$

and for each $x \in S$ and $m \in \omega$,

$$f(x|\bar{m}) = \bigcup_{y \in S} \bigcap_{n \in \omega} g(x|\bar{m}; y|\bar{n}),$$

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where $g(x|\bar{m};\quad)$ is some function on $\bigcup_{n\in\omega} S_n'$ to H. Thus

$$A = \bigcup_{x\in S}\bigcap_{m\in\omega}\bigcup_{y\in S}\bigcap_{n\in\omega} g(x|\bar{m}; y|\bar{n})$$

$$= \bigcup_{x\in S}\bigcup_{\xi\in S^{\omega}}\bigcap_{m\in\omega}\bigcap_{n\in\omega} g(x|\bar{m}; \xi(m)|\bar{n})$$

in accordance with 4.8.

Now let c be such a sequence of natural numbers that for each $n \in \omega$, $c_n \leqslant n$ and $\{i: c_i = n\}$ is infinite. Let the elements of this set be ordered so that

$$\{i: c_i = n\} = \{N_j(n): c_{N_j(n)} = n \text{ for each } j \in \omega\}.$$

Let ϕ be a one-to-one function on ω onto $\omega \times \omega$. For $x \in S$, let x' and x'' be the members of S defined by

$$\phi(x_k) = (x_k', x_k'') \qquad \text{for } k \in \omega.$$

For $n \in \omega$, let k_n be the largest j in ω such that

$$N_j(c_n) \leqslant n$$

and let

$$h(x|\bar{n}) = g(x_0', \ldots, x'_{c_n}; x''_{N_0(c_n)}, \ldots, x''_{N_{k_n}(c_n)}).$$

Then

$$\bigcap_{n\in\omega} h(x|\bar{n}) = \bigcap_{m\in\omega}\bigcap_{\substack{n\in\omega\\ c_n=m}} g(x'|\bar{m}; x''_{N_0(m)}, \ldots, x''_{N_{k_m}(m)})$$

$$\subset \bigcap_{m\in\omega} f(x'|\bar{m}).$$

Thus

$$\bigcap_{x\in S}\bigcup_{n\in\omega} h(x|\bar{n}) \subset A.$$

On the other hand, for $s \in S$ and $\xi \in S^{\omega}$, let x be the member of S defined by

$$\phi(x_n) = (s_n, \xi_i(c_n)) \qquad \text{for } n \in \omega,$$

where i is such that $N_i(c_n) = n$. Then

$$\bigcap_{m\in\omega}\bigcap_{n\in\omega} g(s|\bar{m}; \xi(m)|\bar{n}) \subset \bigcap_{m\in\omega}\bigcap_{\substack{n\in\omega\\ c_n=m}} h(x|\bar{m})$$

$$= \bigcap_{m\in\omega} h(x|\bar{m}),$$

so that

$$A \subset \bigcup_{x\in S}\bigcap_{m\in\omega} h(x|\bar{m}).$$

Thus

$$A = \bigcup_{x\in S}\bigcap_{n\in\omega} h(x|\bar{n}) \in \text{Souslin } H.$$

References for 4.4. This theorem is due to Sierpiński **(16)**. A proof of it can also be found in **(9**, p. 7**)**.

Proof of 4.5. Always

$$\bigcup_{s\in S}\bigcap_{n\in\omega} h(s|\bar{n}) \subset \bigcap_{n\in\omega}\bigcup_{s\in S} h(s|\bar{n}) = \bigcap_{n\in\omega}\bigcup_{x\in S_n'} h(x).$$

Now suppose

$$p \in \bigcap_{n\in\omega}\bigcup_{x\in S_n'} h(x).$$

For each $n \in \omega$ let $x^{(n)}$ be such a member of S_n' that $p \in h(x^{(n)})$. The hypotheses guarantee that

$$x^{(n+1)}|\bar{n} = x^{(n)}.$$

Thus, there is an $s \in S$ such that

$$s|\bar{n} = x^{(n)}$$

and

$$p \in \bigcap_{n\in\omega} h(s|\bar{n}).$$

Proof of 4.6. Always

$$\bigcup_{s\in T}\bigcap_{n\in\omega} h(s|\bar{n}) \subset \bigcap_{n\in\omega}\bigcup_{s\in T} h(s|\bar{n}) = \bigcap_{n\in\omega}\bigcup_{x\in T_n'} h(x).$$

Conversely, suppose

$$p \in \bigcap_{n\in\omega}\bigcup_{x\in T_n'} h(x).$$

It remains to prove the existence of an s in T such that

$$p \in \bigcap_{n\in\omega} h(s|\bar{n}).$$

This can be done as follows.

For n and m in ω, $n \leqslant m$, and $x \in T_n'$, let

$$B_m(x) = \{y \in T: y|\bar{n} = x \text{ and } p \in h(y)\}.$$

Now for each $n \in \omega$ there is $x \in T_n'$ with $B_m(x) \neq 0$ for every $m \in \omega$ and $m \geqslant n$. For otherwise, since T_n' is finite and $B_{m+1}(x) \subset B_m(x)$, there exists $m \in \omega$ with $B_m(x) = 0$ for every $x \in T_n'$. But this is not so because for some $y \in T_m'$, $p \in h(y)$ and $y|\bar{n} \in T_n'$.

Therefore, by recursion, for each $n \in \omega$ there is such an $x^{(n)} \in T_n'$ that $x^{(n+1)}|\bar{n} = x^{(n)}$ and $p \in h(x^{(n)})$. Thus there is s in T with $s|\bar{n} = x^{(n)}$ for each $n \in \omega$, and for this s

$$p \in \bigcap_{n\in\omega} h(s|\bar{n}).$$

It is to be noted that the sole use of the fixed bound on the members of T and T_n' is to ensure that T_n' is finite for each $n \in \omega$.

