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Homogeneous Suslinian Continua

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Abstract. A continuum is said to be Suslinian if it does not contain uncountably many mutually exclusive non-degenerate subcontinua. Fitzpatrick and Lelek have shown that a metric Suslinian continuum X has the property that the set of points at which X is connected im kleinen is dense in X. We extend their result to Hausdorff Suslinian continua and obtain a number of corollaries. In particular, we prove that a homogeneous, non-degenerate, Suslinian continuum is a simple closed curve and that each separable, non-degenerate, homogenous, Suslinian continuum is metrizable.

1 Introduction

Suslinian continua were introduced by A. Lelek in [6]. Fitzpatrick and Lelek [4] have shown that a metric Suslinian continuum is connected im kleinen at each point of a dense set. If it is also atriodic, then the set of points at which it is locally connected is dense. They also construct a non-degenerate rational, therefore, Suslinian dendroid in the plane such that each nonempty, connected, open subset is dense and, hence, the continuum is nowhere locally connected.

In [3], the authors have shown that Suslinian continua are perfectly normal and rim-metrizable, and that locally connected Suslinian continua have weight at most ω_1 and under appropriate set-theoretic conditions are metrizable. In [1], the latter result was improved by showing that under the Suslin Hypothesis, each Suslinian continuum is metrizable.

Herein, we continue our study of Suslinian continua by demonstrating the first result above of Fitzpatrick and Lelek for general continua: the set of points at which a Suslinian continuum is connected im kleinen is dense. As a corollary, we prove that non-degenerate, homogeneous, Suslinian continua are simple closed curves. It follows that each separable, homogeneous, Suslinian continuum is metrizable.

2 Preliminaries

A *compactum* is a compact Hausdorff space. A *continuum* is a connected compactum. A continuum is said to be *Suslinian* if it does not contain uncountably many mutually exclusive non-degenerate subcontinua. An *arc* is a continuum that admits a linear ordering such that the order topology coincides with the given topology. It is well known and easy to prove that each ordered compactum is contained in an ordered continuum. A *Suslin line* is a linearly ordered continuum that is not separable and

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in which each family of pairwise disjoint open subsets is countable. A Hausdorff space X is said to be an *IOK* if it is the continuous image of some compact ordered space K. If K is also connected, then X is said to be *IOC*. A *simple closed curve* is a non-degenerate continuum which is disconnected by each pair of its distinct points.

A topological space X is said to be *homogeneous* if for each $x, y \in X$ there is a surjective homeomorphism $h\colon X \to X$ with $h(x) = y$.

Recall that a space X is *locally connected (resp., connected im kleinen) at a point p* provided that it admits a basis of open connected neighbourhoods (resp., connected neighbourhoods) of p. It is *locally connected* provided that it is locally connected at each point. It is well known and easy to see that if a space is connected im kleinen at each point, then it is locally connected. For other terms and definitions, the reader is referred to [16].

3 Main Results

The proof of the following lemma utilizes the ideas of [4, Theorem 3.1].

Lemma 3.1 *If X is a non-empty Suslinian continuum and U is a non-empty open subset of X, there exists a continuum C with non-empty interior such that $C \subset U$.*

Proof Let U be a non-empty open subset of X, and let O be an open set such that $O \subset Cl(O) \subset U$. Let K denote the set of components of $Cl(O)$ that meet O. By the Boundary Bumping Theorem, each element of K also meets the boundary of O. Then K is a collection of mutually exclusive non-degenerate continua and therefore K is countable. Since K is a closed cover, it follows from the Baire Category Theorem that there is some element $C \in K$ with non-empty interior. ■

Theorem 3.2 *If X is a non-degenerate Suslinian continuum, then every non-empty open set in X contains a Cantor set at every point of which X is connected im kleinen.*

Proof Using a standard recursive argument we obtain for every finite sequence s of 0's and 1's a non-empty open set U_s in X and a continuum C_s such that for every s we have $(Cl(U_{s,0}) \cap Cl(U_{s,1})) = \varnothing$ and $(Cl(U_{s,0}) \cup Cl(U_{s,1})) \subset C_s \subset U_s$. (Here, if n is a natural number and $s\colon \{1, 2, \dots, n\} \to \{0, 1\}$ is a sequence and i is in $\{0, 1\}$, then $s, i\colon \{1, 2, \dots, n, n+1\} \to \{0, 1\}$ is defined by $s, i(m) = s(m)$ if $m \le n$ and $s, i(n+1) = i$.)

