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ON C^*-DIAGONALS

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Preface. The impetus for this study arose from the belief that the structure of a C^*-algebra is illuminated by an understanding of the manner in which abelian subalgebras embed in it. Posed in its full generality, the question concerning abelian subalgebras would seem impossible to answer. A notion of diagonal subalgebra is, however, proposed which has the virtue that one can associate a "topological" invariant to the pair consisting of the diagonal and the ambient algebra, from which these algebras may be retrieved.

In the setting of von Neumann algebras, the analogous question was addressed in the seminal work of Feldman and Moore [**13**]. Their definition of Cartan subalgebra permits the abstraction of a complete invariant consisting of a Borel equivalence relation together with a certain cohomology class on the relation from which the Cartan pair may be recovered. Our development parallels theirs in spirit; differences in substance derive from the topological flavor of C^*-theory.

The invariant associated to a diagonal pair is called a twist. A twist is properly viewed as a groupoid extension of a topological equivalence relation by the unit circle (which in some cases arises from a two-cocycle on the relation).

Renault has applied the techniques of topological groupoids to the study of C^*-algebras with great success (see [**34**]) and, consequently, his work has had considerable influence on the development of our ideas. A notion of Cartan subalgebra appears there (ibid. def. II.4.13), that anticipates the notion of diagonal given here.

A sketch of this work appeared a year ago in the Tübingen Semesterbericht (see [**22**]). Since then Renault has shown that given a diagonal pair the twist may be constructed using the dual groupoid of the ambient algebra (the spectrum of the diagonal is identified with a nice transversal; see [**36**]). Further, the present work is informed by an understanding of diagonals in continuous trace algebras (see [**23**]) acquired by reading a preprint of Raeburn and Taylor [**32**].

The treatment of Morita equivalence of diagonal pairs appearing in Section 5 below is based on related notions in [**20**], the necessary changes being made to reflect the twisting of the relation. In the last section, we consider a class of unital diagonal pairs for which it is possible, in

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principle, to compute invariants. These pairs are characterized by the existence of an ascending chain of continuous trace algebras containing the diagonal whose union is dense in the ambient algebra.

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1. First notions. In the following it is tacitly assumed that C^*-algebras are separable and that topological spaces are second countable, locally compact, and Hausdorff (hence paracompact). If A is a C^*-algebra, let $\tilde{A}$ denote the C^*-algebra obtained by adjoining a unit.

1° *Definition*. Suppose that B is a C^*-subalgebra of a C^*-algebra A. An element $a \in A$ is said to *normalize* B if

i: $a^*Ba \subset B$

ii: $aBa^* \subset B$.

The collection of all such normalizers is denoted $N(B)$. Evidently $B \subset N(B)$; further, $N(B)$ is closed under multiplication and taking adjoints. A normalizer, $a \in N(B)$, is said to be *free* if $a^2 = 0$. The collection of free normalizers is denoted $N_f(B)$.

2° *Example*. Let $A = M_n(C)$, the algebra of complex $n \times n$ matrices. Choose a set of matrix units, $\{e_{ij}:1 \leqq i, j \leqq n\}$ (one has $e_{ik} = e_{ij}e_{jk}$ and $e_{ij}^* = e_{ji}$), and let B denote the diagonal subalgebra (viz. B is spanned by the e_{ii}'s). Then $a = \Sigma\, \lambda_{ij}e_{ij}$ normalizes B if and only if for each i, $\lambda_{ij} \neq 0$ for at most one j, and for each j, $\lambda_{ij} \neq 0$ for at most one i (i.e., at most one entry is non-zero in each row or column). If $i \neq j$, $e_{ij} \in N_f(B)$. Let $P:A \to B$ be given by:

$$P(a) = \Sigma\, e_{ii}ae_{ii}.$$

This defines a faithful conditional expectation for which:

$$\ker P = \operatorname{span} N_f(B).$$

We introduce the notion of diagonal in the setting of C^*-algebras by abstracting the essential ingredients from the example above. A closely related notion appears in **[34]** (cf. def. II.4.13) under the name Cartan subalgebra (the definition is modelled on the analogous notion for von Neumann algebras found in **[13]**).

3° *Definition*. Let B be an abelian subalgebra of a C^*-algebra A. If A is unital, then B is said to be *diagonal* in A if $1 \in B$ and

i: there is a faithful conditional expectation $P:A \rightarrow B$

ii: $\ker P = (\text{span } N_f(B))^-$.

If A is non-unital, B is said to be *diagonal* in A if $\widetilde{B}$ is diagonal in $\widetilde{A}$. We shall refer to (A, B) as a diagonal pair.

For the remainder of this section we assume that A is unital; the loss of generality is only apparent. If B is diagonal in A, we show below that B has the extension property relative to A (the extension property was introduced in **[2]** and studied in **[3]**, **[4]**), that is, each pure state of B extends uniquely to a (necessarily pure) state of A.

4° Proposition. *If B is diagonal in A, then B has the extension property relative to A and B is consequently maximal abelian.*

Proof. By 2.7 of **[4]**, it suffices to show that

$$A = B + \overline{[A, B]}$$

where

$$[A, B] = \text{span}\{ab - ba: a \in A, b \in B\}.$$

By condition (ii) of the definition, we need only show

$$N_f(B) \subset \overline{[A, B]}.$$

Let $a \in N_f(B)$ and note that $a^*a, aa^* \in B$; since $a^2 = 0$ one has

$$(a^*a)(aa^*) = 0.$$

Hence

$$a(a^*a)^{1/n} - (a^*a)^{1/n}a = a(a^*a)^{1/n} \rightarrow a \quad \text{as } n \rightarrow \infty.$$

5° *Remarks*. Let $X = \hat{B}$ denote the collection of pure states of B equipped with the weak* topology ($B \cong C(X)$ and X is compact). For $x \in X$, its unique extension to A is $x \circ P$ (in fact, the extension property alone ensures the existence of such a conditional expectation; see **[3]** Theorem 3.4.) Henceforth, X is to be identified with its image in the pure states of A.

The nature of the normalization condition will be illuminated in the following proposition. Some notation is required: for $a \in N(B)$, put

$$s(a) = \{x \in X: x(a^*a) > 0\},$$

$$I(a) = \{f \in B: x(f) \neq 0 \Rightarrow x \in s(a)\}.$$

Note that $s(a)$ is open in X and $I(a)$ is an ideal in B. Obviously, $a^*a \in I(a)$, in fact, it is strictly positive in $I(a)$.

6° PROPOSITION. *For each* $a \in N(B)$, *there is a homeomorphism*

$$\sigma_a: s(a) \to s(a^*)$$

such that

$$x(a^*fa) = \sigma_a(x)(faa^*) \quad \text{for all } f \in B,\ x \in s(a).$$

Proof. Let $a = v|a|$ be the polar decomposition of a in A^{**} (where $|a| = (a^*a)^{1/2}$). It follows that $v|a| = |a^*|v$ (observe that $aa^* = va^*av^*$).

Claim: i° If $f \in I(a)$, then $vfv^* \in I(a^*)$.

ii° If $f \in I(a^*)$, then $v^*fv \in I(a)$.

To verify the claim it suffices to show that i° holds (since $a^* = v^*|a^*|$). Suppose $f = |a|g|a|$ with $g \in B$; then

$$vfv^* = v|a|g|a|v^* = aga^* \in B \quad \text{(since } a \text{ is a normalizer).}$$

If $g \geqq 0$, then

$$aga^* \leqq \|g\|aa^* \in I(a^*).$$

Since any $f \in I(a)$ may be approximated by linear combinations of such elements (recall that a is strictly positive in $I(a)$), the claim is established. It follows that v determines an isomorphism between $I(a)$ and $I(a^*)$. Let

$$\sigma_a: s(a) \to s(a^*)$$

be the unique partial homeomorphism for which

$$\sigma_a(x)(f) = x(v^*fv) \quad \text{for all } f \in I(a^*),\ x \in s(a).$$

7° COROLLARY. *For* $f \in B \subset N(B)$, *one has*

$$I(f) = I(f^*) \quad \text{and} \quad \sigma_f = \mathrm{id}|_{s(f)}.$$

If $a, b \in N(B)$, *then* $ab \in N(B)$ *and* $\sigma_{ab} = \sigma_a \circ \sigma_b$ *where this composition makes sense.*

Let $(\mathscr{H}_x, \pi_x, \xi_x)$ denote the GNS triple associated to the pure state $x \in X$. The unique extension of π_x to a normal representation of A^{**} will be denoted by the same symbol. Observe that

$$\pi_x(f)\xi_x = x(f)\xi_x \quad \text{for all } f \in B.$$

8° COROLLARY. *Let* $a \in N(B)$ *with* $a = v|a|$ *as above and* $x \in s(a)$. *Then*

$$\sigma_a(x) = (\pi_x(\cdot)\pi_x(v)\xi_x, \pi_x(v)\xi_x);$$

further, $\pi_x(v)\xi_x \perp \xi_x$ *if and only if* $\sigma_a(x) \neq x$.

Proof. By scaling a appropriately, we may assume that $x(a^*a) = 1$. By the proposition,

$$\sigma_a(x)(aa^*) = 1$$

and for all $f \in B$ one obtains

$$x(a^*fa) = \sigma_a(x)(faa^*) = [\sigma_a(x)(f)][\sigma_a(x)(aa^*)] = \sigma_a(x)(f).$$

Thus

$$\begin{aligned}\sigma_a(x)(f) &= (\pi_x(a^*fa)\xi_x, \xi_x)\\ &= (\pi_x(\,|a|v^*fv|a|\,)\xi_x, \xi_x)\\ &= (\pi_x(f)\pi_x(v)\pi_x(\,|a|\,)\xi_x, \pi_x(v)\pi_x(\,|a|\,)\xi_x)\\ &= (\pi_x(f)\pi_x(v)\xi_x, \pi_x(v)\xi_x)\end{aligned}$$

since $\pi_x(\,|a|\,)\xi_x = \xi_x$. By the extension property one infers that this formula holds for arbitrary elements of A. If $\sigma_a(x) \neq x$, it is evident that these are orthogonal pure states; hence, the vectors giving rise to them must be orthogonal.

Since the irreducible representations associated to the pure states, x and $\sigma_a(x)$, are equivalent (they occur on the "same" Hilbert space), the states themselves are unitarily equivalent.

A further technical result is required for later use.

9° LEMMA. *Given $a \in N(B)$ and $x \in s(a)$, the following conditions are equivalent*:

i° $x(a) \neq 0$

ii° $\sigma_a(x) = x$

iii° *there is $f \in B$ with $x(f) \neq 0$ such that $af \in B$.*

Proof. As above, we may assume that $x(a^*a) = 1$. As a first step, we demonstrate the equivalence of the first two conditions. Let $a = v|a|$ be the polar decomposition of a. Realizing x as a vector state, one obtains:

$$\begin{aligned}x(a) &= (\pi_x(a)\xi_x, \xi_x) = (\pi_x(v)\pi_x(\,|a|\,)\xi_x, \xi_x)\\ &= (\pi_x(v)\xi_x, \xi_x) \quad (\text{since } \pi_x(\,|a|\,)\xi_x = \xi_x).\end{aligned}$$

By the preceding corollary, $\sigma_a(x) = x$ if and only if

$$(\pi_x(v)\xi_x, \xi_x) \neq 0.$$

iii $\Rightarrow$ ii: This follows immediately from Corollary 7.

i $\Rightarrow$ iii: By continuity, there is an open set U, with $x \in U \subset s(a)$, such that $y(a) \neq 0$ for all $y \in U$; hence, $\sigma_a(y) = y$ for all $y \in U$. Choose $f \in B$ so that $x(f) = 1$ and $s(f) \subset U$, and set $b = af$; by Corollary 7,

$$\sigma_b = \sigma_a \circ \sigma_f = \mathrm{id}|_{s(b)}.$$

Thus by Proposition 6

$$b^*hb = hb^*b \quad \text{for all } h \in B.$$

A routine calculation verifies that $[h, b]^*[h, b] = 0$ (where $[h, b] = hb - bh$) for all $h \in B$. Hence, $b = af \in B$.

10° *Remark*. The definition of a diagonal subalgebra may be reformulated in a way that does not refer to the presence of an identity. An abelian subalgebra is said to be diagonal if it contains a positive element which is strictly positive in the ambient algebra and if conditions i, ii in the original definition are satisfied.

11° *Example*. In a recent article, **[32]**, Raeburn and Taylor construct a continuous trace algebra from a Čech two-cocycle (with values in $\mathcal{S}$, the sheaf of germs of continuous circle-valued functions), the class of which corresponds to the Dixmier-Douady class of the algebra constructed (under the isomorphism:

$$H^2(\cdot, \mathcal{S}) \cong H^3(\cdot, \mathbf{Z})).$$

This algebra has a natural diagonal isomorphic to $\bigoplus_i C_0(U_i)$, where $\{U_i\}$ is the covering with respect to which the two-cocycle is given. A systematic study of diagonals in continuous trace algebras may be found in **[23]**.

