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Cohomology of Real Diagonal Subspace Arrangements via Resolutions

IRENA PEEVA[1], VIC REINER[2] and VOLKMAR WELKER[3]
[1]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A., e-mail: irena@math.mit.edu*
[2]*Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A., e-mail: reiner@math.umn.edu*
[3]*FB6 Mathematik und Informatik, Universität GH-Essen, 45117 Essen, Germany, e-mail: welker@exp-math.uni-essen.de.*

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Abstract. We express the cohomology of the complement of a real subspace arrangement of diagonal linear subspaces in terms of the Betti numbers of a minimal free resolution. This leads to formulas for the cohomology in some cases, and also to a cohomology vanishing theorem valid for all arrangements.

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1. Introduction

Consider $\mathbf{R}^n$ with coordinates given by $u_1, \ldots, u_n$. A linear subspace of the form $u_{i_1} = \cdots = u_{i_s}$ is called a *diagonal subspace*. In this paper we study arrangements of diagonal subspaces called *diagonal arrangements* (or *hypergraph arrangements* according to other authors).

The following problem has been of interest:

PROBLEM 1.1. Compute the cohomology of the complement $\mathcal{M}_{\mathcal{A}} := \mathbf{R}^n - \mathcal{A}$ of an arrangement $\mathcal{A}$ of linear subspaces.

The usual approach to computing the cohomology $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}}; k)$ is to

- compute the homology of lower intervals in the *intersection lattice* L_A (see Section 5) using techniques such as *nonpure shellability*, and then
- apply a result of Goresky and MacPherson [GM] (or further refinements such as [ZZ, SWe]) which expresses $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}})$ in terms of this data.

See [Bj] for a nice survey of the subject of subspace arrangements. The goal in this paper is to bring to bear algebraic techniques to attack Problem 1.1 for the diagonal arrangements. We will use the following construction.

CONSTRUCTION 1.2. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k. Let I be a *monomial* ideal in S, i.e. an ideal generated by monomials. It has a unique set of minimal generating monomials, and among these let the *squarefree* monomials be $m_1, \ldots, m_s$. For a squarefree monomial m, let U_m be the intersection of the hyperplanes $u_p = u_q$ for monomials $x_p x_q$ dividing m_i. Define the *canonical arrangement* $\mathcal{A}_I$ associated to I to be the union of the diagonal linear subspaces U_{m_i}, $i = 1, 2, \ldots, s$. For example, if $I = (x_1^4,\ x_1x_3,\ x_3x_4^2,\ x_2x_3x_4) \subset k[x_1, x_2, x_3, x_4]$ then $\mathcal{A}_I := \{u_1 = u_3\} \cup \{u_2 = u_3 = u_4\}$.

For every diagonal arrangement $\mathcal{A}$ there exists an ideal I such that $\mathcal{A} = \mathcal{A}_I$. The squarefree generators of I are uniquely determined by the subspaces in $\mathcal{A}$; the nonsquarefree generators can be chosen arbitrarily.

Furthermore, the homology Tor groups $\mathrm{Tor}_*^{S/I}(k, k)$ can be computed from the minimal free resolution of k over S/I. Since S/I carries a natural $\mathbf{N}^n$-grading, this resolution may also be chosen $\mathbf{N}^n$-graded, and for a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we denote by $\mathrm{Tor}_i^{S/I}(k, k)_\alpha$ or $\mathrm{Tor}_i^{S/I}(k, k)_{\mathbf{x}^\alpha}$ the α-graded piece of $\mathrm{Tor}_i^{S/I}(k, k)$. □

Our algebraic approach to solving Problem 1.1 is based upon the following:

THEOREM 1.3. *Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$, and $\mathcal{A}_I$ its canonical arrangement. Then* $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}_I}; k) \cong \mathrm{Tor}_{n-i}^{S/I}(k, k)_{x_1 \cdots x_n}$.

Note that in Construction 1.2 there is a huge choice of adding nonsquarefree monomials in the ideal I without changing the canonical arrangement $\mathcal{A}_I$. This is possible because the nonsquarefree monomials do not affect the multidegree $x_1 \cdots x_n$ component of the Tor groups used in Theorem 1.3. Having such choice of the generators of I is very beneficial: for example for the r-equal arrangement in Example 3.3 the choice allows to take I equal to a power of the maximal ideal (for which the Tor-groups are well known).

The numbers $\dim_k \mathrm{Tor}_{n-i}^{S/I}(k, k)$ are the ranks of the free modules in the minimal free resolution of k over S/I, and are called the *Betti numbers* of k. Thus, Theorem 1.3 links the Betti numbers of $\mathcal{M}_{\mathcal{A}_I}$ and k. The theorem is proved in Section 2, using the Bar resolution of k to compute $\mathrm{Tor}_*^{S/I}(k, k)$ and relying on a specific geometric realization of $\mathcal{M}_{\mathcal{A}_I}$. In Section 3 we demonstrate some applications of the theorem. An example is given which shows how $\dim_k \mathrm{H}^*(\mathcal{M}_{\mathcal{A}}; k)$ can depend upon the characteristic of the field k. We also comment there that the theorem opens up the possibility to compute cohomology in specific examples by the computer the algebra packages MACAULAY and MACAULAY 2.