Every point x in the Cantor set $\{0, 1\}^{\mathbb{N}}$ determines a continuum in X, the intersection $C_x = \cap_n C_{x|_n} = \cap_n U_{x|_n}$. (Here $x|_n$ denotes the first n coordinates of x.) For all but countably many x the continuum C_x consists of one point. Thus, there is a copy D of the Cantor set in $\{0, 1\}^{\mathbb{N}}$ such that C_x is a singleton $\{d_x\}$ for each $x \in D$. The continuum X is connected im kleinen at each d_x as witnessed by the continua $C_{x|_n}$. The map $x \to d_x$ is an embedding of D. ■

Proposition 3.3 *A Suslinian homogeneous continuum is locally connected.*

Proof By Theorem 3.2, it is connected im kleinen at each point. ■

Proposition 3.4 *A Suslinian, non-separable, homogeneous continuum X is rim-finite and is, therefore, the continuous image of an arc.*

Proof In [3], it is shown that X contains a hereditarily locally connected subcontinuum Y with non-degenerate interior in X such that X is rim-finite at each point of Y. By homogeneity, X is rim-finite and is, therefore, the image of an arc ([10, 15]). ■

In [9], Nikiel and Tymchatyn have shown that if a homogeneous compactum is an IOK, then either X is zero-dimensional or one of the following holds:

(i) X is metrizable;

(ii) X is a union of finitely many pairwise disjoint simple closed curves.

It follows that a homogeneous continuum that is an IOK is either metrizable or a simple closed curve. We have the following corollary.

Corollary 3.5 *A Suslinian, non-separable, homogeneous continuum is a simple closed curve.*

We note that an example to which the previous corollary applies can be obtained from a homogeneous Suslin line. For example, see [14].

The proof of the following theorem is necessary because Corollary 3.5 covers only the non-separable case.

Theorem 3.6 *Let X be a homogeneous Suslinian compactum with* $\dim(X) > 0$. *Then X is the pairwise disjoint union of finitely many simple closed curves. If X is also separable then it is metrizable.*

Proof Let X be a homogeneous Suslinian compactum with $\dim(X) > 0$. Each component of X is a non-degenerate homogeneous continuum. Since X is Suslinian, compact, and homogeneous, it can only have finitely many components. Hence, it suffices to suppose X is connected. By Proposition 3.3 X is locally connected.

Suppose X has no local separating point. Let A_0 and B_0 be disjoint, non-degenerate continua in $U_0 = X$. By [13], let U_{00} and U_{01} be connected open sets such that $cl(U_{0i}) \subset U_0$ for $i = 0, 1$, $cl(U_{00}) \cap cl(U_{01}) = \varnothing$ and $U_{0i} \cap A_0 \neq \varnothing$ and $U_{0i} \cap B_0 \neq \varnothing$ for $i = 0, 1$. Let $A_{0i} \subset U_{0i} \cap A_0$ and $B_{0i} \subset U_{0i} \cap B_0$ for $i = 0, 1$ be non-degenerate continua. For $i = 0, 1$, let U_{0ij}, $j = 0, 1$ be connected open sets in U_{0i} with disjoint closures that are contained in U_{0i} such that each U_{0ij} meets both A_{0i} and B_{0i}. Continuing recursively, we obtain an uncountable family of pairwise disjoint continua from A_0 to B_0, which is a contradiction. Hence, X contains a local separating point. By homogeneity each point of X is a local separating point of X.

If X is metrizable, by [16, III.8.4.1] some local separating point x of X is a point of order 2 (*i.e.,* x has a neighbourhood base of open sets with two point boundaries.). Hence, each point of X is order of 2. It follows that X contains no triod. Since it is locally connected, it is a 1-manifold. Since it is homogeneous, it is a simple closed curve. We remark that a separable simple closed curve is metrizable, as each separable arc is metrizable.