12° *Example*. Let G be a discrete abelian group and let χ be a symplectic bicharacter on G; that is, $\chi:G \wedge G \rightarrow \mathbf{T}$, is a character of the skew-symmetric tensor product. Let A_χ denote the universal C^*-algebra spanned by unitaries, $\{u_g:g \in G\}$, satisfying the relations,

$$u_{g+h}\, u_g^* u_h^* \in \mathbf{C} \cdot \mathbf{1} \quad \text{and}$$

$$(\dagger) \quad u_g u_h = \chi(g \wedge h) u_h u_g \quad \text{for } g, h \in G$$

(see **[12]**, **[26]**, **[41]**). Such algebras are characterized by Olesen, Pedersen, and Takesaki (see **[26]**) as those admitting ergodic actions by compact abelian groups. Indeed

$$\alpha:\hat{G} \rightarrow \mathrm{Aut}(A_\chi)$$

by

$$\alpha_s(u_g) = \langle s, g\rangle u_g \quad \text{for } s \in \hat{G},\ g \in G,$$

yields such an action.

Let $H \subset G$ be maximal with the property that $H \wedge H \subset \ker \chi$. Let $B_H \subset A_\chi$ be the abelian subalgebra generated by $\{u_h:h \in H\}$ and $P_H:A \rightarrow B_H$ be the faithful conditional expectation given by

$$P_H(a) = \int_{H^\perp} \alpha_t(a)dt$$

where

$$H^\perp = \{t \in \hat{G}:\langle t, h\rangle = 1 \text{ for all } h \in H\}.$$

We claim that $B_H \cong C(\hat{H})$ is diagonal in A_χ. It follows from the commutation relations (†) that u_g normalizes B_H for all $g \in G$. Observe that the bicharacter induces a map

$$\chi_H:G \to \hat{H}$$

such that

$$\langle \chi_H(g), h \rangle = \chi(g \wedge h) \quad \text{for } g \in G, h \in H.$$

Since H is maximal with the property $\chi(H \wedge H) = 1$, it follows that

$$H = \ker \chi_H.$$

Moreover, Ad $u_g|_B$ is simply translation by $\chi_H(g)$ (clearly, $s(u_g) = \hat{H} = s(u_g^*)$ and $\sigma_{u_g}(x) = x\chi_H(g)$ for all $x \in \hat{H}$). If $g \notin H$, there are a finite number of positive elements $f_k \in B_H$ with $\Sigma f_k = 1$, so that

$$f_k u_g f_k = 0 \quad \text{for each } k,$$

since the collection of open sets

$$\{U \subset \hat{H}:\chi_H(g)U \cap U = \emptyset\}$$

covers $\hat{H}$.

It follows that $u_g f_k \in N_f(B_H)$ (each k) and that $N_f(B_H)$ is total in ker P_H. Thus, B_H is diagonal in A_x as claimed. We shall return to this example in later sections.

2. Twists. We draw upon the theory of continuous groupoids to provide us with a machine for producing diagonal pairs. That all such pairs arise in this fashion will be shown to be the case in the next section. The reader unfamiliar with the theory of locally compact groupoids is urged to consult [**34**] for details. As our notation differs slightly from his, a few comments concerning terminology will aid intelligibility.

1° *Notation.* If Γ is a topological groupoid with unit space X (Renault uses the symbol Γ° for the unit space), we denote the range and source maps by r, $s:\Gamma \to X$; the collection of composible pairs is denoted Γ^2 (recall that

$$\Gamma^2 = \{ (\gamma_1, \gamma_2) \in \Gamma \times \Gamma:s(\gamma_1) = r(\gamma_2) \}$$

with composition $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$); the involution is written $\gamma \mapsto \gamma^*$ (so $\gamma^*\gamma = s(\gamma)$ and $\gamma\gamma^* = r(\gamma)$). The isotropy group bundle, $\{\gamma \in \Gamma:r(\gamma) = s(\gamma)\}$, is written Is($\Gamma$); evidently, $X \subset$ Is(Γ). The groupoid Γ will be called a *relation* if $X =$ Is(Γ) and $s:\Gamma \to X$ is a local homeomorphism; the letter "R" is to reserved for relations (Renault terms relations principal r-discrete groupoids). We collect some elementary properties of a relation R (cf [**38**], Proposition I.2.8).

i° $X \subset R$ is open.

ii° $r:R \to X$ is a local homeomorphism.

iii° $\Omega(R) = \{U \subset R \text{ open}: r|_U, s|_U \text{ are injective}\}$ covers R.

iv° The quotient map $X \to X/R$ is open. It will often be convenient to identify R with its image in $X \times X$ (under (r, s)) though the relative topology is often coarser. If $V \subset X$, set

$$[V] = r(s^{-1}(V));$$

$[V]$ is called the *saturation* of V; if $[V] = X$, then V is said to be *full*. We shall say Γ is a *groupoid* on X when we mean that X is the unit space.

2° *Definition*. Let Γ be a groupoid on X and let G be a locally compact abelian group. Then Γ is said to be a *G-groupoid* if Γ is a free G-space, with Γ/G Hausdorff, for which the following holds: (♀) For all $g_1, g_2 \in G$, and $(\gamma_1, \gamma_2) \in \Gamma^2$, one has

$$(g_1\gamma_1, g_2\gamma_2) \in \Gamma^2 \quad \text{and}$$

$$(g_1\gamma_1)(g_2\gamma_2) = (g_1g_2)(\gamma_1\gamma_2).$$

Further, Γ is said to be a *proper G-groupoid* if it is a locally trivial principal G-bundle over Γ/G, and if every $\gamma \in \text{Is}(\Gamma)$ can be written as gx for some $g \in G$ and $x \in X$.

3° *Facts*. With G and Γ as above, it is immediate that $s(g\gamma) = s(\gamma)$ and $r(g\gamma) = r(\gamma)$; indeed, if $\gamma \in \Gamma$ and $x \in X$, then $(\gamma, x) \in \Gamma^2$ if and only if $x = s(\gamma)$ and by (♀) one has $(g\gamma, x) \in \Gamma^2$; moreover, $(g\gamma)^* = g^{-1}\gamma^*$. If Γ is proper (as a G-groupoid), then $\text{Is}(\Gamma) \cong G \times X$. Observe that Γ/G is a groupoid on X with groupoid structure inherited from Γ. The groupoid Γ/G will be referred to as the *subjacent* groupoid.

4° *Definition*. A proper **T**-groupoid Γ on X is said to be a *twist* if $\Gamma/\mathbf{T}$ is a relation.

Let Γ be a twist over R (i.e., $R \simeq \Gamma/\mathbf{T}$); let $E(\Gamma)$ denote the collection of compactly supported equivariant continuous functions on Γ, that is

$$E(\Gamma) = \{f \in C_c(\Gamma): f(t\gamma) = tf(\gamma) \text{ all } t \in \mathbf{T}, \gamma \in \Gamma\}.$$

It will be convenient to regard elements of $E(\Gamma)$ as sections of the conjugate complex line-bundle $F(\Gamma)$ associated to the principal **T**-bundle Γ (i.e., $F(\Gamma) = \mathbf{C} \times \Gamma/\sim$ where $(z\bar{t}, \gamma) \sim (z, t\gamma)$ for all $t \in \mathbf{T}$, $\gamma \in \Gamma$, $z \in \mathbf{C}$). If $f \in E(\Gamma)$, the corresponding section is given by

$$\dot{\gamma} \mapsto (f(\gamma), \gamma) \quad \text{where } \dot{\gamma} \in R.$$

Note that equivariance ensures that this assignment is well defined. Our purpose is to endow $E(\Gamma)$ with the structure of a *-algebra. We require yet more notation; put

$$N(\Gamma) = \{f \in E(\Gamma):\operatorname{supp} f \subset U \in \Omega(R)\},$$

note that $N(\Gamma)$ spans $E(\Gamma)$; finally, put

$$D(\Gamma) = \{f \in E(\Gamma):\operatorname{supp} f \subset X\}.$$

Since $F(\Gamma)$ is trivial over X, one has that $D(\Gamma) \cong C_c(X)$.

A partially defined product is now introduced on $F(\Gamma)$:

$$(z_1, \gamma_1)(z_2, \gamma_2) = (z_1 z_2, \gamma_1\gamma_2) \quad \text{where } (\gamma_1, \gamma_2) \in \Gamma^2,\ z_1, z_2 \in \mathbf{C}.$$

Define the product of two elements $f, g \in E(\Gamma)$ according to the formula

$$(fg)(\rho) = \sum_{\alpha\beta=\rho} f(\alpha)g(\beta) \quad \text{where } (\alpha, \beta) \in R^2,\ \rho \in R.$$

Note that for each $\rho \in R$ the sum is finite, since $\operatorname{supp} f \times \operatorname{supp} g$ is compact in R^2. If $g \in D(\Gamma)$, then the above formula yields

$$(fg)(\rho) = f(\rho)g(s(\rho)) \quad \text{for } \rho \in R.$$

Involution is defined in the usual manner, namely: $f^*(\gamma) = \overline{f(\gamma^*)}$.

5° Proposition. *With product and involution as above, $E(\Gamma)$ is a $*$-algebra which contains a distinguished abelian subalgebra, $D(\Gamma) \cong C_c(X)$.*

Proof. The tedious verification is left to the reader. We remark that the product above agrees with the usual groupoid convolution in $C_c(\Gamma)$ restricted to the equivariant functions.

Viewing $E(\Gamma)$ as a right $D(\Gamma)$-module, the map

$$P:E(\Gamma) \rightarrow D(\Gamma)$$

given by restricting sections (to X) is clearly a $D(\Gamma)$-module morphism. We shall require further properties.

6° Lemma. *The map $P:E(\Gamma) \rightarrow D(\Gamma)$ satisfies the following conditions for all $f \in E(\Gamma)$, $g \in N(\Gamma)$.*

i° $P(f^*) = P(f)^*$

ii° $P(f^*f) \geqq 0$ *and* $P(f^*f) = 0$ *if and only if* $f = 0$

iii° $P(g^*fg) = g^*P(f)g$.

Proof.

i° This is obvious.

ii° If $f \in E(\Gamma)$ and $x \in X$ (note that if $(\sigma, \rho) \in R^2$ with $x = \sigma\rho$, then $\sigma = \rho^*$), the product formula yields

$$(f^*f)(x) = \sum_{s(\rho)=x} f^*(\rho^*)f(\rho) = \sum_{s(\rho)=x} |f(\rho)|^2 \geqq 0;$$

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if $(f^*f)(x) = 0$ for all $x \in X$, then $f(\rho) = 0$ for all $\rho \in R$.

iii° If $g \in N(\Gamma)$ and $f \in D(\Gamma)$, it follows immediately that $g^*fg \in D(\Gamma)$. For $g \in N(\Gamma)$ and $x \in X$, there is at most one $\rho \in R$ for which $s(\rho) = x$ and $g(\rho) \neq 0$. Now let $f \in E(\Gamma)$; if there is such a ρ, then

$$(g^*fg)(x) = |g(\rho)|^2 f(r(\rho));$$

otherwise the left side of the equation is zero. In either case the assertion follows.

We view $E(\Gamma)$ as a $C_o(X)$-module (on the right) in the obvious way, and use P to define a $C_o(X)$-valued sesquilinear form on $E(\Gamma)$ along the lines of Rieffel's treatment in [**37**] (see also [**18**], [**27**]). For $f, g \in E(\Gamma)$ put

$$\langle f, g \rangle = P(f^*g);$$

note that

$$\langle f, gh \rangle = P(f^*gh) = P(f^*g)h = \langle f, g \rangle h$$

for all $h \in C_o(X)$ and $f, g \in E(\Gamma)$.

7° Proposition. *With the inner product defined above, $E(\Gamma)$ is a (right) pre-Hilbert $C_o(X)$-module. The completion of $E(\Gamma)$ with respect to the norm*

$$\|f\|_\Delta = |\langle f, f \rangle|_\infty^{1/2}$$

is a Hilbert $C_o(X)$-module.

Proof. That $E(\Gamma)$ is a pre-Hilbert module follows directly from the definition (cf [**18**] Definition 2.1) and the above lemma. The second assertion follows from Remark 2.5 of [**27**].

Let $\mathscr{H}(\Gamma)$ denote the completion of $E(\Gamma)$ with the above norm.