In Section 4 we introduce *stable diagonal arrangements* motivated by the fact that the minimal free resolutions of stable ideals are well known. For such arrangements $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}})$ is explicitly computed in Theorem 4.2. This class includes the *r-equal arrangements* $\mathcal{A}_{n,r}$, for which we are able to further refine our results and describe the action of the symmetric group on $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_{n,r}}; \mathbf{C})$ (Theorem 4.4). The

r-equal arrangements have received much attention recently (see [Bj, BLY, BWe, Ko, SWa, SWe]). The proofs of Example 3.3 and Theorem 4.4 are entirely based on the algebraic approach and do not make use of the combinatorially established properties of $\mathcal{A}_{n,r}$.

Section 5 is inspired by a result of Backelin and Eisenbud et al. on the rate of growth of $\mathrm{Tor}_*^{S/I}(k,k)$ based on the minimum degree of minimal generators for I. We prove a sharp lower bound for the vanishing of the homology of the intersection lattice of an arbitrary arrangement of linear subspaces in a vector space, based on the minimum codimension of the maximal subspaces (Theorem 5.2).

2. Resolutions

In this section we prove Theorem 1.3 and discuss some consequences.

Proof of Theorem 1.3. The first part of the proof is a computation of Betti numbers by the Bar resolution. This idea has already been applied in [HRW1, Thm. 3.1] and [PRS]. We present it in detail for the purposes of keeping this paper self-contained, and we recapitulate the argument in a slightly different form here.

Denote $R = S/I$. In order to compute $\mathrm{Tor}_*^R(k,k)$, we resolve k as a trivial R-module $k = R/(x_1,\dots,x_n)$ using the Bar resolution [Ma, Sect. IV.5]:

$$\mathbf{B} : \cdots \to B_i \to \cdots \to B_1 \to B_0 \to k \to 0.$$

This is a free resolution in which B_i is the free R-module having basis indexed by all symbols $[m_1|m_2|\cdots|m_i]$ where m_j, $j = 1,\dots,i$, are monomials in R. We interpret this symbol as 0 if any of the monomials m_j lies in I. The differential $d_i\colon B_i \to B_{i-1}$ is defined R-linearly by

$$\begin{aligned} &d_i[m_1|m_2|\cdots|m_i] \\ &\quad = m_1[m_2|\cdots|m_i] + \sum_{1\leqslant j\leqslant i-1} (-1)^j [m_1|\cdots|m_j\cdot m_{j+1}|\cdots|m_i]. \end{aligned}$$

The free resolution $\mathbf{B}$ is far from minimal. To compute $\mathrm{Tor}_*^R(k,k)$, we tensor $\mathbf{B}$ with k, and then take the homology. $\mathbf{B}\otimes_R k$ is a complex of k-vector spaces with differential

$$\overline{d_i}[m_1|m_2|\cdots|m_i] = \sum_{1\leqslant j\leqslant i-1} (-1)^j [m_1|\cdots|m_j\cdot m_{j+1}|\cdots|m_i].$$

Notice that d_i preserves the product $\prod_i m_i$ of the monomials appearing in square brackets, i.e. it preserves the $\mathbf{N}^n$-grading. This means that $\mathbf{B}\otimes_R k$ decomposes as a direct sum of chain complexes $(\mathbf{B}\otimes_R k)_\alpha$ for $\alpha \in \mathbf{N}^n$, and $\mathrm{Tor}_*^R(k,k)_\alpha$ is the homology of the chain complex $(\mathbf{B}\otimes_R k)_\alpha$.

For $\mathbf{x}^{\alpha} = x_1 \cdots x_n$, the chain complex $(\mathbf{B} \otimes_R k)_{x_1 \ldots x_n} = (\mathbf{B} \otimes_R k)_{\alpha}$ may be further identified with the (augmented) relative chain complex for a certain pair of cell complexes which we now describe. Consider the decomposition of $\mathbf{R}^n$ into cones of various dimensions by the union of all hyperplanes of the form $u_i = u_j$, i.e. the classical *braid arrangement* of Type A_{n-1} [OT, Sect. 1.2]. By restricting this decomposition to the unit sphere $\mathbf{S}^{n-2}$ inside the hyperplane $\sum_i u_i = 0$, one obtains a simplicial decomposition Δ_n of this sphere commonly known as the *Coxeter complex* for Type A_{n-1}.

A typical face in Δ_n is the intersection of the sphere with the cone defined by a sequence of equalities and inequalities relating all the variables $u_1, \ldots, u_n$, such as $u_2 = u_5 = u_7 > u_4 = u_{10} > u_6 > u_1 = u_3 = u_8 > u_9$, for $n = 10$. Identify this face of Δ_n with the k-basis vector $[x_2x_5x_7|x_4x_{10}|x_6|x_1x_3x_8|\ x_9]$ in $(\mathbf{B} \otimes_R k)_{x_1 \cdots x_n}$. Observe that the symbols $[m_1| \cdots |m_n]$ which have been set to 0, namely those in which some $m_j \in I$, exactly correspond to the faces of Δ_n which triangulate the intersection $\mathbf{S}^{n-2} \cap \mathcal{A}_I$. We conclude that

$$(\mathbf{B} \otimes_R k)_{x_1 \cdots x_n} \cong C_*(\mathbf{S}^{n-2}, \mathbf{S}^{n-2} \cap \mathcal{A}_I; k),$$

where $C_*(\mathbf{S}^{n-2}, \mathbf{S}^{n-2} \cap \mathcal{A}_I; k)$ denotes the augmented relative chain complex with coefficients in k for the pair $(\mathbf{S}^{n-2}, \mathbf{S}^{n-2} \cap \mathcal{A}_I)$. Therefore

$$\mathrm{Tor}_i^R(k,k)_{x_1 \cdots x_n} \cong \tilde{\mathrm{H}}_{i-2}(\mathbf{S}^{n-2}, \mathbf{S}^{n-2} \cap \mathcal{A}_I; k).$$