If X is not metrizable, by the remarks preceding Corollary 3.5, it suffices to prove that X is hereditarily locally connected because hereditarily locally connected continua are IOC by [8]. We argue by contradiction. By [11], suppose X contains a continuum of convergence D. Let $\{D_i\}_{i=1}^{\infty}$ be pairwise disjoint continua in $X \setminus D$ with $D \cap D_j = \varnothing$ for each j and such that some subnet of D_i's converges to D. Let

D' be an irreducible continuum from a to b for some distinct pair of points a, b in D. By [5], there exists $\pi\colon D' \to L$ a continuous map of D' to an arc L such that the fibers of π are nowhere dense continua. Now, $\pi(a)$ and $\pi(b)$ are endpoints of L and we give L its cutpoint order with initial point $\pi(a)$. If $c < d$ in L, we let (c, d) denote the open interval from c to d in L. Since X is Suslinian, π has at most countably many non-degenerate fibers. Let $F' = \{x \in D' : \pi^{-1}(\pi(x)) = \{x\}$ and $x \neq a, b\}$. Then F' is a dense G_δ set in D'. Each point of F' disconnects D' between a and b. So F' is naturally linearly ordered and F' is homeomorphic to $\pi(F')$. Each point of F' is a point of order 2 in D'. Since X is perfectly normal ([3]), let $\{U_i\}$ be a countable sequence of connected open sets such that $cl(U_{i+1}) \subset U_i$ and $\bigcap U_i = D'$. For each $x \in F'$, let $D' \setminus \{x\} = A_x \cup B_x$, where A_x and B_x are disjoint connected open sets in D' with $a \in A_x$ and $b \in B_x$.

For each $x \in F'$, let U_x be a connected open neighborhood of x such that $U_x \setminus \{x\} = P_x \cup Q_x$, where P_x and Q_x are disjoint non-empty open sets. Let $a_x, b_x \in F'$ such that $x \in \pi^{-1}((\pi(a_x), \pi(b_x))) \subset U_x$. Without loss of generality, we may suppose the connected set $\pi^{-1}((\pi(a_x), \pi(x))) \subset Q_x$. Let

$$F'' = \left\{ x \in F' \mid \pi^{-1}\big((\pi(x), \pi(b_x))\big) \subset P_x \right\}.$$

For $x \in F' \setminus F''$, let $U_x^* \subset U_x$ be a connected open neighbourhood of x such that $cl(U_x^*) \cap D' \subset \pi^{-1}((\pi(a_x), \pi(b_x)))$. Let $U'_x = (P_x \cap U_x^*) \cup Q_x \cup \{x\}$. Then U'_x is a connected open neighbourhood of x, and $(P_x \cap U_x^*) \cup \{x\}$ is connected, locally connected, and locally compact. The boundary of $P_x \cap U_x^*$ contains x as an isolated point and $D' \cap cl(P_x \cap U_x^*) = \{x\}$. Since $D' = \bigcap U_i$, there exists i such that the boundary of $P_x \cap U_x^*$ except for x is contained in $X \setminus U_i$. Let L_x be a non-degenerate continuum with $x \in L_x \subset cl(P_x \cap U_x^* \cap U_i)$.

If $F' \setminus F''$ were uncountable, then there would exist a positive integer i such that for each of uncountably many $x \in F' \setminus F''$, $P_x \cap U_x^*$ had boundary except for x in $X \setminus U_i$. It would follow that for such x, the collection $\{L_x\}$ would be an uncountable collection of pairwise disjoint, non-degenerate continua contradicting the Suslinian condition on X. Thus, $F' \setminus F''$ is at most countable.

Let $x \in F''$. We prove that for some i, a and b lie in different components of $U_i \setminus U_x$. Let C_i be the component of a in $cl(U_i) \setminus U_x$. Then $C_{i+1} \subset C_i$ and $\bigcap C_i$ is a continuum in $D' \setminus U_x$. So $b \notin \bigcap C_i$. Let j be such that $b \notin C_j$. Since x separates U_x between $\pi^{-1}((\pi(a_x), \pi(x)))$ and $\pi^{-1}((\pi(x), \pi(b_x)))$ and $\bigcap cl(U_i) = D$, it follows that x separates U_k between a and b for some $k \geq j$. Moreover, since F'' is uncountable, there exists an uncountable subset F^* of F'' and a positive integer i such that $U_i \setminus \{x\} = K_x \cup L_x$ where K_x and L_x are disjoint open sets with $a \in K_x$ and $b \in L_x$ for each $x \in F^*$. Since X is Suslinian, we may suppose each point of F^* is a limit point of F^* from both above and below in the natural order on F^*. Note that for each j, $D_j \not\subset U_i$, since each point of F^* separates U_i between a and b.