8° Proposition. *There is a *-homomorphism, $\pi:E(\Gamma) \to L(\mathscr{H}(\Gamma))$, such that*

$$\pi(f)g = fg \quad \textit{for all } f, g \in E(\Gamma).$$

Proof. We must first show that left multiplication by elements in $E(\Gamma)$ is bounded with respect to $\|\cdot\|_\Delta$. This is clear for elements in $D(\Gamma)$. Since $E(\Gamma)$ is spanned by $N(\Gamma)$, it suffices to show that left multiplication is bounded for elements in $N(\Gamma)$. Let $f \in N(\Gamma)$ and note that $f^*f \in D(\Gamma)$; now, putting $h = (f^*f)^{1/2}$, one obtains

$$\langle fg, fg \rangle = P(g^*f^*fg) = P(g^*h^2g) = \langle hg, hg \rangle,$$

for any $g \in E(\Gamma)$, and it follows that left multiplication by f is bounded. Let $\pi(f)$ denote the unique extension to a bounded operator on $\mathscr{H}(\Gamma)$, where $f \in E(\Gamma)$ is arbitrary. That $\pi(f)$ has an adjoint is clear from

$$\langle fg, h\rangle = P(g^*f^*h) = \langle g, f^*h\rangle;$$

in fact, $\pi(f^*) = \pi(f)^*$ for all $f \in E(\Gamma)$. Hence the desired *-homomorphism.

Let $A(\Gamma)$ (resp. $B(\Gamma)$) denote the completion of $\pi(E(\Gamma))$ (resp. $\pi(D(\Gamma))$) in $L(\mathscr{H}(\Gamma))$; it is well known that $L(\mathscr{H}(\Gamma))$ is a C^*-algebra (though generally not separable). Henceforth, we suppress π and regard $B(\Gamma)$ as an abelian subalgebra of the C^*-algebra $A(\Gamma)$. Note that $B(\Gamma) \cong C_o(X)$.

9° THEOREM. *With the situation as above, $B(\Gamma)$ is diagonal in $A(\Gamma)$.*

The proof proceeds through a sequence of lemmas. In showing that P extends to a conditional expectation on $A(\Gamma)$, we invoke Tomiyama's characterization of such maps as projections of norm one onto a subalgebra. The following elementary fact concerning operators on a Hilbert space will be required:

(¶) Let $a_1, \dots, a_n \in B(\mathscr{H})$ be such that

$$\sum_i a_i^* a_i \leqq 1;$$

for any $b \in B(\mathscr{H})$ one has

$$\left|\left|\sum_i a_i^* b a_i\right|\right| \leqq ||b||.$$

10° LEMMA. *For all $f \in E(\Gamma)$, $||P(f)|| \leqq ||f||$.*

Proof. Given $f \in E(\Gamma)$, it will suffice to find $g_1, \dots, g_n \in B(\Gamma)$ with

$$\sum_i g_i^* g_i \leqq 1,$$

such that

$$P(f) = \sum_i g_i^* f g_i.$$

Let $j:R \to X \times X$ be the embedding defined by

$$j(\rho) = (r(\rho), s(\rho)).$$

If $K \subset R$ is compact and $K \cap X = \emptyset$, then the collection of open subsets of X,

$$\mathscr{U}(K) = \{U \subset X: j(K) \cap U \times U = \emptyset\},$$

covers X (indeed, $j(K)$ is compact and its complement, which contains $j(X)$, is open). Choose a partition of unity, $\{h_1, \dots, h_n\} \subset B(\Gamma)$, subordinate to the cover $\mathscr{U}(\text{supp}\, f \backslash X)$, for supp$(P(f))$. Set $g_i = h_i^{1/2}$ and observe that

$$P(f) = \sum_i g_i f g_i.$$

Let P also denote its unique extension to $A(\Gamma)$; by Tomiyama's criterion

$$P:A(\Gamma) \to B(\Gamma)$$

is a conditional expectation (see [**43**]).

11° LEMMA. *P is faithful.*

Proof. Let a be a non-zero element in $A(\Gamma)$; there exists $f \in N(\Gamma)$ such that $af \neq 0$ (since $N(\Gamma)$ is total in $\mathscr{H}(\Gamma)$). One obtains

$$f^*P(a^*a)f = P(f^*a^*af) = \langle af, af \rangle \neq 0,$$

where the first equality follows from Lemma 6(iii) by continuity. Hence $P(a^*a) \neq 0$ and P is faithful.

12° LEMMA. *The free normalizers are total in* ker P.

Proof. Evidently, $E(\Gamma) \cap \ker P$ is dense in ker P. It will, therefore, suffice to verify that for any $f \in E(\Gamma)$ with $P(f) = 0$, there exist $g_1, \dots, g_n \in N(\Gamma)$ with $g_i^2 = 0$ such that

$$f = \sum_i g_i.$$

Given such an f, there are a finite number of open sets, $V_1, \dots, V_n \in \Omega(R)$, such that

$$s(V_i) \cap r(V_i) = \emptyset \quad \text{for each } i, \text{ and } \operatorname{supp} f \subset \bigcup_i V_i.$$

Let $h_1, \dots, h_n \in C_c(R)$ be a partition of unity for supp f subordinate to $\{V_i\}$ and set

$$g_i(\gamma) = f(\gamma)h_i(\dot{\gamma}).$$

The adjunction of a unit (if necessary) does not change the fact that P is faithful; further, the free normalizers remain total in its kernel. Thus $B(\Gamma)$ is diagonal in $A(\Gamma)$, and the theorem is proved.

13° *Remarks.* The correspondence between 1-cocyles on a groupoid and automorphisms of the associated *-algebra is well known (see [**34**] Section II.5 and [**13**] Section II.4). A continuous map $c:R \to \mathbf{T}$ is called a $\mathbf{T}$-valued 1-cocycle if

$$c(\rho\sigma) = c(\rho)c(\sigma) \quad \text{for all } (\rho, \sigma) \in R^2;$$

the collection of all such is denoted $Z^1(R, \mathbf{T})$.

Returning to the situation above with Γ a twist over a relation R, let $c \in Z^1(R, \mathbf{T})$, and define an element $u_c \in L(\mathscr{H}(\Gamma))$ by

$$(u_c f)(\gamma) = c(\dot{\gamma})f(\gamma) \quad \text{for all } f \in E(\Gamma), \gamma \in \Gamma$$

(this is clearly bounded so one extends to all of $\mathscr{H}(\Gamma)$ by continuity). That u_c is unitary follows from the observation that it preserves the inner product (note that $u_c^* = u_{-c}$, where $-c$ is the inverse of $c \in Z^1(R, \mathbf{T})$). One checks that

$$\operatorname{Ad} u_c(A(\Gamma)) = A(\Gamma)$$

and we denote this automorphism of $A(\Gamma)$ by α_c. If $f \in B(\Gamma)$, then $\alpha_c(f) = f$. In fact, this condition will be seen to characterize automorphisms arising from 1-cocycles (see Section 3 below). Note that α_c is inner if and only if there is a continuous function $h:X \to \mathbf{T}$ such that

$$c(\rho) = h(r(\rho))\overline{h(s(\rho))} \quad \text{for all } \rho \in R,$$

that is, $c \in B^1(R, \mathbf{T})$ (cf [**34**] Proposition II.5.3).

3. The twist of a diagonal. In this section, the twist is shown to be a complete invariant of a diagonal pair. This is done by associating a twist to a given diagonal pair and then verifying that the pair constructed from the twist (as in the preceding section) is canonically isomorphic to the given pair.

Let A be a C^*-algebra. The extremal points of the unit ball of A^* may be viewed as a topological groupoid (with respect to the weak* topology) on the collection of pure states. The dual groupoid, as it is called, has been studied in [**1**] and [**36**]. In the latter, Renault shows that the spectrum of a diagonal, as a subset of the pure state space is a "nice" transversal, in the sense that the desired twist is the reduction of the dual groupoid to the spectrum of the diagonal.

For the sake of completeness we provide an independent construction which follows the lines indicated in our original note (cf. [**22**]). Nevertheless, the reader would do well to keep the dual groupoid perspective in mind.

1° THEOREM. *Given a diagonal pair*, (A, B), *there is* (*up to isomorphism*) *a unique twist* Γ *and an isomorphism*

$$\phi:A(\Gamma) \to A$$

such that $\phi(B(\Gamma)) = B$.

Proof. We may assume without loss of generality that A is unital. For, if A is not unital, one adjoins a unit to both A and B, obtains the desired twist for the pair (A, B), and then deletes the point at infinity from the unit space (together with the copy of $\mathbf{T}$ above it). Let $X = \hat{B} \subset A^*$ (where the injection is given by the extension property of B relative to A).

Set

$$D = \{ (d, x) \in N(B) \times X : x(d^*d) > 0\};$$

for each $(d, x) \in D$, let $[d, x]$ be the linear functional on A defined by

$$[d, x](a) = x(d^*a)x(d^*d)^{-1/2} \quad \text{for all } a \in A.$$

Observe that if $f \in B$ with $x(f) > 0$, then $x = [f, x]$. If $(c, x), (d, x) \in D$, then by Lemma 1.9 the following conditions are equivalent:

i° $[c, x] = [d, x]$

ii° $x(c^*d) > 0$

iii° there exist $f, g \in B$ with $x(f), x(g) > 0$ such that $cf = dg$.

Let

$$\Gamma = \{ [d, x] : (d, x) \in D\} \subset A^*.$$

Γ is to be endowed with the structure of a twist. The **T**-action is given by scalar multiplication (that is, $t[d, x] = [\bar{t}d, x]$). The groupoid structure is given by the following formulas

$s[d, x] = x$

$[d, x]^* = [d^*, \sigma_d(x)]$

$([c, y], [d, x]) \in \Gamma^2$ if and only if $y = \sigma_d(x)$, in which case,

$[c, y][d, x] = [cd, x]$.

We define the topology on Γ by giving a neighbourhood system at a point $[d, x] \in \Gamma$. Let $U \subset X$, $V \subset \mathbf{T}$ be open sets such that $x \in U \subset s(d)$ and $1 \in V$; set

$$W(U, V) = \{t[d, y] : y \in U, t \in V\}.$$

One checks that this topology is the same as the relative weak* topology and that the above operations are continuous with respect to it. Since scalar multiplication commutes with multiplication in any algebra, it is clear that Γ is a **T**-groupoid. Let R denote the subjacent groupoid and let

$$q:\Gamma \rightarrow R$$

denote the quotient map. If $r[d, x] = x$, then $[d, x] = tx$ for some $t \in \mathbf{T}$, whence $\mathrm{Is}(R) = X$, and R is a relation (as $q(W(U, V)) \cong U$). Further, each normalizer determines a local section for the circle-bundle map q. Indeed, if $\rho \in R$, then $\rho = q[d, x]$ for some $(d, x) \in D$ and the map

$$(\sigma_d(y), y) \mapsto [d, y]$$

trivializes the circle-bundle over the open set,

$$U(d) = \{ (\sigma_d(y), y) : y \in s(d) \}.$$

Thus, Γ is a twist.

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We claim that, for every $f \in N(\Gamma)$, there is a unique $\phi(f) \in N(B)$ such that

(†) $f(\gamma) = \gamma(\phi(f))$, for all $\gamma \in \Gamma$.

By compactness, there exist $d_1, \ldots, d_n \in N(B)$ such that

$$q(\operatorname{supp} f) \subset \bigcup_k U(d_k).$$

Choose a partition of unity, $\{h_k\}$, for $s(\operatorname{supp} f) \subset X$, subordinate to the cover $\{s(d_k)\}$. Define $g_k \in B$ for $1 \leqq k \leqq n$ by

$$x(g_k) = \begin{cases} f([d_k, x])h_k(x)x(d_k^*d_k)^{-1/2} & \text{for } x \in s(d_k) \\ 0 & \text{elsewhere} \end{cases}$$

and set

$$\phi(f) = \sum_k d_k g_k.$$

Then, (†) holds by construction. Now, ϕ extends linearly to a *-isomorphism from $E(\Gamma)$ onto a dense subalgebra of A (one checks that $\phi(fg) = \phi(f)\phi(g)$ for $f, g \in N(\Gamma)$ and $\phi(f^*) = \phi(f)^*$). It remains to show that ϕ is isometric. Since P is faithful, one has

$$\|a\| = \sup\{ \|P(b^*a^*ab)\|^{1/2} : P(b^*b) \leqq 1\} \quad \text{for all } a \in A,$$

but this is how the C^*-norm was defined on $E(\Gamma)$. Hence,

$$\phi : E(\Gamma) \to A$$

extends to an isomorphism of C^*-algebras.

Suppose Γ is a twist on X.

2° *Definition*. Let $\operatorname{Aut}(\Gamma)$ denote the group of homeomorphisms, $\delta : \Gamma \to \Gamma$, satisfying:

i° for all $(\alpha, \beta) \in \Gamma^2$,

$$(\delta(\alpha), \delta(\beta)) \in \Gamma^2 \quad \text{and} \quad \delta(\alpha\beta) = \delta(\alpha)\delta(\beta)$$

ii° $t\delta(\gamma) = \delta(t\gamma)$ for all $t \in \mathbf{T}$, $\gamma \in \Gamma$.