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Proof of 4.7. Let

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} f(s|\bar{n}),$$

where f is on $\bigcup_{n \in \omega} S_n'$ to H. For $n \in \omega$ and $x \in S_n'$ let

$$g(x) = \bigcap_{k \in \bar{n}} f(x|\bar{k}).$$

Then g is on $\bigcup_{n \in \omega} S_n'$ to H, and if $n \in \omega$ and $x \in S'_{n+1}$,

$$g(x) \subset g(x|\bar{n})$$

and

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} g(s|\bar{n}).$$

For $n \in \omega$, $x \in S_n'$, and $k \in \omega$ let

$$h(x_0, \ldots, x_n, k) = \bigcup_{j \in \bar{k}} g(x_0, \ldots, x_n, j).$$

Then h is on $\bigcup_{n \in \omega} S_n'$ to H. If $n \in \omega$, $x \in S_n'$, and $j \in \omega$, then $h(x_0, \ldots, x_n, j) \subset h(x_0, \ldots, x_n, j+1)$, and if $n \in \omega$ and $x \in S'_{n+1}$, then

$$h(x) \subset h(x|\bar{n}).$$

Now

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} g(s|\bar{n}) \subset \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}).$$

Conversely, for $s \in S$ and $n \in \omega$ let

$$T = \{t \in S: t_i \leqslant s_i \text{ for each } i \in \omega\}$$

and

$$T_n' = \{x \in S_n': x_i \leqslant s_i \text{ for each } i \in \bar{n}\}.$$

By 4.6,

$$\begin{aligned} \bigcap_{n \in \omega} h(s|\bar{n}) &= \bigcap_{n \in \omega} \bigcup_{j \in \bar{s}_n} g(s_0, \ldots, s_{n-1}, j) \\ &\subset \bigcap_{n \in \omega} \bigcup_{x \in T_n'} g(x) = \bigcup_{s \in T} \bigcap_{n \in \omega} g(s|\bar{n}) \subset A. \end{aligned}$$

Thus

$$\bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}) \subset A,$$

and the proof is complete.

5. Relations between Borel, Souslin, and analytic families in topological spaces. This section is concerned with relations between the Borel, Souslin, and analytic families when a topological structure is assumed on the underlying space X.

The definitions of the terms used are collected in 5.1. The theorems are collected in three subsections. Part A complements the results of Section 4 by putting topological conditions on the family H. In Part B some results are given on the separation of Souslin sets by Borel sets which lead to criteria for determining when a Souslin set belongs to a Borel family. Part C contains some results on images of analytic and Borel sets which enable one to determine when an analytic set belongs to a Borel family. Proofs and remarks follow Part C.

5.1. *Definitions.*
5.1.1. $K(X)$ is the family of all closed, compact sets in$X$.
5.1.2. $\mathfrak{F}(X)$ is the family of all closed sets in X.
5.1.3. $\mathfrak{G}(X)$ is the family of all open sets in X.

When there is no danger of ambiguity the variable "X" will be dropped from the above three notations.

5.1.4. X has *property I* if and only if X is Hausdorff and $A - B \in K_\sigma(X)$ for each A and B in $K(X)$.
5.1.5. A and B can be *separated* by F if and only if there exist A' and B' in F such that $A \subset A'$, $B \subset B'$, and $A' \cap B' = 0$.
5.1.6. f is *countable-to-one* if and only if $f^{-1}[\{x\}]$ is countable for each x.

A. General relations.

5.2. THEOREMS.

5.2.1. Souslin $K(X) \subset$ Analytic in X.

5.2.2. *If X is Hausdorff, then* Analytic in $X \subset$ Souslin $\mathfrak{F}(X)$.

5.2.3. *If X is Hausdorff and $X \in K_\sigma(X)$, then* Souslin $K(X) =$ Analytic in $X =$ Souslin $\mathfrak{F}(X)$.

5.3. THEOREM. Borelian Analytic in $X =$ Analytic in X.

5.4. THEOREM. *If X is Hausdorff and A and B belong to K, then $A - B \in$* Analytic in X *if and only if $A - B \in K_\sigma$.*

5.5. THEOREM. *X has property I if and only if X is Hausdorff and* Borelian $K =$ Borel ring K.

5.6. THEOREM. *If X is a metric space, then* Borelian $\mathfrak{F} =$ Borel ring $\mathfrak{F} =$ Borel field $\mathfrak{F}$.

5.7. THEOREM. *If X is a complete, separable metric space, then* Analytic in $X =$ Souslin $\mathfrak{F} = \{A : A = f[J]$ *for some continuous function f on J, the set of irrational numbers, to X*$\}$.

B. Separation theorems.

5.8. THEOREM. *If A and B are disjoint and belong to* Souslin K, *then A and B can be separated by* Borelian K.

5.9. COROLLARY. *A and $X - A$ belong to* Souslin K *if and only if A and $X - A$ belong to* Borelian K.

5.10. THEOREM. *If X is a complete, separable metric space, A and B are disjoint and belong to* Souslin $\mathfrak{F}$ (= Analytic in X), *then A and B can be separated by* Borelian $\mathfrak{F}$(= Borel field $\mathfrak{F}$).

5.11. COROLLARY. *If X is a complete, separable metric space, then A and $X - A$ belong to* Souslin $\mathfrak{F}$ *if and only if A and $X - A$ belong to* Borelian $\mathfrak{F}$ *if and only if A belongs to* Borelian $\mathfrak{F}$.

5.12. *Remark.* Theorems 5.8 and 5.10, in view of their corresponding corollaries, are easily extended to the case of separating a sequence of disjoint Souslin sets by a sequence of disjoint Borelian sets. See (**9**, pp. 393-395).

C. Images of analytic and Borel sets.

5.13. THEOREM. *If $A \in$* Analytic in X *and f is a continuous function on A to Y, then $f[A] \in$ Analytic in Y.*

5.14. THEOREM. *If X and Y are Hausdorff, $X \in K_\sigma(X)$, $A \in$* Analytic in Y, *and f is a continuous function on X to Y, then $f^{-1}[A] \in$* Analytic in X.