On the other hand, $\tilde{\mathrm{H}}_i(\mathbf{S}^{n-2}) = 0$ unless $i = n-2$, so the long exact sequence for the pair, along with Alexander duality gives

$$\begin{aligned} \mathrm{Tor}_i^R(k,k)_{x_1 \cdots x_n} &\cong \tilde{\mathrm{H}}_{i-3}(\mathbf{S}^{n-2} \cap \mathcal{A}_I; k), \\ &\cong \tilde{\mathrm{H}}^{n-i}(\mathbf{S}^{n-2} - (\mathbf{S}^{n-2} \cap \mathcal{A}_I); k). \end{aligned} \tag{2.1}$$

for $i < n$. A similar computation shows that

$$\begin{aligned} \mathrm{Tor}_n^R(k,k)_{x_1 \cdots x_n} &\cong \tilde{\mathrm{H}}^0(\mathbf{S}^{n-2} - (\mathbf{S}^{n-2} \cap \mathcal{A}_I); k) \oplus k, \\ &\cong \mathrm{H}^0(\mathbf{S}^{n-2} - (\mathbf{S}^{n-2} \cap \mathcal{A}_I); k). \end{aligned}$$

It only remains to observe that $\mathbf{S}^{n-2} - (\mathbf{S}^{n-2} \cap \mathcal{A}_I)$ is homotopy equivalent to $\mathcal{M}_{\mathcal{A}_I} = \mathbf{R}^n - \mathcal{A}_I$ for the following reason: one can first project perpendicularly onto the subspace $\sum_i u_i = 0$ in $\mathbf{R}^n$ since every subspace in $\mathcal{A}_I$ contains the kernel $u_1 = \cdots = u_n$ of this projection, and then perform a straight-line homotopy $\mathbf{v} \mapsto (1-t)\mathbf{v} + t\mathbf{v}/|\mathbf{v}|$ to project radially onto the unit sphere $\mathbf{S}^{n-2}$.

This completes the proof of Theorem 1.3 □

The numbers $\mathrm{Tor}_i^{S/I}(k,k)$ are equal to the ranks of the corresponding free modules in the minimal free resolution of k over S/I, and are called the *Betti numbers* of k. The multigraded *Poincaré series* of k is

$$\mathrm{Poin}_{S/I}^k(t, \mathbf{x}) := \sum_{i \geqslant 0, \alpha \in \mathbf{N}^n} \dim_k \mathrm{Tor}_i^{S/I}(k,k)_{\alpha} t^i \mathbf{x}^{\alpha},$$

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where we are abusing notation by using the variables $\mathbf{x} = x_1, \dots, x_n$ as both indeterminates in S and as generating function variables in $\mathrm{Poin}^k_{S/I}(t, \mathbf{x})$.

For a power series f in $\mathbf{Z}[t][[x_1, \dots, x_n]]$ and monomial m in the variables $t, \mathbf{x}$ denote by $\mathrm{coeff}_m(f)$ the coefficient of m in f. In this notation, Theorem 1.3 can be rephrased:

COROLLARY 2.1. *Let I be a monomial ideal and $\mathcal{A}_I$ its canonical arrangement. Then* $\mathrm{Poin}(\mathcal{M}_{\mathcal{A}_I}; k) = t^n \,\mathrm{coeff}_{x_1 \cdots x_n}\big(\mathrm{Poin}^k_{S/I}(t^{-1}, \mathbf{x})\big)$.

Backelin showed in [Ba1] that when I is a monomial ideal, $\mathrm{Poin}^k_{S/I}(t, \mathbf{x})$ can always be written as a rational fraction

$$\mathrm{Poin}^k_{S/I}(t, \mathbf{x}) = \frac{(1 + tx_1) \dots (1 + tx_n)}{K_I(t, \mathbf{x})},$$

where K_I is a polynomial which we call the *I-denominator*. Furthermore, he gave explicit bounds for the maximum degree of t and each x_i in K_I, so that in principle one need only compute a finite number of steps in the minimal free resolution of k as an S/I-module to get enough information for computing K_I.

It was proven by Serre (see [GL]) that

$$\mathrm{Poin}^k_{S/I}(t, \mathbf{x}) \leqslant \frac{(1 + tx_1) \dots (1 + tx_n)}{1 - t^2 Q_I(t, \mathbf{x})}, \tag{2.2}$$

where the above inequality means coefficient-wise comparison of power series, and where $Q_I(t, \mathbf{x})$ is the Poincaré series for the *finite* minimal free resolution of I as an S-module,

$$Q_I(t, \mathbf{x}) := \mathrm{Poin}^I_S(t, \mathbf{x}) = \sum_{i \geqslant 0, \alpha \in \mathbf{N}^n} \dim_k \mathrm{Tor}^S_i(I, k)_\alpha t^i \mathbf{x}^\alpha.$$

We summarize all the above information in the next corollary of Theorem 1.3.