Set

$$\operatorname{Aut}_X(\Gamma) = \{\delta \in \operatorname{Aut}(\Gamma) : \delta(x) = x, \text{ for all } x \in X\}.$$

3° *Remarks*. Condition i° implies that $\delta(X) = X$. One sees immediately that $\operatorname{Aut}_X(\Gamma)$ is a normal subgroup of $\operatorname{Aut}(\Gamma)$. Further, $\operatorname{Aut}_X(\Gamma)$ is isomorphic to $Z^1(R, \mathbf{T})$, where R is the subjacent relation, via the map

$$c \in Z^1(R, \mathbf{T}) \mapsto \delta_c \in \operatorname{Aut}_X(\Gamma),$$

where $\delta_c(\gamma) = c(\dot{\gamma})\gamma$. For analogous results concerning standard Borel equivalence relations consult [**13**] (Section II.4).

4° Corollary. *For each* $\alpha \in \mathrm{Aut}(A)$ *with* $\alpha \circ P = P \circ \alpha$, *there is* $\alpha^* \in \mathrm{Aut}(\Gamma)$ *such that*:

$$\gamma(\alpha(a)) = (\alpha^*\gamma)(a) \quad \textit{for all } a \in A, \gamma \in \Gamma.$$

Further, $\alpha^* \in \mathrm{Aut}_X(\Gamma)$ *if and only if* $\alpha \circ P = P$.

Proof. This follows directly from the construction of $\Gamma \subset A^*$ in Theorem 1.

5° *Example.* Let G be a discrete abelian group with symplectic bicharacter χ, and let $H \subset G$ be maximal with the property that $\chi(H \wedge H) = 1$, as in example 1.12. Let R denote the relation on $\hat{H}$ associated to the diagonal pair, (A_χ, B_H). One sees that

$$R \cong G/H \times \hat{H}$$

where the unit space is identified with $\{0\} \times \hat{H}$ (note that $s([g], x) = x$, and $r([g], x) = x\chi_H(g)$). The associated twist is seen to be topologically trivial as each u_g defines a continuous section on $\{[g]\} \times \hat{H}$.

Suppose H_0, H_1 are abelian groups and

$$\theta: H_0 \to \hat{H}_1$$

is an injective homomorphism. We view $R_\theta = H_0 \times \hat{H}_1$ as a relation on $\hat{H}_1$ via the maps

$$s(h, x) = x$$

$$(h, x)^* = (-h, x\theta(h)).$$

We wish to characterize those twists over R_θ which arise as above for some bicharacter χ on $H_0 \oplus H_1$ for which H_1 is maximal with the property that $\chi(H_1 \wedge H_1) = 1$.

Let Γ be a twist over $R_\theta = H_0 \times \hat{H}_1$, and suppose there is a homomorphism

$$\alpha: \hat{H}_1 \to \mathrm{Aut}(\Gamma)$$

satisfying the conditions

i° $\alpha: \hat{H}_1 \times \Gamma \to \Gamma$ is continuous

ii° $\alpha_x(y) = xy$ for all $x, y \in \hat{H}_1$.

Consider the H_0-valued cocycle on R_θ defined by $p(h, x) = h$ and the associated homomorphism (cf [**34**] Section II.5)

$$\delta: \hat{H}_0 \to \mathrm{Aut}_X(\Gamma)$$

given by

$$\delta_z(\gamma) = \langle z, p(\dot{\gamma}) \rangle \gamma;$$

since $p(\dot{\alpha_x}(\gamma)) = p(\dot{\gamma})$ for all $x \in \hat{H}_1$, $\gamma \in \Gamma$, one obtains

$$\delta_z\alpha_x(\gamma) = \alpha_x\delta_z(\gamma) \quad \text{for all } x \in \hat{H}_1,\ z \in \hat{H}_0,\ \gamma \in \Gamma.$$

Consequently, there is an ergodic action

$$\beta:\hat{H}_0 \times \hat{H}_1 \to \mathrm{Aut}(A(\Gamma))$$

such that

$$\gamma(\beta(z, x)(a)) = (\delta_z\alpha_x(\gamma))(a),$$

for all $(z, x) \in \hat{H}_0 \times \hat{H}_1$, $\gamma \in \Gamma$, $a \in A(\Gamma)$. Let χ denote the associated bicharacter on $H_0 \oplus H_1$ (see [**26**]); it follows that

$$(A_\chi, B_{H_1}) \cong (A(\Gamma), C(\hat{H}_1)).$$

4. Twists and cohomology. The collection of twists over a given relation is to be endowed with the structure of an abelian group. The usual second cohomology of a relation (with coefficients in $\mathbf{T}$) is canonically isomorphic to the subgroup of twists which are topologically trivial (i.e., the quotient map; $q:\Gamma \to R$, admits a continuous section). The connection between second cohomology and extensions, familiar from group theory, is considered in the setting of groupoids by Renault (see [**34**] Proposition I.1.14). Viewing twists as extensions which need not admit continuous sections, it is appropriate to consider the group of twists as a replacement for the usual second cohomology of a relation.

Let R be a relation on X, and let $\mathrm{Tw}(R)$ denote the collection of isomorphism classes of twists over R.

1° *Fact.* There is a bijective correspondence between $H^2(R, \mathbf{T})$ and the subset of $\mathrm{Tw}(R)$ consisting of twists admitting a continuous section.

Proof. Suppose $q:\Gamma \to R$ admits a continuous section $\sigma:R \to \Gamma$. Let

$$c_\sigma:R^2 \to \mathbf{T}$$

be the unique continuous function satisfying the condition

$$c_\sigma(\rho_1, \rho_2)\sigma(\rho_1\rho_2) = \sigma(\rho_1)\sigma(\rho_2) \quad \text{for all } (\rho_1, \rho_2) \in R^2.$$

For all $(\rho_1, \rho_2, \rho_3) \in R^3$ (that is, $s(\rho_1) = r(\rho_2)$, $s(\rho_2) = r(\rho_3)$), one obtains the following by associativity of groupoid composition

$$\begin{aligned}\sigma(\rho_1)\sigma(\rho_2)\sigma(\rho_3) &= c_\sigma(\rho_1, \rho_2)c_\sigma(\rho_1\rho_2, \rho_3)\sigma(\rho_1\rho_2\rho_3)\\ &= c_\sigma(\rho_1, \rho_2\rho_3)c_\sigma(\rho_2, \rho_3)\sigma(\rho_1\rho_2\rho_3);\end{aligned}$$

hence, $c_\sigma \in Z^2(R, \mathbf{T})$. Another choice of section yields a cocycle in the same class. That every class in $H^2(R, \mathbf{T})$ arises from some topologically trivial twist is clear.

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2° *Remark.* Let Γ be a twist over R. If there is a continuous section

$$\sigma:R \to \Gamma,$$

which is a groupoid morphism, then Γ is said to be a trivial twist. The map

$$\tilde{\sigma}:\mathbf{T} \times R \to \Gamma,$$

defined by $\tilde{\sigma}(t, \rho) = t\sigma(\rho)$ is an isomorphism of twists. Let R_o denote the trivial twist $\mathbf{T} \times R$.

Given two twists, $q_i:\Gamma_i \to R$, $i = 1, 2$, we define the product twist $\Gamma_1 * \Gamma_2$ as the quotient of

$$\{ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 : q_1(\gamma_1) = q_2(\gamma_2) \}$$

by the equivalence relation

$$(\gamma_1, \gamma_2) \sim (t\gamma_1, \bar{t}\gamma_2) \quad \text{for } t \in \mathbf{T}.$$

Note that this is the usual product of circle-bundles over a common base space. The groupoid structure is prescribed by the following formulas:

$$s(\gamma_1, \gamma_2) = s(\gamma_1) = s(\gamma_2)$$

$$(\gamma_1, \gamma_2)^* = (\gamma_1^*, \gamma_2^*)$$

$$((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in (\Gamma_1 * \Gamma_2)^2 \quad \text{if and only if } (\alpha_i, \beta_i) \in \Gamma_i^2,\ i = 1, 2$$

$$(\alpha_1, \alpha_2)(\beta_1, \beta_2) = (\alpha_1\beta_1, \alpha_2\beta_2).$$

One verifies that $\Gamma_1 * \Gamma_2$ is a twist over R.

3° PROPOSITION. *When endowed with the above product* $\mathrm{Tw}(R)$ *is an abelian group with neutral element* $[R_o]$, *and the injection*

$$H^2(R, \mathbf{T}) \to \mathrm{Tw}(R)$$

is a homomorphism.

Proof. For any twist Γ, the isomorphism $R_o * \Gamma \cong \Gamma$ is given by

$$((t, \dot{\gamma}), \gamma) \mapsto t\gamma.$$

It remains to show that inverses exist; let Γ^+ be the twist obtained from Γ by inverting the $\mathbf{T}$-action. That is, there is a groupoid isomorphism

$$\gamma \in \Gamma \mapsto \gamma^+ \in \Gamma^+$$

for which $t\gamma^+ = (\bar{t}\gamma)^+$. There is a trivializing section,

$$\sigma:R \to \Gamma * \Gamma^+,$$

given by $\sigma(\dot{\gamma}) = (\gamma, \gamma^+)$; since σ is a groupoid morphism, it follows that $[\Gamma^+] = -[\Gamma]$. A straightforward calculation reveals that multiplication of cocycles corresponds to the above defined product of twists.

Let $\mathscr{S}$ denote the sheaf of germs of continuous circle-valued functions. We construct a natural homomorphism (cf. [**22**, **23**])

$$\zeta : H^1(X, \mathscr{S}) \to \mathrm{Tw}(R),$$

where $H^1(X, \mathscr{S})$ is viewed as the group of (equivalence classes of) circle-bundles over X.

We view the Cartesian product, $X \times X$, as a topological groupoid on X (here the unit space is identified with the diagonal $\Delta = \{(x, x) : x \in X\}$) with respect to the following operations:

$$s(x, y) = y, \ (x, y)^* = (y, x),$$

$((x, y), (w, z))$ is composible if and only if $y = w$, and then

$$(x, y)(y, z) = (x, z).$$

4° Proposition. *There is a bijective correspondence between isomorphism classes of proper* **T**-*groupoids with subjacent groupoid* $X \times X$ (*cf. definition* 2.2) *and isomorphism classes of circle-bundles over* X.

Proof. Let Λ be a circle-bundle over X with bundle map $p : \Lambda \to X$. Let Λ^+ denote the conjugate circle-bundle obtained by inverting the **T**-action (so there is a homeomorphism $\lambda \in \Lambda \mapsto \lambda^+ \in \Lambda^+$ such that $(t\lambda)^+ = \bar{t}\lambda^+$); note $[\Lambda^+] = -[\Lambda]$ in $H^1(X, \mathscr{S})$. Let $\Psi(\Lambda)$ denote the quotient of $\Lambda \times \Lambda^+$ by the equivalence relation

$$(\lambda_1, \lambda_2^+) \sim (t\lambda_1, \bar{t}\lambda_2^+) \quad \text{for all } t \in \mathbf{T}.$$

Observe that $\Psi(\Lambda)$ is a circle-bundle over $X \times X$, in fact

$$\Psi(\Lambda) = r^*(\Lambda) * s^*(\Lambda^+) \quad \text{(product of pull-back circle-bundles)}.$$

We claim that $\Psi(\Lambda)$ is a proper **T**-groupoid on X with unit space inclusion

$$p(\lambda) \in X \mapsto (\lambda, \lambda^+) \in \Psi(\Lambda).$$

The source map is given by $s(\lambda_1, \lambda_2^+) = p(\lambda_2)$, involution by $(\lambda_1, \lambda_2^+)^* = (\lambda_2, \lambda_1^+)$. Now

$$((\lambda_1, \lambda_2^+), (\lambda_3, \lambda_4^+)) \in \Psi(\Lambda)^2$$

if and only if

$$p(\lambda_2) = p(\lambda_3);$$

in this case, there is a unique $t \in \mathbf{T}$ such that $\lambda_3 = t\lambda_2$ and one writes

$$(\lambda_1, \lambda_2^+)(\lambda_3, \lambda_4^+) = t(\lambda_1, \lambda_4^+).$$

Composition commutes with the **T**-action by construction and $\Psi(\Lambda)$ is thus a proper **T**-groupoid. Conversely, suppose we are given a proper **T**-groupoid Ψ over $X \times X$. For $x \in X$, put $\Lambda_x = s^{-1}(x)$ and note that $r:\Lambda_x \to X$ is a circle-bundle map where the **T**-action is inherited from Ψ. We claim that $\Psi \cong \Psi(\Lambda_x)$; indeed,

$$\Lambda_x^+ \cong \Lambda_x^* = r^{-1}(x).$$

Further the composite

$$\Lambda_x \times \Lambda_x^* \subset \Psi^2 \to \Psi$$

is surjective by the transitivity of Ψ. Hence, Ψ is the quotient of $\Lambda_x \times \Lambda_x^+$ by the equivalence relation

$$(\lambda, \mu^+) \sim (t\lambda, \bar{t}\mu^+),$$

as desired. This does not depend on the specific choice of x, since for any $y \in X$, there exists $\gamma \in \Psi$ with $r(\gamma) = x$ and $s(\gamma) = y$; hence, the map

$$\lambda \in \Lambda_x \mapsto \lambda\gamma \in \Lambda_y$$

is a circle-bundle isomorphism.