5.15. THEOREM. *If X has property I, $A \in$* Borel ring $K(X)$, *f is a continuous and countable-to-one function on A to some Hausdorff space Y, and $Y \in K_\sigma(Y)$, then $f[A] \in$* Borelian $K(Y)$ *and $f[A]$ has property I.*

5.16. THEOREM. *If X and Y are complete, separable metric spaces, $A \in$* Analytic in X (= Souslin $\mathfrak{F}(X)$) *and f is a Borel function on A to Y* (i.e., $f^{-1}[\alpha] \in$ Borelian $\mathfrak{F}(X)$ *for each* $\alpha \in \mathfrak{G}(Y)$), *then* $f[A] \in$ Analytic in Y (= Souslin $\mathfrak{F}(Y)$).

5.17. THEOREM. *If X and Y are complete, separable metric spaces, $A \in$* Borelian $\mathfrak{F}(X)$ *and, f is a Borel and countable-to-one function on A to Y, then $f[A] \in$ Borelian $\mathfrak{F}(Y)$.*

Proofs and remarks. The following lemma is useful here and in Section 6.

5.18. LEMMA. *Let $D \subset X'$, f be a continuous function on D to X, and A be a descending sequence of closed, compact sets in X' such that $\bigcap_{n\in\omega} A_n \subset D$.*

5.18.1. *If G is open in X and $f[\bigcap_{n\in\omega} A_n] \subset G$, then for some $n \in \omega$, $f[D \cap A_n] \subset G$, and*

5.18.2. *If X is Hausdorff, then*

$$f\Big[\bigcap_{n\in\omega} A_n\Big] = \bigcap_{n\in\omega} f[D \cap A_n] = \bigcap_{n\in\omega} \text{closure} f[D \cap A_n].$$

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Proof of 5.18. Suppose G is open in X and let $G' = f^{-1}[G]$. Then G' is open in D, i.e., $G' = G'' \cap D$, where G'' is open in X', and

$$\bigcap_{n \in \omega} A_n \subset G' \subset G''.$$

Hence for some $n \in \omega$, $A_n \subset G''$ and therefore $D \cap A_n \subset G'' = G'$ so that $f[D \cap A_n] \subset G$.

Next suppose X is Hausdorff and $y \notin f[\bigcap_{n \in \omega} A_n]$. Since $\bigcap_{n \in \omega} A_n$ is compact, so is $f[\bigcap_{n \in \omega} A_n]$, and there is an open set G contained in X such that

$$f\Big[\bigcap_{n \in \omega} A_n\Big] \subset G \qquad \text{and } y \notin \text{closure } G.$$

By 5.18.1 for some $n \in \omega$,

$$\text{closure } f[D \cap A_n] \subset \text{closure } G \qquad \text{and} \qquad y \notin \text{closure } f[D \cap A_n].$$

Thus

$$\bigcap_{n \in \omega} \text{closure } f[D \cap A_n] \subset f\Big[\bigcap_{n \in \omega} A_n\Big].$$

On the other hand one always has

$$f\Big[\bigcap_{n \in \omega} A_n\Big] \subset \bigcap_{n \in \omega} f[D \cap A_n] \subset \bigcap_{n \in \omega} \text{closure } f[D \cap A_n].$$

Proof of 5.2.1. This theorem was first proved by Choquet **(6)** for the case where X is Hausdorff.

Order the elements of S as follows:

If s and t are in S and $s \neq t$, let n be the least natural number for which $s_n \neq t_n$ and then define

$$s < t \text{ if and only if } s_n < t_n.$$

For $x \in S_n'$, let

$$\mathfrak{S}(x) = \{s \in S \colon s|\bar{n} = x\}.$$

Then in the topology on S induced by the above ordering,

$$\mathfrak{S}(x) \in K_\sigma(S).$$

Now if $A \in$ Souslin $K(X)$, then, in accordance with 4.7.

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}),$$

where for each $s \in S$ and $n \in \omega$,

$$h(s|\overline{n+1}) \subset h(s|\bar{n}) \in K(X).$$

Let $X' = X \times S$, with the product topology, let

$$B_n = \bigcup_{x \in S_n'} (h(x) \times \mathfrak{S}(x)),$$

and let

$$D = \bigcap_{n \in \omega} B_n.$$

Since S_n' is countable, $D \in K_{\sigma\delta}(X')$.

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Let P be the projection function on pairs such that $P(x, y) = x$. The proof will be completed by showing that

$$A = P[D]$$

so that $A \in$ Analytic in X.

If $p \in A$, then for some $s \in S$,

$$p \in \bigcap_{n \in \omega} h(s|\bar{n}),$$

and

$$(p, s) \in \bigcap_{n \in \omega} (h(s|\bar{n}) \times \mathfrak{S}(s|\bar{n})) \subset D.$$

If $p \in P[D]$, then, by 4.5,

$$\begin{aligned} p &\in P\Big[\bigcap_{n \in \omega} \bigcup_{x \in S_n'} (h(x) \times \mathfrak{S}(x))\Big] \\ &= P\Big[\bigcup_{s \in S} \bigcap_{n \in \omega} (h(s|\bar{n}) \times \mathfrak{S}(s|\bar{n}))\Big] \\ &\subset \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}) \\ &= A. \end{aligned}$$

Proof of 5.2.2. This result is due to Sion **(19)**.

Let $A = f[D]$, where f is a continuous function on D,

$$D = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j),$$

and $d(i, j) \in K(X)$ for each i and j in ω. For $n \in \omega$ and $x \in S_n'$ let

$$B(x) = f\Big[D \cap \bigcap_{i \in \bar{n}} d(i, x_i)\Big].$$

Then closure $B(x) \in \mathfrak{F}(X)$, and

$$\begin{aligned} A &= f\Big[\bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j)\Big] \\ &= f\Big[\bigcup_{s \in S} \bigcap_{i \in \omega} d(i, s_i)\Big] \\ &= \bigcup_{s \in S} f\Big[\bigcap_{i \in \omega} d(i, s_i)\Big] \\ &= \bigcup_{s \in S} f\Big[\bigcap_{n \in \omega} \bigcap_{i \in \bar{n}} d(i, s_i)\Big] \\ &= \bigcup_{s \in S} \bigcap_{n \in \omega} f\Big[D \cap \bigcap_{i \in \bar{n}} d(i, s_i)\Big], \qquad \text{by 5.18.2,} \\ &= \bigcup_{s \in S} \bigcap_{n \in \omega} \text{closure } B(s|\bar{n}) \\ &\in \text{Souslin } \mathfrak{F}(X). \end{aligned}$$

Theorem 5.2.3 follows readily from 5.2.2.

