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# Corrigenda

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'On the mobility of bodies in $\mathbb{R}^n$'

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The proof of the following theorem which I gave in the above paper is incorrect. A correct proof is given here.

THEOREM 2·1. *Let $\mathscr{B}$ be an intersection regular, finite collection of bodies, and $\mathscr{F}$ a manoeuvre on $\mathscr{B}$. $\mathscr{F}$ is strongly separating if and only if every component of $S(\mathscr{B},\mathscr{F})$ is positive; and, if any $S_{(i,j,p)}(\mathscr{B},\mathscr{F})$ is negative, $\mathscr{F}$ is colliding.*

*Proof*. We will first prove this for the case where $\mathscr{B} = \{B_0, B_1\}$; applying this to every pair of bodies gives the general case. Suppose that, for some $p$, $S_{(0,1,p)}(\mathscr{B},\mathscr{F})$ is negative. Without loss of generality, suppose $B_0$ to be regular at $p$. Then

$$(\mathbf{d}f_1(p) - \mathbf{d}f_0(p)) \cdot \mathbf{n}_0(p) < 0,$$

and so, for small enough positive $t$, $f_1^t(p) \in \operatorname{int} f_0^t(B_0)$. Hence, some interior points of $f_1^t(B_1)$ are also interior to $f_0^t(B_0)$, and $\mathscr{F}$ is colliding.

Suppose that, for some $p$, $S_{(0,1,p)}(\mathscr{B},\mathscr{F}) = 0$. Then, by Lemma 1·2,

$$0 = \lim_{t\downarrow 0}\frac{1}{t}D(f_1^t(p), f_0^t(B_0))$$

$$\geqslant \lim_{t\downarrow 0}\frac{1}{t}D(f_1^t(B_1), f_0^t(B_0));$$

and, by definition, $\mathscr{F}$ is not strongly separating.

To complete the proof, it suffices to show that, if $S_{(0,1,p)}(\mathscr{B},\mathscr{F})$ is positive, there exists a neighbourhood $U_p$ of $p$ and positive constants $a_p$, $T_p$ such that

$$x \in B_1 \cap U_p,\ 0 < t < T_p \Rightarrow D(f_1^t(x), f_0^t(B_0)) > a_p t. \tag{1}$$

As $B_1$ is compact, a finite subset of the neighbourhoods $\{U_p\}$ cover it, and, taking $a$ and $T$ to be the (positive) minima of the corresponding $\{a_p\}$ and $\{T_p\}$:

$$0 < t < T \Rightarrow D(f_1^t(B_1), f_0^t(B_0)) > at,$$

which implies the desired result that $\mathscr{F}$ is strongly separating.

It remains only to prove that, for suitable $U_p$, $a_p$, and $T_p$, (1) holds. As we wish to use only the compactness of one of the bodies, say $B_1$, we must consider two cases.

*Case* I. *$B_0$ is regular at $p$*

We will use coordinates as in the proof of Lemma 1·2, and represent the boundary of $B_0$ as the graph of $x^0 = g(\tilde{\mathbf{x}})$ for some suitable differentiable function $g: \mathbb{R}^{n-1} \to \mathbb{R}$. This coordinate system has been chosen so that $g(0) = 0$, $\nabla g(0) = 0$, and so that $B_0$ is immobile with $\mathbf{p}$ at the origin. $B_1$ is acted upon by a motion $f$; we may factor this as $f = \tau \circ \rho$, where $\rho$ fixes $\mathbf{p}$ and $\tau^t(\mathbf{x}) = \mathbf{x} + f^t(\mathbf{p})$. $f$ and $\tau$ are differentiable, so $\rho$ must also be differentiable; hence,

$$\|\mathbf{d}\rho\| = \max_{\|\mathbf{x}\|=1} \lim_{t\to 0} \frac{\|\rho^t(\mathbf{x}) - \mathbf{x}\|}{t}$$

is finite. Because $\rho^t$ is a linear transformation,

$$\|\mathbf{d}\rho\| = \max_{\mathbf{x}\neq 0} \lim_{t\to 0} \frac{\|\rho^t(\mathbf{x}) - \mathbf{x}\|}{t\,\|\mathbf{x}\|};$$

so there exists $T > 0$ such that for $0 < t < T$, $\mathbf{x} \neq 0$:

$$\|\mathbf{x}\|\,\|\mathbf{d}\rho\| > \frac{\|\rho^t(\mathbf{x}) - \mathbf{x}\|}{2t}.$$

For any $\epsilon > 0$, let $U = \{\mathbf{x}\colon 2\|\mathbf{x}\|\,\|\mathbf{d}\rho\| < \epsilon\}$. Then, for $x \in U$, $0 < t < T$,

$$\|f^t(\mathbf{x}) - \tau^t(\mathbf{x})\| = \|\rho^t(\mathbf{x}) - \mathbf{x}\| < \epsilon t;$$

it is thus enough to prove (1) in the case where $B_0$ is immobile and $B_1$ is translated.