With the situation as above, let $q:\Psi(\Lambda) \to X \times X$ denote the quotient map.

5° Corollary. *Let R be a relation on X and Λ a circle-bundle over X. The groupoid*

$$(r, s)^*(\Psi(\Lambda)) = \{ (\rho, \gamma) \in R \times \Psi(\Lambda):(r, s)(\rho) = q(\gamma) \}$$

is a twist over R. Further, the map

$$\zeta:H^1(X, \mathscr{S}) \to \mathrm{Tw}(R),$$

given by

$$\zeta[\Lambda] = [(r, s)^*(\Psi(\Lambda))]$$

is a group homomorphism.

5. Morita equivalence. Rieffel's theory of strong Morita equivalence of C^*-algebras (developed in [**37**]) is here adapted to the setting of diagonal pairs. A criterion for strong Morita equivalence of diagonal pairs will be given in terms of the associated twists (via the linking algebra characterization appearing in [**6**]). We introduce the notion of Γ-structure (patterned on the notion of Ω-manifold appearing in [**20**]), which enables one to transplant twist structure from one space to another in a manner that ensures the strong Morita equivalence of the associated C^*-algebras. A related but more general notion is that of equivalence of groupoids (cf [**36**], [**35**]).

1° *Definition*. Two diagonal pairs, (A_i, B_i) $i = 1, 2$, are said to be *Morita equivalent* (write $(A_1, B_1) \approx (A_2, B_2)$), if there is a C^*-algebra A and two projections, $p_1, p_2 \in M(A)$, with $p_1 + p_2 = 1$, such that $p_i A p_i$ is not contained in any proper ideal ($i = 1, 2$), as well as isomorphisms

$$\phi_i : A_i \to p_i A p_i \quad \text{for } i = 1, 2$$

such that $B = \phi_1(B_1) \oplus \phi_2(B_2)$ is diagonal in A.

2° *Remarks*. This definition is a modification of the linking algebra characterization of strong Morita equivalence (see **[6]** Theorem 1.1). In particular, if $(A_1, B_1) \approx (A_2, B_2)$, then A_1 and A_2 are strongly Morita equivalent, where the A_1-A_2-equivalence bimodule is $p_1 A p_2$ (with the obvious inner products). One should check that this defines an equivalence relation on diagonal pairs. The requirement that $B = \phi_1(B_1) \oplus \phi_2(B_2)$ be diagonal ensures that:

$$N_f(B) \cap p_1 A p_2 \text{ is total in } p_1 A p_2.$$

Suppose (A_i, B_i) are diagonal pairs for $i = 1, 2, 3$. If J_1 is an A_1-A_2-equivalence bimodule and J_2 is an A_2-A_3-equivalence bimodule, then $J_1 \otimes_{A_2} J_2$ (as defined in **[37]**) is an A_1-A_3-equivalence bimodule. If both J_1 and J_2 arise, as above, in diagonal preserving linking algebras (i.e., $N_f \cap J_i$ is total in J_i), then the same can be said of $J_1 \otimes_{A_2} J_2$.

3° *Definition*. Let Γ be a twist on X and $U \subset X$ an open set; the twist

$$\Gamma_U = \{\gamma \in \Gamma : r(\gamma), s(\gamma) \in U\},$$

is called the *reduction* of Γ to U. If U is full (i.e., $[U] = X$) the reduction is said to be full.

Note that $A(\Gamma_U)$ embeds naturally in $A(\Gamma)$ as an hereditary subalgebra, which is full, that is, not contained in any proper ideal, exactly when the reduction Γ_U is full. When this is the case, the associated pairs are Morita equivalent, that is

$$(A(\Gamma), B(\Gamma)) \approx (A(\Gamma_U), B(\Gamma_U)).$$

The smallest hereditary subalgebra of $M_2(A)$ containing $e_{11} \otimes A(\Gamma)$ and $e_{22} \otimes A(\Gamma_U)$ is the desired linking algebra.

4° PROPOSITION. *Suppose Γ_i is a twist on X_i for $i = 1, 2$; one has*

$$(A(\Gamma_1), B(\Gamma_1)) \approx (A(\Gamma_2), B(\Gamma_2))$$

if and only if there is a twist Γ on the disjoint union $X_1 \perp\!\!\!\perp X_2$ such that $\Gamma_i \cong \Gamma_{X_i}$ and X_i is full for $i = 1, 2$.

Proof. This follows immediately from the definition of Morita equivalence for diagonal pairs and the characterization of the twist as a complete invariant of a diagonal pair (see Theorem 3.1 above).

5° *Definition*. With Γ_1 and Γ_2 as in the above proposition, we say the twists are *Morita equivalent* and write

$$\Gamma_1 \approx \Gamma_2 \text{ via } \Gamma;$$

the twist Γ is referred to as a *linking twist* (note that this need not be unique).

6° *Definition*. A continuous open map $\psi:Z \to X$ is called a *quasilocal homeomorphism* (qlh), if it is locally injective (that is, the collection of open sets $U(\psi) = \{U \subset Z:\psi|_U \text{ is injective}\}$ covers Z), and a local homeomorphism, if, in addition, it is surjective.

Such maps are useful in inducing Morita equivalent twists to other spaces, as is seen in the next proposition.

7° PROPOSITION. *Let Γ be a twist on X and $\psi:Z \to X$ be a quasilocal homeomorphism for which $\psi(Z)$ is full. Let Γ^ψ denote the space*

$$Z * \Gamma * Z = \{ (z_1, \gamma, z_2):\psi(z_1) = r(\gamma), \psi(z_2) = s(\gamma) \} \\ \subset Z \times \Gamma \times Z.$$

Then Γ^ψ may be endowed with the structure of a twist on Z in such a way that $\Gamma \approx \Gamma^\psi$.

Proof. One checks first that Γ^ψ is a $\mathbf{T}$-groupoid on Z (where the unit space is identified with $\{ (z, \psi(z), z):z \in Z\}$) under the operations

$$s(z_1, \gamma, z_2) = z_2, \quad (z_1, \gamma, z_2)^* = (z_2, \gamma^*, z_1)$$

$$(z_1, \gamma_1, z_2)(z_2, \gamma_2, z_3) = (z_1, \gamma_1\gamma_2, z_3)$$

and the $\mathbf{T}$-action is given by

$$t(z_1, \gamma, z_2) = (z_1, t\gamma, z_2);$$

the subjacent relation is denoted R^ψ where R is the subjacent relation for Γ:

$$R^\psi = \{ (z_1, \rho, z_2):\psi(z_1) = r(\rho), \psi(z_2) = s(\rho) \} \subset Z \times R \times Z.$$

The local triviality of the circle-bundle Γ^ψ over R^ψ follows from that of Γ over R and the fact that ψ is locally injective (so there exist local sections). We proceed now to the construction of the linking twist. Let

$$\phi:Z \sqcup X \to X$$

denote the local homeomorphism given by

$$\phi(y) = \begin{cases} \psi(y) & \text{if } y \in Z \\ y & \text{if } y \in X. \end{cases}$$

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Then $\Gamma^{\psi} \approx \Gamma$ via Γ^{ϕ} (note that $(\Gamma^{\phi})_Z \cong \Gamma^{\psi}$ and $(\Gamma^{\phi})_X \cong \Gamma$).

Observe that there is a distinguished copy of $R(\psi)$ in Γ^{ψ}, where

$$R(\psi) = \{ (z_1, z_2) \in Z \times Z: \psi(z_1) = \psi(z_2) \}.$$

The following proposition provides a converse of sorts to the preceding one:

8° Proposition. *Let Γ be a twist on Z, and let $\psi:Z \to X$ be a local homeomorphism. If there is a groupoid morphism $j:R(\psi) \to \Gamma$ which identifies unit spaces (i.e., $j(z) = z$), then there is a unique (up to isomorphism) twist Λ on X and a surjective twist morphism $\pi:\Gamma \to \Lambda$ satisfying the following conditions*:

i° $\pi|_Z = \psi$

ii° $\text{Im}(j) = \pi^{-1}(X)$

iii° $\Gamma \cong \Lambda^{\psi}$.

Proof. The twist Λ is constructed as the quotient of Γ by the equivalence relation

$\alpha \sim \beta$ if there exist $\rho, \sigma \in R(\psi)$ such that $\alpha j(\rho) = j(\sigma)\beta$.

This is clearly reflexive and symmetric; transitivity remains to be checked. With $\alpha \sim \beta$ as above, suppose $\beta \sim \gamma$ with $\beta j(\lambda) = j(\mu)\gamma$. One sees that

$$\alpha j(\rho\lambda) = j(\sigma)\beta j(\lambda) = j(\sigma\mu)\gamma.$$

The quotient Λ is then a twist on X (note that the quotient map $\pi:\Gamma \to \Lambda$ is itself a local homeomorphism). Conditions i° and ii° hold by construction. Let $\phi:\Gamma \to Z * \Lambda * Z$ be given by

$$\phi(\gamma) = (r(\gamma), \pi(\gamma), s(\gamma)).$$

One checks that ϕ defines the isomorphism indicated in iii°.

Viewing Λ as the quotient of Γ by $j:R(\psi) \to \Gamma$, one writes $\Lambda = \Gamma/j$. It is perhaps worth noting that the isomorphism class of Λ depends on the choice of j. It is shown below that any twist Morita equivalent to a given one can be obtained via the constructions delineated in the two preceding propositions. Towards this end, we require the notion of Γ-cocycle.

9° *Definition.* Let Γ be a twist on X with quotient map $q:\Gamma \to R$ and let $\mathscr{U} = \{U_i : i \in 1\}$ be a countable locally finite cover of a space Z. A collection of continuous maps

$$\theta_{ij}:U_{ij} \to \Gamma \quad (\text{where } U_{ij} = U_i \cap U_j)$$

is called a Γ-*cocycle on Z relative to $\mathscr{U}$* if

i° $q \circ \theta_{ij}:U_{ij} \to R$ is a homeomorphism onto some $\omega_{ij} \in \Omega(R)$

ii° for each $z \in U_{ijk}$,

$$(\theta_{ij}(z), \theta_{jk}(z)) \in \Gamma^2 \quad \text{and}$$

$$\theta_{ij}(z)\theta_{jk}(z) = \theta_{ik}(z).$$

Observe that $\operatorname{Im}(\theta_{ii}) \subset X$ by ii° and $\operatorname{Im}(\theta_{ii})$ is open by i°. We say that θ is *true* if $\cup_i \operatorname{Im}(\theta_{ii})$ is full.

10° PROPOSITION. *With notation as above, let θ be a true Γ-cocycle on Z relative to $\mathcal{U}$. There is a twist $\Gamma\langle\theta\rangle$ on Z which is Morita equivalent to Γ via a linking twist Υ over Q for which there exist sections μ_i defined on $W_i \subset Q$ satisfying the conditions*:

i° $r|_{W_i}: W_i \cong U_i$

ii° *If $\rho \in W_i$, $\sigma \in W_j$ with $z = r(\rho) = r(\sigma)$, then*

$$\theta_{ij}(z) = \mu_i(\rho)^*\mu_j(\sigma).$$

Proof. The twist $\Gamma\langle\theta\rangle$ is constructed directly from the Γ-cocycle θ using the methods of Propositions 7 and 8. Let

$$U = \coprod_i U_i$$

and define

$$\psi: U \to X \text{ by } \psi(z) = \theta_{ii}(z) \text{ if } z \in U_i \text{ and}$$

$$\phi: U \to Z \text{ by inclusion } (z \in U_i \mapsto z \in Z).$$

Note that ψ and ϕ are quasilocal homeomorphisms and that $\psi(U)$ is full (because θ is true), while ϕ is surjective. Define a groupoid morphism

$$\iota: R(\phi) \to \Gamma^{\psi} \quad \text{(identifying the unit spaces)},$$

where

$$R(\phi) = \coprod_{i,j} U_{ij} \quad \text{and} \quad \Gamma^{\psi} = \coprod_i U_j * \Gamma * \coprod_i U_i,$$

as follows. If $z \in U_{ij}$ put

$$\iota(z) = (z, \theta_{ij}(z), z) \in U_i \times \Gamma \times U_j \cap \Gamma^{\psi}.$$