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Proof of 5.3. This result is due to Choquet **(5)**.

For each $n \in \omega$ let $A_n \in$ Analytic in X. Thus for $n \in \omega$

$$A_n = f_n[D_n],$$

where

$$f_n \text{ is a continuous function on } D_n,$$

and

$$D_n \in K_{\sigma\delta}(X^{(n)}) \qquad \text{for some } X^{(n)}.$$

Without loss of generality it can be assumed that the spaces $X^{(n)}$ are disjoint.

Let

$$B = \bigcup_{n \in \omega} A_n.$$

To see that $B \in$ Analytic in X, let Y be the topological union space with components $X^{(n)}$ for $n \in \omega$, and let

$$E = \bigcup_{n \in \omega} D_n.$$

Then

$$E \in K_{\sigma\delta}(Y).$$

Let g be on E to X such that for each $n \in \omega$

$$g|D_n = f_n.$$

Then g is continuous on E, and

$$B = g[E] \in \text{Analytic in } X.$$

Let

$$C = \bigcap_{n \in \omega} A_n.$$

To see that $C \in$ Analytic in X, let Z be the topological cartesian product space with components $\tilde{X}^{(n)}$, the one-point compactifications of the $X^{(n)}$, for $n \in \omega$, and let F be the cartesian product of the D_n. Then F is the intersection of the cylinders

$$d_n = \{x \in Z : x_n \in D_n\} \qquad \text{for } n \in \omega.$$

Since each $d_n \in K_{\sigma\delta}(Z)$, the same is true of F.

Let

$$H = \{x \in F : f_n(x_n) = f_0(x_0) \text{ for each } n \in \omega\}.$$

Then H is the intersection of F and a closed subset of Z so that $H \in K_{\sigma\delta}(Z)$.

Let $h(x) = f_0(x_0)$ for each $x \in H$. Then h is continuous on H, and it is easy to verify that

$$C = h[H] \in \text{Analytic in } X.$$

Thus, Analytic in X is closed to countable union and intersection, and

$$\text{Borelian Analytic in } X = \text{Analytic in } X.$$

Proof of 5.4. This result is due to Sion **(21)**. The proof uses Theorem 6.6 given in the next section. There is no circularity in the argument, however.

Clearly, if $A - B \in K_\sigma$, then $A - B \in$ Analytic in X.

Suppose $A - B \in$ Analytic in X. By 6.6 $A - B$ is Lindelöf (i.e., each open covering of $A - B$ can be reduced to a countable open covering of $A - B$). Let

$$G = \{\beta : \beta \in \mathfrak{G} \text{ and } B \subset \beta\},$$

and let

$$F = \{\alpha : \alpha = X - \text{closure } \beta \quad \text{for some } \beta \in G\}.$$

Since X is Hausdorff, F is an open covering of $X - B$, and hence a countable subfamily F' covers $A - B$. Let G' be such a countable subfamily of G that

$$F' = \{\alpha : \alpha = X - \text{closure } \beta \quad \text{for some } \beta \in G'\},$$

and let

$$H = \{\gamma : \gamma = A - \beta \quad \text{for some } \beta \in G'\}.$$

Then H is a countable family of compact sets whose union is $A - B$ so that $A - B \in K_\sigma$.

Proof of 5.5. This result is due to Sion **(21)**.

Suppose X has property I. Let H be a maximal family such that $K \subset H$ and if A and B are in H, then A and $A - B$ are in Borelian K. It is easily checked that H is closed to countable union and intersection so that $H =$ Borelian K. Thus, if A and B are in Borelian K, then $A - B \in$ Borelian K. Hence

$$\text{Borel ring } K \subset \text{Borelian } K \text{ and so Borel ring } K = \text{Borelian } K.$$

Suppose X is Hausdorff and Borel ring $K =$ Borelian K. If A and B are in K, then $A - B \in$ Borelian K, and, in view of 5.3, $A - B \in$ Analytic in X. Hence, by 5.4, $A - B \in K_\sigma$.

Theorem 5.6 is a well-known elementary result.

Proof of 5.7. Let X be a complete, separable, metric space and let

$$C(J) = \{A : A = f[J] \text{ for some continuous } f \text{ on } J \text{ to } X\}.$$

It is enough to show that

$$C(J) \subset \text{Analytic in } X \subset \text{Souslin } \mathfrak{F} \subset C(J).$$

The first inclusion follows from the fact that J is a $K_{\sigma\delta}$. The second follows from 5.2.2. The third inclusion is due to Lusin **(10)** and is proved as follows.

For A in Souslin $\mathfrak{F}$ it is always possible to put

$$A = \bigcup_{s \in S} \bigcap_{m \in \omega} d(s|\bar{m}),$$

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where for $s \in S$ and $m \in \omega$

$$d(s|\overline{m+1}) \subset d(s|\bar{m}) \in \mathfrak{F}$$

and

$$\lim_n \operatorname{diam} d(s|\bar{n}) = 0.$$

Let

$$Z = \{x : x \in S \text{ and } d(x|\bar{m}) \neq 0 \text{ for each } m \in \omega\}.$$

Then for $x \in Z$, $\bigcap_{m \in \omega} d(x|\bar{m})$ is a singleton, and g can be defined on Z by

$$g(x) \in \bigcap_{m \in \omega} d(x|\bar{m}).$$

It follows that

$$A = g[Z].$$

A metric can be introduced on S so that S is a complete, separable metric space. Under this metric Z is closed in S and g is continuous on Z. Hence (**9**, p. 343)

$$Z = h[J]$$

for some continuous function h on J, and

$$A = g[h[J]] \in C(J).$$

Remark on 5.7. The fact that, in any separable metric space X,

$$\text{Analytic in } X = C(J)$$

has been proved independently by Choquet (**6**) and Sion (**20**). This can be seen with the help of 6.6. If $A \in$ Analytic in X, then by 6.6, A is separable and, by 5.2.2. $A \in$ Souslin $\mathfrak{F}$. Hence A is a continuous image of J.