COROLLARY 2.2. *Let I, $K_I(t, \mathbf{x})$, $Q_I(t, \mathbf{x})$, $\mathcal{M}_{\mathcal{A}_I}$ be as above. For $i \geqslant 1$ we have*

$$\begin{aligned} \dim_k \mathrm{H}^i(\mathcal{M}_{\mathcal{A}_I}, k) &= \dim_k \mathrm{Tor}^{S/I}_{n-i}(k, k)_{x_1 \cdots x_n} \\ &= \mathrm{coeff}_{t^{n-i} x_1 \cdots x_n} \frac{(1 + tx_1) \dots (1 + tx_n)}{K_I(t, \mathbf{x})} \\ &\leqslant \mathrm{coeff}_{t^{n-i} x_1 \cdots x_n} \frac{(1 + tx_1) \dots (1 + tx_n)}{1 - t^2 Q_I(t, \mathbf{x})}. \end{aligned}$$

3. Applications

In this section we demonstrate how to apply Theorem 1.3.

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EXAMPLE 3.1. Let $\mathcal{A}$ be a hyperplane arrangement of diagonal hyperplanes $u_i = u_j$. Then we can choose a monomial ideal I generated by quadratic monomials so that $\mathcal{A} = \mathcal{A}_I$. By [Fr], the minimal free resolution of k over S/I is linear. Hence $\mathrm{Tor}_i^{S/I}(k,k)_{x_1\cdots x_n}$ vanishes for $i \neq n$. This corresponds to the fact that $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_I}; k)$ simply counts the connected components of $\mathcal{M}_{\mathcal{A}_I}$. □

Among other things, Theorem 1.3 opens up the possibility of calculating $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_I}; k)$ by computer (via *Gröbner bases*). The Betti numbers $\dim_k \mathrm{Tor}_i^{S/I}(k,k)_\alpha$ can be computed in the computer algebra package MACAULAY by D. Bayer and M. Stillman [BS] using a script for $\mathbf{N}^n$-homogeneous calculations by A. Reeves. Alternatively, the computations can be done by MACAULAY 2 [GS]. The minimal free resolution of k is infinite, however note that $\mathrm{Tor}_i^R(k,k)_{x_1\cdots x_n}$ vanishes for $i > n$, so only the first n Betti numbers need to be computed.

Next we illustrate how to apply results from commutative algebra in order to obtain formulas for the cohomology of $\mathcal{M}_{\mathcal{A}}$.

DEFINITION 3.2. A ring is called *Golod* if equality holds in Serre's upper bound (2.2). It was shown by Golod, cf. [GL], that this happens exactly when certain homology operations (*Massey operations*) vanish in the Koszul complex computing $\mathrm{Tor}_*^S(k, S/I) \cong \mathrm{Tor}_*^S(S/I, k)$. Thus, Golodness is encoded in finite data. It can be used, via Corollary 2.2, to compute $\dim_k \mathrm{H}^i(\mathcal{M}_{\mathcal{A}_I}, k)$.

EXAMPLE 3.3. One class of subspace arrangements which have received a great deal of attention recently are the *r-equal arrangements* $\mathcal{A}_{n,r}$. This arrangement has been studied extensively in recent years, see [BLY, BWe, Kh, Ko, SWa, SWe] and see [Bj] for its history. The arrangement $\mathcal{A}_{n,r}$ in $\mathbf{R}^n$ is the union of all subspaces $u_{i_1} = \cdots = u_{i_r}$ defined by setting r coordinates equal. Equivalently, this is the arrangement $\mathcal{A}_{\mathfrak{m}^r}$ associated to the rth power $\mathfrak{m}^r$ of the irrelevant ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. For any field k, we will prove that

$$\begin{aligned}&\dim_k \mathrm{H}^{s(r-2)}(\mathcal{M}_{\mathcal{A}_{n,r}}; k)\\ &\quad= \mathrm{Tor}_{n-s(r-2)}^{S/\mathfrak{m}^r}(k,k)_{x_1\cdots x_n}\\ &\quad= \sum_{\substack{(i_1,\ldots,i_s)\\ sr+\sum_j i_j \leqslant n}} \binom{n}{r+i_1\ r+i_2\ \cdots\ r+i_s} \prod_j \binom{r-1+i_j}{r-1},\end{aligned}$$

and all other cohomology groups vanish. This formula can also be deduced from [Bj, second formula in Equation 2.4].

Proof. For $r \geqslant 2$ it was first proved by Golod [GL] and is well known that $R = k[x_1, \ldots, x_n]/\mathfrak{m}^r$ is a Golod ring. Hence the $\mathfrak{m}^r$-denominator is

$$1 - t^2 \left(\sum_{i \geqslant 0} \dim(\mathrm{Tor}_i^S(\mathfrak{m}^r, k)_m)\, t^i m \right).$$

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Here $\mathrm{Tor}_i^S(\mathfrak{m}^r, k)_m$ are the Betti numbers of the minimal free resolution $\mathbf{F}_r$ of $\mathfrak{m}^r$ over the polynomial ring. This resolution is also well known, cf. [EK]: the elements $\{(m; 1 \leqslant i_1 < \cdots < i_s) \mid m$ is a monomial of degree $r, i_j \in \mathbf{N}, i_s <$ (maximal variable dividing $m)\}$ denote a basis for the free module in homological degree s of $\mathbf{F}_r$. The desired formula follows from a simple computation of the Betti numbers of $\mathbf{F}_r$ and applying Theorem 1.3 (cf. also Remark 4.5(2)). □

Another class of Golod squarefree monomial ideals are the Stanley–Reisner ideals of the complexes dual to sequentially Cohen–Macaulay complexes, as shown in [HRW2].

4. Stable Diagonal Arrangements

In this section we compute $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}}; k)$ for what we will call *stable diagonal* arrangements, which include all r-equal arrangements $\mathcal{A}_{n,r}$. We refine these results to give a description of the representation of the symmetric group Σ_n on $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_{n,r}}; \mathbf{C})$.