For each $\mathbf{x}$, select $\mathbf{q}(\mathbf{x})$, a point in bdy $(B_0)$ at minimum distance from $\mathbf{x}$. (This cannot in general be chosen continuously.) As bdy $(B_0)$ contains the origin,

$$\|\mathbf{q}(\mathbf{x})\| \leqslant 2\|\mathbf{x}\|. \tag{2}$$

For any $\epsilon > 0$, we can select a neighbourhood $U$ of $\mathbf{p}$ and a positive constant $T$ such that

$$\mathbf{x} \in U,\ 0 < t < T \Rightarrow \max\{\|\mathbf{x}\|, \|\tau^t(\mathbf{x})\|\} < \epsilon. \tag{3}$$

As $g$ is continuously differentiable in a neighbourhood of the origin, for any $\epsilon > 0$ there exists a disc $D$ about the origin in $\mathbb{R}^{n-1}$ such that $\|\nabla g\| < \epsilon$ in $D$. Using (2) and (3), we can find a neighbourhood $U$ and positive $T$ such that, for $\mathbf{x} \in U$, $0 < t < T$, both $\tilde{\mathbf{q}}(\mathbf{x})$ and $\tilde{\mathbf{q}}(\tau^t(\mathbf{x}))$ are in $D$. As $D$ is convex,

$$\begin{aligned}\epsilon\|\tilde{\mathbf{q}}(\mathbf{x}) - \tilde{\mathbf{q}}(\tau^t(\mathbf{x}))\| &\geqslant \|g(\tilde{\mathbf{q}}(\mathbf{x})) - g(\tilde{\mathbf{q}}(\tau^t(\mathbf{x})))\| \\ &= |q^0(\mathbf{x}) - q^0(\tau^t(\mathbf{x}))|.\end{aligned} \tag{4}$$

Now, as $\mathbf{q}(\mathbf{x})$ is selected to be at minimum distance from $\mathbf{x}$, if we are close enough to the origin that bdy $(B_0)$ is smooth, $\mathbf{q}(\mathbf{x})$ must be the foot of a perpendicular from $\mathbf{x}$ to bdy $(B_0)$. Thus, $\tilde{\mathbf{q}}(\mathbf{x}) - \tilde{\mathbf{x}} = (x^0 - q^0(\mathbf{x}))\,\nabla g(\tilde{\mathbf{q}}(\mathbf{x}))$; and, using (2) and the continuity of $g$, we can find, for any $\epsilon > 0$, some $\delta > 0$ such that

$$\begin{aligned}\|x\| < \delta &\Rightarrow \|\nabla g(q^0(\mathbf{x}))\| < \epsilon \\ &\Rightarrow \|\tilde{\mathbf{q}}(\mathbf{x}) - \tilde{\mathbf{x}}\| < \epsilon|x^0 - q^0(\mathbf{x})| \leqslant \epsilon\|\mathbf{x}\|.\end{aligned} \tag{5}$$

From the fact that $\tau$ is differentiable and a translation, we know that for all $a > 0$ there exists $T > 0$ such that

$$\begin{aligned}0 < t < T &\Rightarrow \|\mathbf{x} - \tau^t(\mathbf{x})\| = \|\mathbf{p} - \tau^t(\mathbf{p})\| < at \\ &\Rightarrow \|\tilde{\mathbf{x}} - (\tau^t(\mathbf{x}))\| < at.\end{aligned}$$

Combining this with (5), we can find positive $a$, $T$, and a neighbourhood $U$ of the origin such that

$$0 < t < T,\ \mathbf{x} \in U \Rightarrow \|\tilde{\mathbf{q}}(\mathbf{x}) - \tilde{\mathbf{q}}(\tau^t(\mathbf{x}))\| < at;$$

substituting this into (4), for $\epsilon > 0$ we can find $T > 0$ and $U$ such that

$$0 < t < T,\ \mathbf{x} \in U \Rightarrow |q^0(x) - q^0(\tau^t(\mathbf{x}))| < \epsilon t. \tag{6}$$

By hypothesis, if $\mathbf{x} \in B_1$, $\mathbf{x} \notin \operatorname{int} B_0$. Hence, within a region where bdy $(B_0)$ is a graph, $q^0(\mathbf{x}) \leqslant x^0$, and so

$$\begin{aligned}(\tau^t(\mathbf{x}))^0 - q^0(\mathbf{x}) &\geqslant (\tau^t(\mathbf{x}))^0 - x^0 \\ &= (\tau^t(\mathbf{p}))^0 \quad (\text{as } \tau^t \text{ is a translation}).\end{aligned}$$

As $S_{(0,1,p)}(\mathscr{B}, \mathscr{F}) > 0$, by Lemma 1·2 we can find positive $a, T$ such that

$$0 < t < T \Rightarrow (\tau^t(\mathbf{p})) > at.$$

Thus, for small enough $\mathbf{x}$ in $B_1$,

$$(\tau^t(\mathbf{x}))^0 - q^0(\mathbf{x}) > at.$$

Combining this with (6), we find that there exist positive constants $a, T$, and a neighbourhood $U$ of $\mathbf{p}$, such that $0 < t < T$, $\mathbf{x} \in B_i \cap U \Rightarrow (\tau^t(\mathbf{x}))^0 - q^0(\tau^t(\mathbf{x})) > at$. As $D(\tau^t(\mathbf{x}), B_0) \geqslant (\tau^t(\mathbf{x}))^0 - q^0(\tau^t(\mathbf{x}))$, we are done.

*Case* II. *$B_1$ is regular at $p$*

By the proof of Case I, there exists a neighbourhood $U_0$ of $p$, and constants $a, T_0$ such that

$$0 < t < T_0,\ \mathbf{x} \in U_0 \cap B_0 \Rightarrow D(\mathbf{x}, f^t(B_1)) > at.$$

We can find $U_1 \subset U_0$, and positive $T_1$, such that

$$0 < t < T_1,\ \mathbf{x} \in U_1 \Rightarrow \mathbf{q}(f^t(\mathbf{x})) \in U_0.$$

Therefore, taking $T = \min\{T_0, T_1\}$,

$$\begin{aligned}0 < t < T,\ \mathbf{x} \in U_1 \cap B_1 \Rightarrow D(f^t(\mathbf{x}), B_0) &\geqslant D(\mathbf{q}(f^t(\mathbf{x})), B_1) \\ &> at.\end{aligned}$$

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