Proof of 5.8. In the case of a compact Hausdorff space this result is due to Šneider (**24**, **25**, **26**). The present form is due to Sion. See also (**19**, Theorem 4.3).

Let

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} g(s|\bar{n}) \quad \text{and} \quad B = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n})$$

with $g(s|\bar{n})$ and $h(s|\bar{n})$ in K and (in view of 4.7)

$$g(s|\overline{n+1}) \subset g(s|\bar{n}) \quad \text{and} \quad h(s|\overline{n+1}) \subset h(s|\bar{n})$$

for each $s \in S$ and $n \in \omega$.

For $n \in \omega$ and $p \in S_n'$ let

$$\mathfrak{S}'(p) = \{p' : p' \in S'_{n+1} \text{ and } p'|\bar{n} = p\},$$

$$\mathfrak{S}(p) = \{s : s \in S \text{ and } s|\bar{n} = p\},$$

$$G(p) = \bigcup_{s \in \mathfrak{S}(p)} \bigcap_{m \in \omega} g(s|\bar{m}),$$

$$H(p) = \bigcup_{s \in \mathfrak{S}(p)} \bigcap_{m \in \omega} h(s|\bar{m}).$$

The proof of the theorem can be obtained with the aid of the following statement.

Statement. Under the hypotheses of the theorem, if $n \in \omega$, $p \in S_n'$, $q \in S_n'$, and $G(p)$ and $H(p)$ cannot be separated by Borelian K, then for some $p' \in \mathfrak{S}'(p)$ and $q' \in \mathfrak{S}'(q)$, $G(p')$ and $H(q')$ cannot be separated by Borelian K.

If the statement is false, then for each $p' \in \mathfrak{S}'(p)$ and $q' \in \mathfrak{S}'(q)$, there are $\alpha(p', q')$ and $\beta(p', q')$ in Borelian K such that

$$G(p') \subset \alpha(p', q'), \qquad H(q') \subset \beta(p', q'),$$
$$\alpha(p', q') \cap \beta(p', q') = 0.$$

Let

$$\alpha' = \bigcup_{p' \in \mathfrak{S}'(p)} \bigcap_{q' \in \mathfrak{S}'(q)} \alpha(p', q')$$

and

$$\beta' = \bigcup_{q' \in \mathfrak{S}'(q)} \bigcap_{p' \in \mathfrak{S}'(p)} \beta(p', q').$$

Then $\alpha' \cap \beta' = 0$ and since $\mathfrak{S}'(p)$ and $\mathfrak{S}'(q)$ are countable, α' and β' belong to Borelian K. Moreover,

$$G(p) = \bigcup_{p' \in \mathfrak{S}'(p)} G(p') \subset \alpha'$$

and

$$H(q) = \bigcup_{q' \in \mathfrak{S}'(q)} H(q') \subset \beta'$$

in contradiction to the hypotheses of the statement.

Returning to the proof of the theorem, if A and B cannot be separated by Borelian K, then there are s and t in S such that for each $n \in \omega$,

$$G(s|\bar{n}) \text{ and } H(t|\bar{n}) \text{ cannot be separated by Borelian } K.$$

Let

$$G' = \bigcap_{n \in \omega} g(s|\bar{n}) \quad \text{and} \quad H' = \bigcap_{n \in \omega} h(t|\bar{n}).$$

Then G' and H' belong to K, $G' \subset A$, and $H' \subset B$ so that

$$G' \cap H' = 0.$$

Consequently, for some $m \in \omega$,

$$g(s|\bar{m}) \cap H' = 0,$$

and for some $n \in \omega$,

$$h(t|\bar{n}) \cap g(s|\bar{m}) = 0,$$

so that for some $k \in \omega$,

$$g(s|\bar{k}) \cap h(t|\bar{k}) = 0.$$

But

$$G(s|\bar{k}) = \bigcup_{x \in \mathfrak{S}(s|\bar{k})} \bigcap_{n \in \omega} g(x|\bar{n}) \subset g(s|\bar{k})$$

and

$$H(t|\bar{k}) = \bigcup_{x \in \mathfrak{S}(t|\bar{k})} \bigcap_{n \in \omega} h(x|\bar{n}) \subset h(t|\bar{k}),$$

in contradiction to the above, and this completes the proof.

Proposition 5.9 is an immediate corollary of 5.8.

Theorem 5.10 is due to Lusin (**11**, p. 52; **12**, p. 155). See also (**9**, p. 393). The proof is similar to that of 5.8.

Theorem 5.11 is due to Souslin (**27**). See also (**9**, p. 395). This result is a corollary of 5.10.

Theorem 5.13 follows immediately from the transitivity of continuity.

Proof of 5.14. In view of 5.2.2, $A \in$ Souslin $\mathfrak{F}(Y)$ so that

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}),$$

where $h(s|\bar{n}) \in \mathfrak{F}(Y)$ for each $s \in S$ and $n \in \omega$. Consequently

$$f^{-1}[A] = \bigcup_{s \in S} \bigcap_{n \in \omega} f^{-1}[h(s|\bar{n})] \in \text{Souslin } \mathfrak{F}(X) = \text{Analytic in } X,$$

in accordance with 5.2.3.

Theorem 5.15 was first given by Lusin for the case of a countable-to-one projection of a Borel set contained in Euclidean space; see (**12**, p. 178). The present form is due to Sion (**21**, Theorem 4.7). The details of the proof involve technical notions which are not introduced here. Under the hypotheses of the theorem, it is argued in (**21**) that $f[A]$ is a continuous and countable-to-one image of a $K_{\sigma\delta}(X')$ set for some X' which has property I. From this it follows that $f[A] \in$ Borelian $K[Y]$ and also, being Hausdorff, that $f[A]$ has property I.