Let $x < x^* < x'$ in F^* and let W be a connected neighborhood of x^* in $K_{x'} \cap L_x \cap U_i$. Since $x^* \in \lim D_j$, there is a continuum in W from x^* to D_j for some large j. Since the continuum $D_j \not\subset U_i$, there is a continuum in $cl(K_{x'}) \cap cl(L_x)$ from x to $X \setminus U_i$. Since x is a limit point of F^* from both above and from below, it follows that there is a continuum M_x from x to $X \setminus U_i$ such that $M_x \cap D' = \{x\}$ for each

$x \in F^*$ and such that $M_x \cap M_y = \varnothing$ for each distinct pair of points $x, y \in F^*$. This is again a contradiction to the Suslinian hypothesis on X. It follows that X contains no continuum of convergence. Hence it is hereditarily locally connected. This completes the proof of Theorem 3.6. ■

References

[1] T. Banakh, V. V. Fedorchuk, J. Nikiel, and M. Tuncali, *The Suslinian number and other cardinal invariants of continua.* Fund. Math. **209**(2010), no. 1, 43–57. doi:10.4064/fm209-1-4

[2] H. Cook and A. Lelek, *On the topology of curves. IV.* Fund. Math. **76**(1972), no. 2, 167–179.

[3] D. Daniel, J. Nikiel, L. B. Treybig, H. M. Tuncali, and E. D. Tymchatyn, *On Suslinian continua.* Canad. Math. Bull. **48**(2005), no. 2, 195–202.

[4] B. Fitzpatrick, Jr. and A. Lelek, *Some local properties of Suslinian compacta.* Colloq. Math. **31**(1974), 189–197.

[5] G. R. Gordh Jr., *Monotone decompositions of irreducible Hausdorff continua.* Pacific J. Math. **36**(1971), 647–658.

[6] A. Lelek, *On the topology of curves. II.* Fund. Math. **70**(1971), no. 2, 131–138.

[7] A. Lelek and L. Mohler, *On the topology of curves. III.* Fund. Math. **71**(1971), no. 2, 147–160.

[8] J. Nikiel, *The Hahn-Mazurkiewicz theorem for hereditarily locally connected continua.* Topology Appl. **32**(1989), no. 3, 307–323. doi:10.1016/0166-8641(89)90037-0

[9] J. Nikiel and E. D. Tymchatyn, *On homogeneous images of compact ordered spaces.* Canad. J. Math. **45**(1993), no. 2, 380–393.

[10] B. J. Pearson, *Mapping arcs and dendritic spaces onto netlike continua.* Colloq. Math. **34**(1975/76), no. 1, 39–48.

[11] J. Simone, *Concerning hereditarily locally connected continua.* Colloq. Math. **39**(1978), no. 2, 243–251.

[12] L. B. Treybig, *Concerning continua which are continuous images of compact ordered spaces.* Duke Math. J. **32**(1965), 417–422. doi:10.1215/S0012-7094-65-03241-2

[13] E. D. Tymchatyn, *Some n-arc theorems.* Pacific J. Math. **66**(1976), no. 1, 291–294.

[14] J. van Mill, *On the character and π-weight of homogeneous compacta.* Israel J. Math. **133**(2003), 321–338. doi:10.1007/BF02773072

[15] L. E. Ward Jr., *The Hahn-Mazurkiewicz theorem for rim-finite continua.* General Topology and Appl. **6**(1976), no. 2, 183–190. doi:10.1016/0016-660X(76)90031-3

[16] G. T. Whyburn, *Analytic topology.* American Mathematical Society Colloquium Publications, 28, American Mathematical Society, New York, 1942.

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