A large source of Golod monomial ideals are the stable monomial ideals $I \subset S$. A monomial ideal I is called *stable* if it satisfies the following property: if m is a monomial in I and x_i is the variable of largest index i dividing m, then $x_j m/x_i \in I$ for all $1 \leqslant j < i$. It is enough if this property is satisfied by all minimal generators of I. Such ideals play an important role in Gröbner bases theory: they appear as initial ideals in generic coordinates [Ei, Chapt. 15]. The minimal free resolution of a stable ideal as an S-module was constructed in [EK]. The Golodness property for stable monomial ideals is established in [AH].

Motivated by this, we define an arrangement of subspaces $\mathcal{A}$ to be a *stable diagonal* arrangement if $\mathcal{A} = \mathcal{A}_I$ for some stable monomial ideal $I \subset S$ (the ideal I will in general not be unique). It is easy to check that this is equivalent to the following condition on $\mathcal{A}$: all maximal subspaces in $\mathcal{A}$ are of the form $u_{i_1} = \cdots = u_{i_r}$ with $i_1 < \cdots < i_r$, and whenever such a maximal subspace is in $\mathcal{A}$ and we have $j < i_r$ and $j \notin \{i_1, \ldots, i_r\}$, then $u_{i_1} = \cdots u_{i_{r-1}} = u_j$ is also contained in some subspace of $\mathcal{A}$.

To describe the results of [EK] on $\mathrm{Tor}^S(I, k)$ succinctly, we introduce the terminology of *partitions* and *Young tableaux* (see e.g. [Sa]). A partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0)$ is a weakly decreasing sequence of nonnegative integers λ_i. We say that λ has *weight* $|\lambda| := \sum_i \lambda_i$ and *length* $l(\lambda) := r$. The *Ferrers diagram* for λ is simply a set of boxes in the plane which is left-justified and has λ_i boxes in row i for each i. Partitions of the form $(r, 1^{n-r})$ are called *hooks* because of the shape of their Ferrers diagrams. A *(reverse) column-strict tableau* of shape λ is an assignment of positive integers to the boxes in the Ferrers diagram for λ so that the entries weakly decrease from left to right in each row and strictly decrease from top to bottom in each column. A tableau is called *standard* if it contains each of the entries $1, 2, \ldots, n-1, n = |\lambda|$ exactly once. Given a tableau T, let $\mathbf{x}^T := \prod_{i=1}^n x_i^{e_i}$

where e_i is the number of occurrences of the entry i in T. We will also use *skew Ferrers shapes* $\lambda_1 * \cdots * \lambda_t$ obtained by placing the Ferrers diagrams for each of the λ_i in disjoint rows and columns in the plane. Tableaux filling skew shapes are defined similarly to tableaux of Ferrers shapes.

When dealing with a stable monomial ideal I, given a column-strict tableaux T filling some hook Ferrers shape $(r, 1^s)$, we say that T is I-*appropriate* if the values $i_1, \ldots, i_r$ occurring in the horizontal row of the hook form the indices of some monomial $x_{i_1} \cdots x_{i_r}$ which is a minimal generator of I. Similarly for a stable diagonal arrangement $\mathcal{A}$, we say that T is $\mathcal{A}$-*appropriate* if the values $i_1, \ldots, i_r$ occurring in the horizontal row of the hook are all distinct and form the indices of some maximal subspace $u_{i_1} = \cdots = u_{i_r}$ in $\mathcal{A}$.

The Betti numbers in the minimal free resolution of a stable ideal were given in [EK] and we interpret this result as follows:

THEOREM 4.1. *For a stable monomial ideal $I \subset S = k[x_1, \ldots, x_n]$ and any field k, the Poincaré series $Q_I(t, \mathbf{x})$ for the finite minimal free resolution of I as an S-module is $Q_I(t, \mathbf{x}) = \sum_T \mathbf{x}^T t^{l(T)-1}$, where the sum ranges over all column-strict tableaux T of hook shapes having entries bounded by n and which are I-appropriate. Here $l(T)$ denotes the length of the partition which T fills.*

From Theorem 4.1 and Golodness, we will deduce

THEOREM 4.2. *For any stable diagonal arrangement $\mathcal{A}$, we have that* $\dim_k \mathrm{H}^i(\mathcal{M}_{\mathcal{A}}; k)$ *is the number of standard tableaux filling skew shapes of the form* $1^{i_0} * (r_1, 1^{i_1}) * \cdots * (r_s, 1^{i_s})$ *for which*

- *the skew shape has n boxes, i.e. $i_0 + \sum_{j=1}^{s}(r_j + i_j - 1) = n$,*
- *$i_0 + \sum_{j=1}^{s} (i_j + 2) = n - i$,*
- *every hook shape is filled $\mathcal{A}$-appropriately.*

Proof. Let I be any stable monomial ideal whose canonical arrangement $\mathcal{A}_I$ is equal to $\mathcal{A}$. Using the fact that S/I is Golod, along with Corollary 2.1, Definition 3.2 and Theorem 4.1 one concludes that

$$\mathrm{Poin}^k_{S/I}(t, \mathbf{x}) = \prod_{j=1}^{n} \frac{(1 + tx_j)}{1 - t^2 \sum_T \mathbf{x}^T t^{l(T)-1}},$$

where T ranges over the set of tableaux described in Theorem 4.1. By Theorem 1.3, $\dim_k \mathrm{H}_i(\mathcal{M}_{\mathcal{A}}; k)$ is the coefficient of $t^{n-i} x_1 \cdots x_n$ on the right-hand side in this equation. This is exactly counted by the set of tableaux in the corollary: the entries filling the leftmost (single-column) Ferrers shape correspond to a choice of a monomial from the numerator, while the fillings of the remaining hook Ferrers shapes correspond to a choice of monomials from the denominator after it is expanded as a geometric series. □

EXAMPLE 4.3. Let $n = 4$ and $\mathcal{A} = \{u_1 = u_2\} \cup \{u_1 = u_3 = u_4\}$. The diagonal arrangement $\mathcal{A}$ is stable. There are four tableaux satisfying the conditions in Theorem 4.2 (Figure 1).