Proof of 5.16. This theorem was first given by Lusin (**10**, Theorem III) for the case of real analytic sets. For further results of this type see (**9**, §34, III and §35, VII).

Let P_X and P_Y be the projection functions on $X \times Y$ to X and Y respectively. (For $(x, y) \in X \times Y$, $P_X(x, y) = x$ and $P_Y(x, y) = y$.)

Since f is a Borel function on A to Y,

$$f = \{(x, y) : y = f(x)\} \in \text{Analytic in } (X \times Y).$$

By 5.14,

$$A \times Y = P_X^{-1}[A] \in \text{Analytic in } (X \times Y),$$

so that

$$f \cap (A \times Y) \in \text{Analytic in } (X \times Y).$$

Thus, by 5.13,

$$f[A] = P_Y(f \cap (A \times Y)] \in \text{Analytic in } Y.$$

Proof of 5.17. According to a theorem of Braun (**3**, Theorem 3)

$$f[A] = Y - B \text{ for some } B \in \text{Analytic in } Y.$$

According to Lusin's Theorem 5.16

$$f[A] \in \text{Analytic in } Y.$$

Thus by Souslin's Theorem 5.11

$$f[A] \in \text{Borelian } \mathfrak{F}(Y).$$

Remarks on the proof of 5.17. The result of Braun which was used in the proof of 5.17 is a result of the theory of sieve operations (*opérations des cribles*). For the definitions and theory of sieve operations see (**11**, **12**, **18**).

In the case of a one-to-one Borel function Theorem 5.17 was first given by Lusin (**11**, pp. 59–60; **12**, p. 259).

6. Approximation from below. This section is concerned with what we believe to be the key property of analytic sets which makes them useful in analysis. Many seemingly unrelated theorems are consequences of it. Stated informally it is essentially the following: if an analytic set has a property P satisfying certain conditions, then there exists a compact set contained in it which has a property Q closely related to P. Frequently $Q = P$. Thus, in a very broad sense, analytic sets can be approximated from below by compacta.

In Part A this general result is stated formally. In Part B are collected several theorems, each of which is a consequence of the general approximation theorem 6.3.

A. A general approximation theorem.

6.1. Definition. P is a *capacitance in* X if and only if P is a family of subsets of X such that:

1. If A is an ascending sequence and $\bigcup_{n\in\omega} A_n \in P$, then for some $n \in \omega$, $A_n \in P$, and
2. If $A \in P$ and $A \subset B \subset X$, then $B \in P$.

6.2. Definition. H is (f, P, Q)-monotone if and only if for each descending sequence α, if $\alpha_n \in H$ and $f[\alpha_n] \in P$ for each $n \in \omega$ and $\bigcap_{n\in\omega} \alpha_n \subset$ domain f, then $f[\bigcap_{n\in\omega} \alpha_n] \in Q$.

H is (P, Q)-monotone if and only if H is (f, P, Q)-monotone where $f(x) = x$ for each x.

6.3. Theorem. *Let H be a family of subsets of X which is closed to finite union and intersection, and let $A \in$ Souslin H. If P is a capacitance in X and $A \in P$, then there exists a descending sequence α such that*

$$\alpha_n \in H \cap P \text{ for each } n \in \omega$$

and

$$\bigcap_{n\in\omega} \alpha_n \subset A.$$

6.4. COROLLARY. *Let H be a family of subsets of X which is closed to finite union and intersection, and let $A \in$* Souslin H. *If P is a capacitance in X, H is (P, Q)-monotone and $A \in P$, then there exists $\alpha \in H_\delta$ such that $\alpha \subset A$ and $\alpha \in Q$.*

6.5. COROLLARY. *Let K' be the family of closed, compact sets in some space X'. Let $D \in K'_{\sigma\delta}$ and let f be a continuous function on D to X. (Thus, $f[D]$ is* Analytic in X.) *If P is a capacitance in X, K' is (f, P, Q)-monotone and $f[D] \in P$, then there exists a compact set α such that $\alpha \subset f[D]$ and $\alpha \in Q$.*

Proofs and remarks.

Proof of 6.3. This result is due to Sion (**22**, Theorem 5.6).

Let

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s|\bar{n}),$$

where, for each $n \in \omega$ and $x \in S_n'$, $h(x) \in H$. In view of 4.7 it can be assumed that for each $n \in \omega$ and $x \in S'_{n+1}$,

$$h(x) \subset h(x|\bar{n}).$$

For any two sequences x and y let $x < y$ if and only if for some $n \in \omega$, $x|\overline{n-1} = y|\overline{n-1}$ and $x_n < y_n$. For $x \in S_n'$ let

$$T_x = \{s \in S: s \leqslant x\}$$

and

$$B(x) = \bigcup_{s \in T_x} \bigcap_{n \in \omega} h(s|\bar{n}).$$

Then

$$A = \bigcup_{j \in \omega} B(j)$$

and

$$B(x) = \bigcup_{j \in \omega} B(x_0, \ldots, x_n, j).$$

Suppose P is a capacitance and $A \in P$. Since $B(j) \subset B(j+1)$ for $j \in \omega$, there is $k_0 \in \omega$ such that $B(k_0) \in P$. Then, since

$$B(x_0, \ldots, x_n, j) \subset B(x_0, \ldots, x_n, j+1)$$

for $x \in S_n'$ and $j \in \omega$, there is $k_1 \in \omega$ such that $B(k_0, k_1) \in P$. Proceeding by recursion, there is a sequence $k \in S$ such that $B(k|\bar{n}) \in P$ for each $n \in \omega$. Let

$$U_n = \{x \in S_n': x_i \leqslant k_i \text{ for } i = 0, \ldots, n\}$$

and

$$\alpha_n = \bigcup_{x \in U_n} h(x).$$

Then, $\alpha_{n+1} \subset \alpha_n \in H$ and since $B(k|\bar{n}) \subset \alpha_n$, $\alpha_n \in P$. Finally, by Lemma 4.6

$$\bigcap_{n\in\omega}\alpha_n = \bigcap_{n\in\omega}\ \bigcup_{x\in U_n} h(x)$$
$$= \bigcap_{n\in\omega}\ \bigcup_{s\in T_k|\bar{n}} h(s|\bar{n})$$
$$= \bigcup_{s\leqslant k}\ \bigcap_{n\in\omega} h(s|\bar{n}) \subset A.$$

Proposition 6.4 is an immediate consequence of 6.3 and 6.2.