That ι is a groupoid morphism follows directly from the cocycle property. Set $\Gamma\langle\theta\rangle = \Gamma^{\psi}/\iota$ and note that $\Gamma\langle\theta\rangle$ is Morita equivalent to Γ. The linking twist Υ is constructed, in like manner, as the twist induced by a Γ-cocycle on $Z \coprod X$. The collection $\tilde{\mathcal{U}} = \mathcal{U} \cup \{X\}$ constitutes an open covering of $Z \coprod X$; let $\tilde{\theta}$ denote the Γ-cocycle defined relative to $\tilde{\mathcal{U}}$ which restricts to θ_{ij} on each U_{ij} and for which $\tilde{\theta}_*(x) = x$ for each $x \in X$ (note that $X \cap U_i = \emptyset$ for each i). Let $\tilde{U} = U \coprod X$ and define, by analogy with the above,

$$\widetilde{\psi}:\widetilde{U} \to X$$

$$\widetilde{\phi}:\widetilde{U} \to Z \coprod X$$

$$\widetilde{\iota}:R(\widetilde{\phi}) \to \Gamma^{\widetilde{\psi}}.$$

It follows that $\Upsilon = \Gamma\langle\widetilde{\theta}\rangle$ is the linking twist, that is, $\Gamma\langle\theta\rangle \approx \Gamma$ via $\Gamma\langle\widetilde{\theta}\rangle$. Now, for each i, define a continuous map $\nu_i:U_i \to \Gamma^{\widetilde{\psi}}$ by

$$\nu_i(z) = (z, \widetilde{\theta}_{ii}(z), \widetilde{\theta}_{ii}(z)) \in (U_i \times \Gamma \times X) \cap \Gamma^{\widetilde{\psi}};$$

put $\widetilde{\nu}_i = \pi_{\widetilde{\iota}} \circ \nu_i$, where $\pi_{\widetilde{\iota}}:\Gamma^{\widetilde{\psi}} \to \Gamma^{\widetilde{\psi}}/\widetilde{\iota}$ (cf. Proposition 8 above). Observe that

$$W_i = \mathrm{Im}(q\widetilde{\nu}_i) \in \Omega(Q)$$

(where Q is the subjacent relation for Υ). Define sections $\mu_i: W_i \to \Upsilon$ by

$$\mu_i(\rho) = \widetilde{\nu}_i(r(\rho)).$$

It is a routine matter to verify that conditions i° and ii° hold for these sections.

11° *Remark.* The linking twist $\Upsilon = \Gamma\langle\widetilde{\theta}\rangle$ in the above proposition is unique subject to the requirement that there exist sections $\{\mu_i\}$ which implement the cocycle (see Theorem 13 below).

We introduce an equivalence relation on the collection of Γ-cocycles on a given space Z (by analogy with the usual notion of equivalence of Čech cocycles). An equivalence class of Γ-cocycles will be termed a Γ-structure on Z.

12° *Definition.* Suppose θ^1 and θ^2 are Γ-cocycles on Z relative to the same covering $\mathscr{U} = \{U_i:i \in I\}$. Then θ^1 and θ^2 are said to be equivalent (write $\theta^1 \sim \theta^2$), if there exist continuous functions $h_i:U_i \to \Gamma$ such that for each $z \in U_{ij}$

$$(h_i(z), \theta^1_{ij}(z)), (\theta^2_{ij}(z), h_j(z)) \in \Gamma^2 \quad \text{and}$$

$$h_j(z)\theta^1_{ij}(z) = \theta^2_{ij}(z)h_j(z).$$

If they are given relative to different covers, then they are said to be equivalent if their restrictions to some common refinement are equivalent (in the above sense). An equivalence class of Γ-cocycles on a space is termed a Γ-structure on that space. If Θ is a Γ-structure on Z, then $\theta \in \Theta$ will be construed as a Γ-cocycle representative (relative to some covering).

In the following result we establish that the twist structure induced by a Γ-cocycle only depends on its equivalence class; furthermore, every Morita equivalent twist arises in this fashion.

13° THEOREM. *Suppose* Λ *and* Γ *are twists on* Z *and* X, *respectively, which are Morita equivalent via a linking twist* Υ. *There is, then, a unique* Γ-*structure* Θ *on* Z *such that if* $\theta \in \Theta$ *is any* Γ-*cocycle in this class, one has a twist isomorphism* $\Gamma\langle\tilde{\theta}\rangle \cong \Upsilon$.

Proof. Let Q be the subjacent relation for the linking twist Υ. For each $z \in Z$, there is $\rho_z \in Q$ with $r(\rho_z) = z$ and $V_z \in \Omega(Q)$ with $\rho_z \in V_z$ such that $r(V_z) \subset Z$ and $s(V_z) \subset X$. By the paracompactness of Z, we may assume there to be a countable family of open sets $\{V_i: i > 0\} \subset \Omega(Q)$ with $s(V_i) \subset X$ and $r(V_i) \subset Z$ for each $i > 0$, such that the collection

$$\mathcal{U} = \{r(V_i): i > 0\}$$

covers Z. We may further assume there to be a family of continuous sections

$$\sigma_i: V_i \to \Upsilon \quad \text{for } i > 0.$$

We define a Γ-cocycle θ relative to $\mathcal{U}$ using these sections as follows. For each $z \in r(V_i) \cap r(V_j)$, there are unique $\rho_i \in V_i$ and $\rho_j \in V_j$ such that

$$z = r(\rho_i) = r(\rho_j);$$

put

$$\theta_{ij}(z) = \sigma_i(\rho_i)^*\sigma_j(\rho_j).$$

This is to be the desired Γ-cocycle. Set $V_0 = X$, and note that there is a canonical section σ_0 defined on X (recall that X is identified with the unit space of $\Gamma \subset \Upsilon$). Put

$$V = \coprod_{i \geqq 0} V_i,$$

and define two local homeomorphisms

$$\psi: V \to X$$

$$\phi: V \to Z \coprod X,$$

by $\psi(\rho_i) = s(\rho_i)$ and $\phi(\rho_i) = r(\rho_i)$ for $\rho_i \in V_i \subset V$. We define a groupoid morphism

$$\iota: R(\phi) \to \Gamma^{\psi},$$

by

$$\iota(\rho_i, \rho_j) = (\rho_i, \sigma_i(\rho_i)^*\sigma_j(\rho_j), \rho_j),$$

for $\rho_i \in V_i$, $\rho_j \in V_j$, $r(\rho_i) = r(\rho_j)$. The linking twist associated to the Γ-cocycle θ is then given as the quotient of Γ^{ψ} by this embedding, that is,

$$\Gamma\langle\tilde{\theta}\rangle \cong \Gamma^{\psi}/\iota.$$

Consider the map $\Phi:\Gamma^{\psi} \to \Upsilon$ given by

$$\Phi(\rho_i, \gamma, \rho_j) = \sigma_i(\rho_i)\gamma\sigma_j(\rho_j)^* \quad \text{where } s(\rho_i) = r(\gamma),\ s(\rho_j) = s(\gamma).$$

The map Φ factors through $\Gamma\langle\widetilde{\theta}\rangle$ in the evident way (N.B. $\iota(R(\phi)) = \Phi^{-1}(Z \amalg X)$), and one obtains the isomorphism $\Gamma\langle\widetilde{\theta}\rangle \cong \Upsilon$. To complete the proof, it remains to show that if θ' is a Γ-cocycle on Z equivalent to θ, then there exist implementing sections as above. It suffices to consider two cases. First, if θ' is obtained from θ by restriction to some refining cover, then the implementing sections are obtained from those of θ by restriction. We may therefore assume that θ' is given relative to $\{r(V_i)\}$ and that there exist continuous functions

$$h_i:r(V_i) \to \Gamma$$

satisfying the condition (as in definition 12)

$$(\dagger) \quad h_i(z)\theta_{ij}(z) = \theta'_{ij}(z)h_j(z) \quad \text{for all } z \in r(V_i) \cap r(V_j),$$

where it is implicit that the compositions make sense. There exist open sets $U_i \in \Omega(Q)$ with $r(U_i) = r(V_i)$, and continuous sections $\sigma'_i:U_i \to \Upsilon$ such that for each $i > 0$, and $\rho'_i \in U_i$

$$\sigma'_i(\rho'_i) = \sigma_i(\rho_i)h_i^*(r(\rho_i)), \quad \text{where } r(\rho_i) = r(\rho'_i).$$

It follows from ($\dagger$) that if $\rho'_i \in U_i$, $\rho'_j \in U_j$ with $r(\rho'_i) = r(\rho'_j) = z$, then

$$\theta'_{ij}(z) = \sigma'_i(\rho'_i)^*\sigma'_j(\rho'_j).$$

14° *Remark*. Let θ be a Γ-cocycle on X (where X is the unit space of Γ) given relative to some cover $\mathcal{U} = \{U_i:i \in I\}$. Suppose that there are open sets $V_i \in \Omega(R)$ with $s(V_i) = U_i$ for each $i \in I$, such that

$$\theta_{ii}(s(\rho)) = r(\rho) \quad \text{for each } i \in I \text{ and } \rho \in V_i.$$

Then $\Gamma\langle\theta\rangle$ is a twist over R; moreover, $[\Gamma\langle\theta\rangle] - [\Gamma] \in \operatorname{Im}\zeta$ (cf. Section 4.4). Conversely, if Λ is a twist over R with $[\Lambda] - [\Gamma] \in \operatorname{Im}\zeta$, there is a Γ-cocycle θ as above such that $[\Lambda] = [\Gamma\langle\theta\rangle]$. Let $M(R)$ denote the quotient $\mathrm{Tw}(R)/\operatorname{Im}\zeta$; we view $M(R)$ as the group of Morita classes of twists over R.

6. Hyperfinite relations. In this section the group of twists is computed for a certain class of relations on compact spaces. A characterization of the associated diagonal pairs is presented together with a number of examples. An analogous discussion of relations on locally compact spaces is possible, but in the interests of brevity and smoothness of exposition, we defer the imperative towards greatest generality. In contrast to the situation considered in [**13**], where the second cohomology of hyperfinite Borel relations is shown to be trivial, the group of twists (read second cohomology) of the hyperfinite relations considered here need not vanish.

1° *Definition*. Let R be a relation on X; then R is said to be *finite* if it is compact (as a topological space).

N.B. If R is finite then X is necessarily compact and X/R is Hausdorff (see [**19**] Theorem 3.11).

2° PROPOSITION. *If R is a finite relation on X, then the quotient map $\psi:X \to X/R$ is a covering map and $R \cong R(\psi)$.*

Proof. That ψ is an open map is immediate (see 2.1.iv). We show that ψ is locally injective; that is, for each $x \in X$, we produce an open neighbourhood U of x such that $\psi|_U$ is injective (i.e., $U \in U(\psi)$, cf. Definition 5.6). Since the unit space X is open in R, and R is homeomorphic to its image under the embedding

$$(r, s):R \to X \times X,$$

there is an open set $V \subset X \times X$ such that

$$(r, s)^{-1}(V) = X.$$

Hence, given $x \in X$, there is an open neighbourhood U of x such that $U \times U \subset V$; it follows that $\psi|_U$ is injective. By compactness, X may be covered by a finite number of open sets in $U(\psi)$; therefore, each element of X/R has a finite preimage.

Claim. For any $z \in X/R$, there is an open neighbourhood W of z such that $\psi^{-1}(W)$ is homeomorphic to the disjoint union of a finite number of copies of W. Suppose that this is not the case for some $z \in X/R$. Let

$$\psi^{-1}(z) = \{x_1, \dots, x_n\},$$

and choose disjoint open sets U_i with $x_i \in U_i \in U(\psi)$, for each i. Set

$$U = \bigcap_i \psi(U_i),$$

and note that if V is an open set with $z \in V \subset U$, then

$$V \simeq \psi^{-1}(V) \cap U_i \quad \text{for each } i.$$

Thus, by supposition, $\psi^{-1}(V)$ is not contained in $\cup_i U_i$. There is, then, a sequence $\{y_j\} \subset X$ for which $y_j \notin \cup_i U_i$ and $\psi(y_j) \to z$. Let x_0 be a limit point of the sequence $\{y_j\}$; by continuity $\psi(x_0) = z$, but this is impossible as

$$\psi^{-1}(z) \subset \cup_i U_i.$$

The claim is hereby established. It is evident that $R \simeq R(\psi)$.

3° *Remarks*. Suppose that $Q \subset R$ are finite relations on X; then the quotient map of R factors through that of Q (cf. [**23**] Section 4). More precisely, if $\psi_1:X \to X/Q$ and $\psi_2:X \to X/R$ are the associated quotient maps, then there is a covering map $\phi: X/Q \to X/R$ such that $\psi_2 = \phi \circ \psi_1$. Note further that Q is (topologically) a compact-open subset of R. To avoid cumbersome notation, we shall often identify a relation R with its image in $X \times X$.

4° *Definition*. Let R be a relation on a compact space X. Then R is said to be *hyperfinite* if there is an increasing sequence of finite relations $R_n \subset R$, $n \geqq 1$, such that $R = \cup_n R_n$.