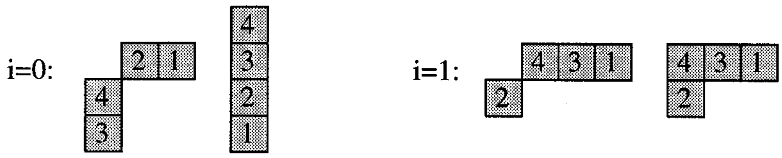

Figure 1. The tableaux contributing to $\mathrm{H}^i(\mathcal{M}_A; k)$.

These tableaux enumerate the dimensions of $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}; k)$ therefore $\mathrm{H}^0(\mathcal{M}_{\mathcal{A}}; k) = k^2$, $\mathrm{H}^1(\mathcal{M}_{\mathcal{A}}; k) = k^2$ and all other cohomology groups vanish. This is consistent with the fact that $\mathcal{M}_{\mathcal{A}}$ is homotopy equivalent to a disjoint union of two circles. □

This result raises two natural questions:

Questions. Is the intersection lattice of a stable diagonal arrangement shellable? Can one use this to give a proof of Theorem 4.2 which uses the more standard approach?

We next study the case of the real r-equal arrangement $\mathcal{A}_{n,r}$, where the above result can be refined to account for the action of the symmetric group Σ_n. Note that $\mathcal{A}_{n,r}$ is a stable diagonal arrangement since $\mathcal{A}_{n,r} = \mathcal{A}_{\mathfrak{m}^r}$, where $\mathfrak{m}$ is the irrelevant ideal $(x_1, \ldots, x_n)$. Note that the symmetric group Σ_n acting on $\mathbf{R}^n$ by permuting coordinates preserves $\mathcal{A}_{n,r}$ and, hence, acts on its complement $\mathcal{M}_{\mathcal{A}_{n,r}}$. In [BWe], recursive formulas are given for the cohomology $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_{n,r}})$ of the complement, and the authors ask whether one can describe explicitly the representation of Σ_n on $\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_{n,r}})$ in general. Such a description was given in [SWe], based on results from [SWa] and our next theorem shows how one can apply the present techniques to recover a different form of this result. First, we need to review some notions from the representation theory of the symmetric group Σ_n and general linear group $\mathrm{GL}(n, \mathbf{C})$ (see [Sa], [FH]).

The irreducible finite dimensional complex representations of Σ_n are indexed by partitions μ of the number n, and we let $\mathcal{S}_\mu$ denote the irreducible representation indexed by μ. The irreducible finite dimensional complex representations of $\mathrm{GL}(n, \mathbf{C})$ are also indexed by partitions μ of any number, and we let $\mathcal{V}_\mu$ denote the irreducible indexed by μ. Let $\mathbf{x}$ be the diagonal matrix in $\mathrm{GL}(n, \mathbf{C})$ with eigenvalues $(x_1, \ldots, x_n)$, i.e. a typical element of a *maximal torus* in $\mathrm{GL}(n, \mathbf{C})$. One can decompose a $\mathrm{GL}(n, \mathbf{C})$-representation $\mathcal{W}$ into its *weight spaces* $\mathcal{W} = \oplus_\nu \mathcal{W}_\nu$ where ν runs over all vectors in $\mathbf{N}^n$, and $\mathcal{W}_\nu$ is defined to be the $\mathbf{x}^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$-eigenspace for the matrix representing $\mathbf{x}$ in the $\mathrm{GL}(n, \mathbf{C})$-action. If μ happens to be a partition of n, then the $(1, \ldots, 1)$-weight space $\mathcal{V}_{\mu,(1,\ldots,1)}$ of $\mathcal{V}_\mu$ is invariant under the subgroup $\Sigma_n \hookrightarrow \mathrm{GL}(n, \mathbf{C})$. Furthermore, this representation of S_n on $\mathcal{V}_{\mu,(1,\ldots,1)}$ is isomorphic to the irreducible representation $\mathcal{S}_\mu$. Given any tuple $(\mu_1, \ldots, \mu_t)$ of

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partitions, the tensor product $\mathcal{V}_{\mu_1} \otimes \cdots \otimes \mathcal{V}_{\mu_t}$ is isomorphic to a special case of what is called a *skew representation* $\mathcal{V}_{\mu_1 * \cdots * \mu_t}$ of $\mathrm{GL}(n, \mathbf{C})$ corresponding to the *skew shape* $\mu_1 * \cdots * \mu_t$.