Proof of 6.5. This result is due to Sion **(22)**.

Let $P' = \{\alpha' : \alpha' \subset X'$ and $f[\alpha'] \in P\}$.

Since P is a capacitance in X, P' is a capacitance in X'. Also $D \in K'_{\sigma\delta} \subset$ Souslin K'. Suppose $f[D] \in P$. Then $D \in P'$ and by 6.3 there is a descending sequence α' such that for each $n \in \omega$

$$\alpha_n' \in K' \cap P', \qquad \text{i.e., } f[\alpha_n'] \in P,$$

and

$$\bigcap_{n\in\omega} \alpha_n' \subset D.$$

Let

$$\beta = \bigcap_{n\in\omega} \alpha_n' \quad \text{and} \quad \alpha = f[\beta].$$

Then $\beta \in K'$ and $\alpha \subset f[D]$. Since f is continuous on D, α is compact. Since K' is (f, P, Q)-monotone, $\alpha \in Q$.

B. Miscellaneous results. In this part are listed several results, most of which are well known. They lie in various fields and were proved at different times independently of each other. It is shown that they are all consequences of 6.4 or 6.5. The definitions of the concepts involved in the theorems together with some remarks are given in the proofs.

6.6. THEOREM. *If* A $\in$ Analytic in X, *then* A *is Lindelöf.*

6.7. THEOREM. *If* $A \in$ Analytic in X *and* μ *is a capacity of order* (1a) *on* X, *then* A *is* μ*-capacitable.*

6.8. THEOREM. *If* μ *is a Carathéodory outer measure on* X, H *is a family of* μ*-measurable sets which is closed to finite union and intersection,* $A \in$ Souslin H, *and* $\mu A < \infty$, *then for each* t *less than* μA *there exists* $\alpha \in H_\delta$ *such that* $\alpha \subset A$ *and* $t \leqslant \mu\alpha$.

6.9. THEOREM. *If* μ *is a Carathéodory outer measure and* H *is the family of* μ*-measurable sets and* $A \in$ Souslin H, *then* A *is* μ*-measurable.*

6.10. THEOREM. *Let* F *be the family of closed sets in* X, *and let* Y *be a space having a countable base. Let* $A \in$ Souslin F, *and let* f *be such a function on* A *to* Y *that for each open set* U *in* Y, $f^{-1}[U] \in$ Souslin F *and* $A - f^{-1}[U] \in$ Souslin F. (*Thus* f *is a Borel function on* A.) *If* μ *is a Carathéodory outer measure on* X

such that $\mu A < \infty$ *and closed sets in* X *are* μ*-measurable, then for each* $\epsilon > 0$ *there exists a closed set* C *contained in* A *such that* $\mu(A - C) < \epsilon$ *and* f *is continuous on* C.

6.11. THEOREM. *Let* X *be Hausdorff,* $A \in$ Analytic in X,

$$B = \{\alpha : \alpha \subset A, \alpha \in \text{Analytic in } X \text{ and } A - \alpha \in \text{Analytic in } X\},$$

and let μ *be a Carathéodory outer measure on* A *such that* $\mu A = 1$ *and closed sets are* μ*-measurable. Then* (A, B, μ) *is a perfect probability space; i.e., if* Y *has a countable base,* f *is a* B*-measurable function on* A *to* Y ($f^{-1}[U] \in B$ *for each open* U *in* Y), $E \subset Y$, *and* $f^{-1}[E] \in B$, *then there exists* F *such that* F *is a countable union of compacta,* $F \subset E$, *and* $\mu(f^{-1}[F]) = \mu(f^{-1}[E])$.

6.12. REMARKS. The proof of Theorem 6.3 can be modified (**23**, Theorem 4.1) to prove theorems which yield directly the separation Theorem 5.8 (see **22**, Theorems 5.8 and 5.13) as well as the following results.

6.12.1. *Let* X *be a complete, separable metric space,* $s \geqslant 0$, *and* μ_s^* *be Hausdorff s-dimensional outer measure. If* $A \in$ Analytic in X *and* A *has non* σ*-finite* μ_s^**-measure, then there is a compact set* α *contained in* A *which has non* σ*-finite* μ_s^**-measure.*

This theorem was first given by Davies (**7**).

6.12.2. *If* X *is a complete, separable metric space,* $A \in$ Analytic in X *and* A *is not countable, then* A *contains a non-empty, perfect subset and therefore has the power of the continuum.*

For real analytic sets this theorem is due to Souslin (**10**).

Proofs and remarks.

Proof of 6.6. This result is due to Sion (**20**). A set is Lindelöf if and only if each open covering of it can be reduced to a countable open covering. Suppose $A \in$ Analytic in X and there is an open covering G of A that cannot be reduced to a countable open covering. Let

$$P = Q = \{\alpha \subset X : \alpha \text{ cannot be covered by a countable subfamily of } G\}.$$

Then P is a capacitance in X and $A \in P$. Using Lemma 5.18.1 to check that 6.5 is applicable, one concludes that there is a compact set α contained in A such that $\alpha \in Q$. This is a contradiction.

Proof of 6.7. This result, for the case where A is contained in a countable union of compacta, was first proved by Choquet (**5**). The theorem as stated is due to Sion (**20**).

μ is a capacity of order (1a) on X if and only if μ is such a function on the family of all subsets of X to the reals that

(i) for each ascending sequence A,

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$$\mu A_n \leqslant \mu A_{n+1} \leqslant \lim \mu A_n = \mu\Big(\bigcup_{n\in\omega} A_n\Big),$$

and

(ii) for each compact set A and for each $\epsilon > 0$, there exists an open set B such that $A \subset B$ and $\mu B \leqslant \mu A + \epsilon$.