5° *Remarks*. With R as in the definition, set $X_0 = X$ and $X_n = X/R_n$. By the above remark, one obtains a sequence of covering maps

$$X_0 \underset{\phi_0}{\to} \ldots \to X_n \underset{\phi_n}{\to} X_{n+1} \to \ldots$$

Note that $\psi_n = \phi_{n-1} \circ \ldots \circ \phi_0$, where ψ_n is the quotient map associated to R_n. If $\psi: X \to Z$ is a covering map, write

$$\mathrm{Tw}(\psi) = \mathrm{Tw}(R(\psi)) \quad \text{and} \quad H^1(\psi) = H^1(R(\psi), \mathbf{T}).$$

Recall the six-term exact sequence associated to ψ (see [**23**]):

$$0 \to H^1(\psi) \underset{\eta}{\to} H^1(Z, \mathcal{S}) \underset{\psi^*}{\to} H^1(X, \mathcal{S}) \underset{\zeta}{\to} \mathrm{Tw}(\psi)$$

$$\underset{\delta}{\to} H^2(Z, \mathcal{S}) \underset{\psi^*}{\to} H^2(X, \mathcal{S}).$$

As the restriction of twists to subrelations (on the same unit space) defines a homomorphism between the associated groups of twists, it is of some interest to consider how the sequences associated to a pair of covering maps intertwine.

6° LEMMA. *Suppose one has a commuting triangle of covering maps*:

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & Y \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & Z & \end{array}$$

Then one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc}
0 & \to & H^1(\psi) & \underset{\eta}{\to} & H^1(Z, \mathcal{S}) & \underset{\psi^*}{\to} & H^1(X, \mathcal{S}) & \underset{\zeta}{\to} & \mathrm{Tw}(\psi) & \underset{\delta}{\to} & H^2(Z, \mathcal{S}) & \underset{\psi^*}{\to} & H^2(X, \mathcal{S}) \\
 & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\phi^*} & & \| & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\phi^*} & & \| \\
0 & \to & H^1(\sigma) & \underset{\eta}{\to} & H^1(Y, \mathcal{S}) & \underset{\sigma^*}{\to} & H^1(X, \mathcal{S}) & \underset{\zeta}{\to} & \mathrm{Tw}(\sigma) & \underset{\delta}{\to} & H^2(Y, \mathcal{S}) & \underset{\sigma^*}{\to} & H^2(X, \mathcal{S})
\end{array}$$

where ν is the map given by restriction of twists and λ is given by restriction of cocycles.

Proof. That the rows are exact follows from **[23]**; that δ intertwines ν and ϕ^* follows from Proposition 4.1 (ibid.). That η intertwines λ and ϕ^* results from the fact that a cocycle on $R(\sigma)$ extends if and only if its class maps into Im ϕ^*.

We shall require some standard facts relating to the inverse limit functor for abelian groups (cf **[16]** Section 2, **[17]** Section 1). By inverse system of groups, we shall mean a sequence indexed by the positive integers

$$G_0 \underset{f_0}{\leftarrow} G \underset{f_1}{\leftarrow} G \underset{f_2}{\leftarrow} \dots .$$

7° *Facts.* The functor $\varprojlim$ defined on inverse systems of abelian groups is left exact. The right derived functor, denoted $\varprojlim^1$ has an explicit description. Let $\{ (G_n, f_n): n \geqq 0\}$ be an inverse system of abelian groups; put

$$G = \prod_n G_n$$

and define an endomorphism f of G as follows

$$f(g_0, g_1, g_2, \dots) = (g_0 - f_0 g_1, g_1 - f_1 g_2, \dots).$$

Then

$$\varprojlim G_n \simeq \ker f,$$

while

$$\varprojlim{}^1 G_n \simeq \operatorname{coker} f.$$

If each f_n is surjective, then so is f, whence

$$\varprojlim{}^1 G_n = 0.$$

If, in addition, there exist morphisms, $s_n: G_n \to G_{n+1}$, such that $f_n \circ s_n = \mathrm{id}_{G_n}$, then one has:

$$\varprojlim G_n = G_0 \oplus \prod_n \ker f_n.$$

To say that $\varprojlim^1$ is the right derived functor of $\varprojlim$ means that given an inverse system of short exact sequences:

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & H_n & \to & G_n & \to & K_n & \to 0, & n \geqq 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

one obtains a six-term exact sequence involving the two functors:

$$0 \to \varprojlim H_n \to \varprojlim G_n \to \varprojlim K_n \to \varprojlim{}^1 H_n$$

$$\to \varprojlim{}^1 G_n \to \varprojlim{}^1 K_n \to 0.$$

Finally, if $G_n = G$, for all n, the sequence simplifies:

$$0 \to \varprojlim H_n \to G \to \varprojlim K_n \to \varprojlim{}^1 H_n \to 0.$$

8° COROLLARY. *With notation as in the remarks above, consider the inverse system of restriction maps*:

$$\mathrm{Tw}(\psi_1) \underset{\nu_1}{\leftarrow} \mathrm{Tw}(\psi_2) \underset{\nu_2}{\leftarrow} \mathrm{Tw}(\psi_3) \leftarrow \dots .$$

One obtains the following short exact sequence:

$$0 \to \varprojlim \mathrm{Im}(\zeta_n) \to \varprojlim \mathrm{Tw}(\psi_n) \to \varprojlim \ker(\psi_n^*) \to 0,$$

where

$$\mathrm{Im}(\zeta_n) \simeq \mathrm{coker}(\psi_n^*) = H^1(X, \mathcal{S})/\mathrm{Im}(\psi_n^*), \text{ and}$$

$$\ker(\psi_n^*) \subset H^2(X_n, \mathcal{S}).$$

Proof. It follows from the above lemma that

$$\{0 \to \mathrm{Im}(\zeta_n) \to \mathrm{Tw}(\psi_n) \to \ker(\psi_n^*) \to 0 : n \geqq 0\}$$

is an inverse system of short exact sequences. To complete the proof it suffices to show that the maps

$$\mathrm{Im}(\zeta_{n+1}) \to \mathrm{Im}(\zeta_n)$$

are surjective; but this follows from the fact that

$$\mathrm{Im}(\psi_{n+1}^*) \subset \mathrm{Im}(\psi_n^*).$$

Thus,

$$\varprojlim{}^1 \mathrm{Im}(\zeta_n) = 0.$$

Our object is to determine the group of twists over a hyperfinite relation in terms of the groups associated to the approximating finite relations.

9° LEMMA. *There is a surjective homomorphism*

$$\mu : \mathrm{Tw}(R) \to \varprojlim \mathrm{Tw}(\psi_n)$$

which maps the class of a twist to the classes of its restrictions.

Proof. Let

$$\mu_n : \mathrm{Tw}(R) \to \mathrm{Tw}(\psi_n)$$

denote the map given by restriction. Since the order in which restrictions are taken is irrelevant, one has:

$$\mu_n = \nu_n \circ \mu_{n+1}.$$

Hence, there is a homomorphism

$$\mu : \mathrm{Tw}(R) \to \varprojlim \mathrm{Tw}(\psi_n)$$

through which each μ_n factors. The surjectivity of μ remains to be verified. Let Γ_n be a twist over $R(\psi_n)$ for each n, so that

$$\nu_n[\Gamma_{n+1}] = [\Gamma_n].$$

Choose twist morphisms $\alpha_n : \Gamma_n \to \Gamma_{n+1}$ which identify Γ_n with the restriction of Γ_{n+1} to $R(\psi_n)$. Let Γ denote the quotient of the disjoint union $\coprod_n \Gamma_n$ by the smallest equivalence relation which respects the above identifications. Then Γ is a twist over R; further, $\mu_n[\Gamma] = [\Gamma_n]$, by construction.

The kernel of μ consists of twists for which each restriction is trivial. Such twists may be obtained by pasting trivial twists together in non-trivial ways. Recall (Section 3.3) that unit space preserving automorphisms of a twist are in bijective correspondence with circle-valued 1-cocycles on the subjacent relation. Write

$$Z^1(\psi) = Z^1(R(\psi), \mathbf{T})$$

if ψ is a local homeomorphism. Suppose $R(\psi_1) \subset R(\psi_2)$ and let $c \in Z^1(\psi_1)$; define an embedding of trivial twists

$$\tilde{c} : R(\psi_1) \times \mathbf{T} \to R(\psi_2) \times \mathbf{T}$$

by

$$\tilde{c}(\rho, t) = (\rho, c(\rho)t).$$

Evidently, any embedding which respects the inclusion of relations must be of this form.

10° *Fact.* If R is a hyperfinite relation with covering maps $\{\psi_n\}$, there is a homomorphism

$$\xi : \prod_n Z^1(\psi_n) \to \mathrm{Tw}(R)$$

such that $\mathrm{Im}\ \xi = \ker \mu$.

Proof. Given $c_n \in Z^1(\psi_n)$, $n > 0$, consider the sequence of twist embeddings:

$$R(\psi_1) \times \mathbf{T} \underset{\tilde{c}_1}{\rightarrow} R(\psi_2) \times \mathbf{T} \underset{\tilde{c}_2}{\rightarrow} R(\psi_3) \times \mathbf{T} \underset{\tilde{c}_3}{\rightarrow} \dots,$$

and form the inductive limit twist as above (by taking the appropriate quotient of the disjoint union). The twist so obtained is denoted $\Gamma(c_n)$; set

$$\xi(c_n) = [\Gamma(c_n)].$$

It is evident that a twist is isomorphic to one of this form if and only if its restrictions are all trivial, that is, $\operatorname{Im} \xi = \ker \mu$.

It remains to determine when two such twists are isomorphic. This will enable us to express $\operatorname{Tw}(R)$ as an extension of certain related groups.

11° THEOREM. *Let R be a hyperfinite relation with covering maps ψ_n. There is a short exact sequence*:

$$0 \rightarrow \underset{\leftarrow}{\lim}{}^1\, H^1(\psi_n) \underset{\xi'}{\rightarrow} \operatorname{Tw}(R) \underset{\mu}{\rightarrow} \underset{\leftarrow}{\lim} \operatorname{Tw}(\psi_n) \rightarrow 0.$$

Proof. Two sequences (c_n), $(d_n) \in \Pi\, Z^1(\psi_n)$ give rise to isomorphic twists if and only if there is a third sequence (e_n) which makes the following diagram commute:

$$\begin{array}{ccccccc}
R(\psi_1) \times \mathbf{T} & \overset{\tilde{c}_1}{\text{———}} & R(\psi_2) \times \mathbf{T} & \overset{\tilde{c}_2}{\text{———}} & R(\psi_3) \times \mathbf{T} & \text{———} & \dots \\
\tilde{e}_1 \Big| & & \tilde{e}_2 \Big| & & \tilde{e}_3 \Big| & & \\
R(\psi_1) \times \mathbf{T} & \underset{\tilde{d}_1}{\text{———}} & R(\psi_2) \times \mathbf{T} & \underset{\tilde{d}_2}{\text{———}} & R(\psi_3) \times \mathbf{T} & \text{———} & \dots
\end{array}$$

Thus, $\Gamma(c_n)$ is isomorphic to the trivial twist if and only if there are $e_n \in Z^1(\psi_n)$ such that

$$c_n(\rho)e_{n+1}(\rho) = e_n(\rho) \quad \text{for all } n > 0,\ \rho \in R(\psi_n).$$

As the existence of such depends solely on the class of the cocycles involved, the twist $\Gamma(c_n)$ is trivial if and only if there are $h_n \in H^1(\psi_n)$ such that

$$[c_n] = h_n - \lambda_n(h_{n+1}) \quad \text{for all } n > 0.$$

It follows from the explicit description of $\underset{\leftarrow}{\lim}{}^1$ given in fact 7 that

$$\ker \mu \simeq \underset{\leftarrow}{\lim}{}^1\, H^1(\psi_n).$$

12° *Remarks.* An analogous result holds for $H^1(R, \mathbf{T})$. Let $U(X)$ denote the unitary group of $C(X)$ (i.e., $U(X)$ is the group of all circle-valued continuous functions on X). The sequence:

$$0 \to \varprojlim{}^1 U(X_n) \to H^1(R, \mathbf{T}) \underset{\pi}{\to} \varprojlim H^1(\psi_n) \to 0$$

is exact. This follows by formal manipulation (see fact 7) from the fact that ker π is isomorphic to the cokernel of the map

$$U(X) \to \varprojlim U(X)/U(X_n).$$

If

$$\ker H^1(\psi_n, \mathcal{S}) = \ker H^2(\psi_n, \mathcal{S}) = 0,$$

that is, ψ_n^* is injective on $H^1(\cdot, \mathcal{S})$ and $H^2(\cdot, \mathcal{S})$ for all n, then

$$\mathrm{Tw}(R) \simeq \varprojlim \operatorname{coker} \psi_n^*.$$

Moreover, by fact 7,

$$M(R) \simeq \varprojlim{}^1 H^1(X_n, \mathcal{S}) \quad \text{(see 5.14)}.$$

We characterize those diagonals which arise from hyperfinite relations. Note that if Λ is a twist over $R(\psi)$ with $\psi{:}X \to Z$ a covering map and X compact, then $A(\Lambda)$ is a unital continuous trace algebra with diagonal $C(X)$. Thus if R is a hyperfinite relation on a space X and Γ is a twist over R, there is an ascending sequence of unital continuous trace algebras $B_n \subset A(\Gamma)$ with $B_0 \simeq C(X)$, whose union is dense. Such a diagonal will be said to be *hyperfinite* (that is, a diagonal B_0 in a C^*-algebra A with unit is hyperfinite if there is such an approximating sequence of continuous trace algebras).