A is μ-capacitable if and only if

$$\mu A = \sup\{\mu A : \alpha \subset A \text{ and } \alpha \text{ is closed and compact}\}.$$

Let $A \in$ Analytic in X. To see that A is μ-capacitable, let $t < \mu A$ and let

$$P = \{\alpha : \alpha \subset X \text{ and } \mu\alpha > t\}.$$

and

$$Q = \{\alpha : \alpha \subset X \text{ and } \mu A \geqslant t\}.$$

Use (i) to check that P is a capacitance in X, and use (ii) with 5.18.1 to check that K' is (f, P, Q)-monotone if K' is the family of closed compact sets of an arbitrary space X' and f is an arbitrary continuous function with domain contained in X'. Since $A = f[D]$ for some $D \in K_{\sigma\delta}(X')$, for some space X' and some continuous function f on D, by 6.5 there is a compact set α contained in A such that $\alpha \in Q$. That is $\mu\alpha \geqslant t$.

Proof of 6.8. Let $t < \mu A$, and let

$$\begin{aligned} \mu^*\alpha &= \inf\{\mu\beta : \beta \text{ is } \mu\text{-measurable and } \alpha \subset \beta\}, \\ P &= \{\alpha : \alpha \subset X \text{ and } t < \mu^*(\alpha \cap A)\}. \\ Q &= \{\alpha : \alpha \subset X \text{ and } t \leqslant \mu^*\alpha\}. \end{aligned}$$

Then P is a capacitance in X, $A \in P$, and H is (P, Q)-monotone. By 6.4 there is α in H_δ such that $\alpha \subset A$ and $\alpha \in P$, i.e., $t \leqslant \mu^*\alpha = \mu\alpha$.

Proof of 6.9. This is a classical result due to Lusin (**10**; see also **13**, **15**, p. 50).

μ is a Carathéodory outer measure on X if and only if μ is such a function on the family of all subsets of X that

$$\mu 0 = 0,$$

and for each sequence A of subsets of X

$$0 \leqslant \mu\Big(\bigcup_{n\in\omega} A_n\Big) \leqslant \sum_{n\in\omega} \mu A_n.$$

A is μ-measurable if and only if $\mu T = \mu(T \cap A) + \mu(T - A)$ for each $T \subset X$.

Let H be the family of all μ-measurable sets and let $A \in$ Souslin H.

For $T \subset X$ and $\alpha \subset X$ let

$$\mu_T(\alpha) = \inf\{\mu(T \cap \beta) : \beta \text{ is } \mu\text{-measurable and } \alpha \subset \beta\}.$$

Then A is μ-measurable if and only if A is μ_T-measurable for each $T \subset X$ with $\mu T < \infty$.

Let $T \subset X$ with $\mu T < \infty$ and let $t < \mu_T(A)$. By 6.8 there is $\alpha \in H_\delta$ $(=H)$ such that $\alpha \subset A$ and $t \leqslant \mu_T(A)$. It follows that A is μ_T-measurable.

Proof of 6.10. This is an analogue of the classical Vitali-Lusin theorem. The restrictions on the function are stronger, but the conditions on the measure are weaker.

Let $\epsilon > 0$, and let $\{U_i\colon i \in \omega\}$ be a base for the topology of Y. Then by 6.8, for each $i \in \omega$ there are C', α_i, and β_i in F such that

$$C' \subset A \quad\text{and}\quad \mu(A - C') < \tfrac{1}{2}\epsilon,$$
$$\alpha_i \subset f^{-1}[U_i] \quad\text{and}\quad \mu(f^{-1}[U_i] - \alpha_i) < \epsilon/2^{i+3},$$

and

$$\beta_i \subset A - f^{-1}[U_i] \quad\text{and}\quad \mu(A - f^{-1}[U_i] - \beta_i) < \epsilon/2^{i+3}.$$

Let

$$C = C' \cap \bigcap_{i\epsilon\omega} (\alpha_i \cup \beta_i).$$

Then $C \in F$, $C \subset A$,

$$\mu(A - C) \leqslant \mu(A - C') + \sum_{i\epsilon\omega} \mu(A - \alpha_i - \beta_i)$$
$$\leqslant \frac{\epsilon}{2} + \sum_{i\epsilon\omega} (\mu(f^{-1}[U_i] - \alpha_i) + \mu(A - f^{-1}[U_i] - \beta_i)) < \epsilon,$$

and, for each $i \in \omega$, $C \cap f^{-1}[U_i] = C \cap (X - \beta_i)$. Thus, for each $i \in \omega$, $f^{-1}[U_i]$ is open in C, and therefore f is continuous on C.

Proof of 6.11. This is a slight generalization of a result due to Blackwell **(1)**. The method of proof is different.

Let $E \subset Y$, $f^{-1}[E] \in B$, $\epsilon > 0$ and let

$P = \{\alpha\colon \alpha \subset f^{-1}[E]$ and $\mu(\alpha - f^{-1}[F]) > \epsilon$ for each F such that F is a countable union of compacta and $F \subset E\}$,

and

$Q = \{\alpha\colon \alpha \subset f^{-1}[E]$ and $\mu(\alpha - f^{-1}[F]) \geqslant \epsilon$ for each F such that F is a countable union of compacta and $F \subset E\}$.

Then P is a capacitance in X and, by 5.18.2, K' is (g, P, Q)-monotone whenever K' is the family of closed compact sets in a space X' and g is a continuous function to X with domain in $K'_{\sigma\delta}$ $(=K_{\sigma\delta}(X'))$. Then by 6.5, if $f^{-1}[E] \in P$, there is a compact set α contained in $f^{-1}[E]$ such that $\alpha \in Q$. Using 5.2.2 and 6.10, let C be closed and such that $C \subset A$, $\mu(A - C) < \epsilon$, and f is continuous on C. If $D = f[\alpha \cap C]$, then D is compact, $D \subset E$, and

$$\mu(\alpha - f^{-1}[D]) \leqslant \mu(\alpha - C) \leqslant \mu(A - C) < \epsilon,$$

which contradicts the fact that $\alpha \in Q$. Hence $f^{-1}[E] \notin P$.

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