13° PROPOSITION. *Let B_0 be a hyperfinite diagonal in a unital C^*-algebra A with approximating continuous trace subalgebras B_n. Then the subjacent relation of the associated twist is hyperfinite, where the covering maps are given as the spectral maps of the pairs (B_n, B_0) (see* [**23**] *Theorem* 2.2).

Proof. Since B_0 has the extension property relative to A, it has the extension property relative to B_n for each n; hence, B_0 is diagonal in B_n (each n) by Theorem 2.2 of [**23**]. Let $\psi_n{:}\hat{B}_0 \to \hat{B}_n$ denote the associated spectral maps; by Section 4 of [**23**] we may conclude that ψ_{n+1} factors through ψ_n for each n. The subjacent relation

$$R = \bigcup_n R(\psi_n)$$

is consequently hyperfinite.

14° *Remarks*. With the situation as in the above proposition, let $C_n = B_n'$ be the relative commutant of B_n in A. Write $\hat{B}_n = X_n$, and note that C_n ($\simeq C(X_n)$) is the center of B_n ($B_0 \subset B_n \Rightarrow B'_n \subset B'_0 = B_0 \subset B_n$). We show below that $B_n = C'_n = B''_n$. Observe that a C^*-algebra containing a hyperfinite diagonal is ultraliminary (as defined in Section 4 of [**23**]).

15° PROPOSITION. *With notation as above, there is a (unique) conditional expectation* $P_n:A \to B_n$, *for each* $n \geqq 0$, *such that*:

i° $P_n = P_n \circ P_m$, *if* $m > n$

ii° $a = \lim_n P_n(a)$, *for all* $a \in A$

iii° *For each* $a \in A$, $\epsilon > 0$, $n \geqq 0$, *there are* $h_1, h_2, \ldots, h_k \in C_n$ *such that*

$$\|P_n(a) - \Sigma h_j^* a h_j\| < \epsilon.$$

Furthermore, $B_n = B''_n$.

Proof. Let Γ be the twist on X_0 associated to the pair (A, B_0) with subjacent relation

$$R = \bigcup_n R(\psi_n),$$

where $\psi_n:X_0 \to X_n$, $n \geqq 0$, are the associated covering maps. We define P_n on the dense subalgebra $E(\Gamma)$ by

$$P_n(f)(\rho) = \begin{cases} f(\rho) & \text{if } \rho \in R(\psi_n) \\ 0 & \text{otherwise} \end{cases}$$

where $f \in E(\Gamma)$ is viewed as a compactly supported continuous section of the associated complex line bundle (thus, $P_n(f)$ is simply the restriction of the section f to the compact open subset $R(\psi_n)$).

Claim. $\|P_n(f)\| \leqq \|f\|$, for all $f \in E(\Gamma)$.

Given $f \in E(\Gamma)$, there is $m \geqq 0$ such that supp $f \subset R(\psi_m)$ (by compactness). If $m \leqq n$, then $P_n(f) = f$, and there is nothing to prove. If $m > n$, let $\phi:X_n \to X_m$ be the unique map for which $\psi_m = \phi \circ \psi_n$. Choose a finite subcover $\{U_1, \ldots, U_k\} \subset U(\phi)$ (of X_n) and $h_j \in C_n \simeq C(X_n)$ with $h_j \geqq 0$, supp $h_j \subset U_j$, for each j, so that $\Sigma h_j^2 = 1$. One obtains

$$(\sigma)\ P_n(f) = \Sigma h_j f h_j$$

(if $g \subset E(\Gamma)$ with supp $g \subset R(\psi_m)\backslash R(\psi_n)$, then $h_j g h_j = 0$ for each j). By fact (¶), the claim is established and P_n extends uniquely to a conditional expectation. Conditions i°, ii° are trivial to verify. We verify iii°; given $a \in A$, $\epsilon > 0$, $n \geqq 0$, choose $f \in E(\Gamma)$ such that

$$\|f - a\| < \epsilon/2.$$

Now, supp $f \subset R(\psi_m)$ for some $m > n$. Choose $h_j \in C_n$ as above so that (σ) holds. It follows that

$$\|P_n(a) - \Sigma\, h_j a h_j\| \leqq \|P_n(a) - P_n(f)\| + \|\Sigma\, h_j(f - a)h_j\| < \epsilon.$$

Since C_n is the center of B_n, one has $B_n \subset C_n' = B_n''$. Given $a \in B_n''$, we wish to show that $a = P_n(a)$, and we will be done as $P_n(a) \in B_n$. But $P_n(a)$ may be approximated by sums of the form $\Sigma\, h_j a h_j$ with $h_j \in C_n$. Therefore, $P_n(a) = a$, as desired (since $\Sigma\, h_j a h_j = a(\Sigma\, h_j^2) = a$).

16° *Remarks*. The foregoing treatment of hyperfinite diagonals owes much to the diagonalization of AF algebras appearing in [**42**] (see Section 1.1). Briefly, given an ascending sequence of finite-dimensional subalgebras A_n which generate an AF algebra A (with $1 \in A_n \subset A$), one constructs an increasing sequence of masas $B_n \subset A_n$ inductively. Given B_n, let B_{n+1} be generated by B_n and a masa in the relative commutant of A_n in A_{n+1}. Let B be the closure of $\cup_n B_n$; that B is a hyperfinite diagonal is immediate (take the approximating continuous trace subalgebras to be those generated by B and A_n).

The associated relation is treated in detail in [**34**] (see Chapter III), where it is shown to be a complete isomorphism invariant of the algebra. Note that the group of twists is trivial as the spectrum of the diagonal is totally disconnected (ibid. III.1.3). A finite relation on a totally disconnected space is necessarily an elementary groupoid in Renault's sense (ibid. p. 123).

17° *Example*. We consider an example arising from Blackadar's twisted double embedding for which the relation in question may be viewed as a skew product of relations (cf [**5**]). Let

$$E = \{e:\mathbf{Z}^+ \to \mathbf{Z}_2\} \simeq \prod^{\infty} \mathbf{Z}_2$$

be the Cantor set and $\phi:E \to E$ be the continuous map defined by

$$(\phi(e))_n = e_{n+1} \quad \text{for each } n.$$

Let F be the hyperfinite relation on E generated by ϕ and its iterates (note that ϕ is a covering map), that is,

$$F = \bigcup_n R(\phi^n);$$

note that the associated C^*-algebra $A(F_\circ)$ (where $F_\circ$ denotes the trivial twist over F) is the Fermion algebra, that is, UHF of type 2^∞.

Now, let σ be a homeomorphism on a compact space Z and consider the two-fold covering map $\phi_\sigma:E \times Z \to E \times Z$ defined by

$$\phi_\sigma(e, z) = \begin{cases} (\phi(e), z) & \text{if } e_0 = 0 \\ (\phi(e), \sigma(z)) & \text{if } e_0 = 1. \end{cases}$$

Let F_σ denote the hyperfinite relation on $E \times Z$ generated by ϕ_σ and its iterates, that is,

$$F_\sigma = \bigcup_n R(\phi_\sigma^n)$$

(observe that F_σ may be viewed as a skew product of F and the trivial relation Z with respect to an appropriate cocycle; see [**5**] for details). Since ϕ_σ admits a continuous section, one has that ϕ_σ^* is injective on all cohomology groups (the same holds for $(\phi_\sigma^n)^*$). Hence,

$$\mathrm{Tw}(\phi_\sigma^n) = \mathrm{Im}\ \zeta_n \simeq \mathrm{coker}(\phi_\sigma^n)^* = H^1(E \times Z, \mathscr{S})/\mathrm{Im}(\phi_\sigma^n)^*;$$

furthermore

$$\mathrm{Tw}(F_\sigma) \simeq \lim_{\leftarrow} \mathrm{Im}\ \zeta_n.$$

The existence of a continuous section for ϕ_σ entails the existence of sections for the bonding maps in the inverse system $\{\mathrm{Im}\ \zeta_n\}$. By fact 7 one may infer (note that the bonding maps are surjective):

$$\mathrm{Tw}(F_\sigma) \simeq \prod^{\infty} H^1(E \times Z, \mathscr{S}).$$

N.B. $H^1(E \times Z, \mathscr{S}) \simeq H^\circ(E, H^1(Z, \mathscr{S}))$.

18° *Example*. Fix $k > 0$, and consider the following inverse system of monomorphisms (of abelian groups):

$$\mathbf{Z}^k \underset{\lambda_0}{\leftarrow} \mathbf{Z}^k \underset{\lambda_1}{\leftarrow} \mathbf{Z}^k \underset{\lambda_2}{\leftarrow} \dots$$

Let $\mu_n:\mathbf{Z}^k \to \mathbf{Z}^k$ denote the composite $\lambda_0 \circ \dots \circ \lambda_{n-1}$; note that μ_n is a monomorphism as well. By dualizing, one obtains a direct system of compact groups:

$$\mathbf{T}^k \underset{\hat{\lambda_0}}{\rightarrow} \mathbf{T}^k \underset{\hat{\lambda_1}}{\rightarrow} \mathbf{T}^k \underset{\hat{\lambda_2}}{\rightarrow} \dots,$$

where each map is surjective. As the kernel of each map is finite, they are, in fact, covering maps when viewed as maps of topological spaces. We wish to compute the group of twists for the associated hyperfinite relation

$$R = \bigcup_n R(\hat{\mu_n}) \quad \text{where } \hat{\mu_n} = \hat{\lambda}_{n-1} \circ \dots \circ \hat{\lambda_0}.$$

N.B. Let $G_n = \ker \hat{\mu_n}$, and $G = \cup_n G_n$; then G is a torsional subgroup of $\mathbf{T}^k$, as G_n is finite for each n. Let $i:G \to \mathbf{T}^k$ be the inclusion map and note that

$$R \simeq R_i = G \times \mathbf{T}^k \quad \text{(cf. Section 3.5).}$$

The long exact sequence of sheaf theory yields natural isomorphisms

$$H^n(\cdot, \mathscr{S}) \simeq H^{n+1}(\cdot, \mathbf{Z}) \quad \text{for } n > 0,$$

which we shall, for computational reasons, avail ourselves of implicitly.

It is known that the cohomology ring of $\mathbf{T}^k$ may be identified (in a way that preserves grading) with the exterior algebra on $\mathbf{Z}^k$, that is

$$H^*(\mathbf{T}^k, \mathbf{Z}) \simeq \Lambda\mathbf{Z}^k.$$

Viewing $\mathbf{Z}^k$ as the dual group, the identification $\mathbf{Z}^k \simeq H^1(\mathbf{T}^k, \mathbf{Z})$ arises by identifying a character with its homotopy class (as a continuous function from $\mathbf{T}^k$ to $\mathbf{T}$). Under this identification one has for each n

$$(\hat{\lambda}_n)^* = \lambda_n \quad (\text{on } H^1(\mathbf{T}^k, \mathbf{Z})\,);$$

moreover, by the functoriality of the cup-product (cf. **[14]** Proposition 24.4)

$$(\hat{\lambda}_n)^*: H^*(\mathbf{T}^k, \mathbf{Z}) \to H^*(\mathbf{T}^k, \mathbf{Z})$$

is a graded ring homomorphism. Hence, it is prescribed by its values on the first cohomology group. Viewing the cohomology ring as the exterior algebra on $\mathbf{Z}^k$, the above ring homomorphism may be written

$$\wedge\lambda_n: \Lambda\mathbf{Z}^k \to \Lambda\mathbf{Z}^k,$$

where

$$\wedge\lambda_n(e_1 \wedge \ldots \wedge e_j) = \lambda_n(e_1) \wedge \ldots \wedge \lambda_n(e_j).$$

It follows that $\wedge\lambda_n$ and, hence, $\wedge\mu_n$ is injective for each n. Hence

$$\mathrm{Tw}(R) \simeq \varprojlim \operatorname{coker}(\mu_n \wedge \mu_n), \quad \text{and}$$

$$M(R) \simeq \varprojlim{}^1(\mathbf{Z}^k \wedge \mathbf{Z}^k, \lambda_n \wedge \lambda_n),$$

by Remark 12 above.

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