Similarly, if the sum of the numbers partitioned by the μ_i happens to be n, then restricting $\mathcal{V}_{\mu_1 * \cdots * \mu_t}$ to the $(1, \ldots, 1)$ weight space $\mathcal{V}_{\mu_1 * \cdots * \mu_t, (1, \ldots, 1)}$ gives a special case of what is called a *skew representation* $\mathcal{S}_{\mu_1 * \cdots * \mu_t}$ of Σ_n. Lastly, we recall that a finite-dimensional complex (rational) representation of $\mathrm{GL}(n, \mathbf{C})$ is completely determined up to isomorphism by its *formal character* which is the polynomial in $x_1, \ldots, x_n$ obtained by taking the trace of the matrix acting on $\mathcal{V}$ which represents $\mathbf{x}$. For the skew representations $\mathcal{V}_D$, this character is the *Schur function* $s_D(x_1, \ldots, x_n)$ which has the formula $s_D(x_1, \ldots, x_n) = \sum_T \mathbf{x}^T$ as T ranges over all *column-strict tableaux* of shape D with entries in $1, 2, \ldots, n$, and $\mathbf{x}^T$ is the product of x_i as i ranges over the entries of T. Analogously, the dimension of a skew representation $\mathcal{S}_D$ for Σ_n is the number of *standard Young tableaux* of shape D, where a column-strict tableaux is standard if $\mathbf{x}^T = x_1 \ldots x_n$

THEOREM 4.4. *As Σ_n-representations we have the isomorphisms*

$$\mathrm{Tor}^{S/\mathfrak{m}^r}_{n-s(r-2)}(\mathbf{C}, \mathbf{C})_{x_1 \cdots x_n} = \bigoplus_{\substack{(i_0, i_1, \ldots, i_s) \\ sr + \sum_j i_j = n}} \mathcal{S}_{(1^{i_0}) * (r, 1^{i_1}) * \cdots * (r, 1^{i_s})},$$

$$\mathrm{H}^{s(r-2)}(\mathcal{M}_{n,r}; \mathbf{C}) = \bigoplus_{\substack{(i_0, i_1, \ldots, i_s) \\ sr + \sum_j i_j = n}} \mathcal{S}_{(i_0) * (i_1 + 1, 1^{r-1}) * \cdots * (i_s + 1, 1^{r-1})}.$$

Proof. In this case, Theorem 4.1 can be rephrased as $Q_I(t, \mathbf{x}) = \sum_{i=0}^{n-1} s_{(r, 1^i)}(\mathbf{x}) t^i$, where $s_{(r,1^i)}(\mathbf{x})$ is the Schur function (defined earlier) corresponding to the shape $(r, 1^i)$. Therefore by Corollary 2.1 and Definition 3.2,

$$\begin{aligned}
\mathrm{Poin}^k_R(t, \mathbf{x}) &= \frac{\prod_{i=0}^n (1 + t x_i)}{1 - t^2 \sum_{i=0}^{n-1} s_{(r,1^i)}(\mathbf{x}) t^i} \\
&= \frac{\sum_{j=0}^n s_{(1^j)}(\mathbf{x}) t^j}{1 - t^2 \sum_{i=0}^{n-1} s_{(r,1^i)}(\mathbf{x}) t^i} \\
&= \sum_{i \geqslant 0} t^i \sum_{\substack{(i_0, i_1, \ldots, i_s) \\ i_0 + \sum_{p \geqslant 1} (i_p + 2) = i}} s_{(1^{i_0})}(\mathbf{x}) s_{(r, 1^{i_1})}(\mathbf{x}) \cdots s_{(r, 1^{i_s})}(\mathbf{x}) \\
&= \sum_{i \geqslant 0} t^i \sum_{\substack{(i_0, i_1, \ldots, i_s) \\ i_0 + \sum_p (i_p + 2) = i}} s_{(1^{i_0}) * (r, 1^{i_1}) * \cdots * (r, 1^{i_s})}(\mathbf{x})
\end{aligned}$$

independent of the field k. If we choose $k = \mathbf{C}$, we can interpret the previous equation in terms of $\mathrm{GL}(n, \mathbf{C})$-representations. Note that $\mathrm{GL}(n, \mathbf{C})$ acts on

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$\mathbf{C}[x_1,\ldots,x_n]$ by invertible linear substitutions of the variables, and leaves $\mathfrak{m}$ and $\mathfrak{m}^r$ invariant. Therefore $\mathrm{GL}(n,\mathbf{C})$ acts on $R = \mathbf{C}[x_1,\ldots,x_n]/\mathfrak{m}^r$, and on $\mathrm{Tor}^R_*(\mathbf{C},\mathbf{C})$. Since $\mathrm{GL}(n,\mathbf{C})$-representations are determined by their characters, we conclude from the last equation above the following isomorphism of $\mathrm{GL}(n,\mathbf{C})$-representations:

$$\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C}) \cong \bigoplus_{\substack{(i_0,i_1,\ldots,i_s)\\ i_0+\sum_{p\geqslant 1}(i_p+2)=i}} \mathcal{V}_{(1^{i_0})*(r,1^{i_1})*\cdots*(r,1^{i_s})}.$$

Note that by definition, $\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C})_{x_1\cdots x_n}$ is the $(1,\ldots,1)$-weight space of $\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C})$. Hence we deduce the following isomorphism of Σ_n-representations:

$$\begin{aligned}\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C})_{x_1\cdots x_n} &\cong \bigoplus_{\substack{(i_0,i_1,\ldots,i_s)\\ i_0+\sum_{p\geqslant 1}(i_p+2)=i,\, i+s(r-2)=n}} \mathcal{V}_{(i_0)*(r,1^{i_1})*\cdots*(r,1^{i_s}),(1,\ldots,1)},\\ &\cong \bigoplus_{\substack{(i_0,i_1,\ldots,i_s)\\ sr+\sum_{p\geqslant 0} i_p=n,\, i+s(r-2)=n}} \mathcal{S}_{(1^{i_0})*(r,1^{i_1})*\cdots*(r,1^{i_s})}.\end{aligned}$$

which is equivalent to the assertion for $\mathrm{Tor}^R_*(\mathbf{C},\mathbf{C})_{x_1\cdots x_n}$ in the theorem. The assertion for $\mathrm{H}_*(\mathcal{M}_{n,r})$ then follows from the following facts:

- the nondegenerate Alexander duality pairing from Theorem 1.3

  $$\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C})_{x_1\cdots x_n} \otimes \mathrm{H}^{n-i}(\mathcal{M}_{n,r};\mathbf{C}) \to \mathrm{H}_{n-2}(\mathbf{S}^{n-2};\mathbf{C})$$

  establishes an isomorphism of Σ_n-representations

  $$\mathrm{H}^{n-i}(\mathcal{M}_{n,r};\mathbf{C}) \cong \left(\mathrm{Tor}^R_i(\mathbf{C},\mathbf{C})_{x_1\cdots x_n}\right)^{\check{}} \otimes \mathrm{H}_{n-2}(\mathbf{S}^{n-2};\mathbf{C})$$

- where $\check{}$ denotes the *contragredient* or *dual* of a representation.
- Complex representations of Σ_n are all self-dual.
- $\mathrm{H}_{n-2}(\mathbf{S}^{n-2};\mathbf{C})$ carries the one-dimensional *sign* representation of Σ_n, since any transposition in Σ_n acts by a reflection in $\mathbf{R}^{n-1}$ and hence acts by -1 on the fundamental cycle of the sphere $\mathbf{S}^{n-2}$.
- When one tensors a skew representation $\mathcal{S}_D$ by the sign representation of Σ_n, one obtains the skew representation $\mathcal{S}_{D^t}$ corresponding to the *transposed* diagram D^t obtained from D by flipping across the diagonal. □

Remarks 4.5. (1) The description of the Σ_n-action in Theorem 4.4 could also be deduced from the results of [BWa, SWa, SWe], although this computation is not carried out in any of these three references. In fact, it is interesting to compare

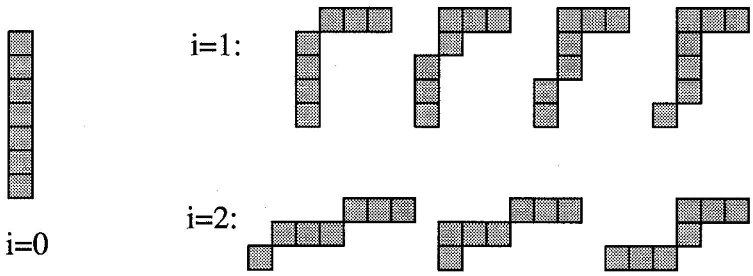

Figure 2. The skew representations appearing in $\mathrm{H}^i(\mathcal{M}_{7,3}; \mathbf{C})$.

Theorem 4.3 with the case $d = 1$ in [SWe, Thm. 4.4] since one obtains a nontrivial representation-theoretic identity by setting the two answers equal.

(2) The formula for the dimension of $\mathrm{H}^{n-i}(\mathcal{M}_{n,r}; \mathbf{C})$ in Example 3.3 comes from the fact that the skew representation $\mathcal{S}_D$ has dimension equal to the number of standard Young tableaux of shape D. For $D = (1^{i_0}) * (r, 1^{i_1}) * \cdots * (r, 1^{i_s})$ the number of such tableaux is easily seen to be

$$\binom{n}{r+i_1 \; r+i_2 \; \cdots \; r+i_s} \prod_j \binom{r-1+i_j}{r-1}.$$

EXAMPLE 4.6. Let $n = 7, r = 3$, then we obtain the following formulas:

$$\dim_k \mathrm{H}^i(\mathcal{M}_{7,3}; k) = \begin{cases} 1, & \text{for } i = 0, \\ 351, & \text{for } i = 1, \\ 350, & \text{for } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This coincides with the values given in Table 1 in [BWe], where the computations are done using recursive relations. Furthermore, we conclude from Theorem 4.4 that as a representation of Σ_7, the vector spaces $\mathrm{H}^i(\mathcal{M}_{7,3}; \mathbf{C})$ for $i = 0, 1, 2$ are isomorphic to the direct sum of representations corresponding to the skew shapes shown in Figure 2.

5. A Vanishing Theorem for Intersection Lattices

The main result of this section (Theorem 5.2) is a vanishing theorem for the homology of the intersection lattice associated to any arrangement of subspaces in a vector space over any field, given a lower bound on the codimension of the maximal subspaces in the arrangement. The theorem was inspired by a special case (Corollary 5.1) that follows from a result of Backelin and Eisenbud et al. on the rate of growth of $\mathrm{Tor}_*^{S/I}(k, k)$.

We begin by reviewing the notion of intersection lattices. For any field $\mathbf{F}$, let $\mathcal{A}$ denote an arrangement of subspaces in $\mathbf{F}^n$. The *intersection lattice* $L_{\mathcal{A}}$ is the poset whose elements correspond to all intersections of the subspaces, ordered by *reverse* inclusion, with top and bottom elements $\hat{1}, \hat{0}$ added on corresponding to the $\mathbf{0}$-subspace and the whole space $\mathbf{F}^n$ respectively. Note that this means that in the case when all of the subspaces in $\mathcal{A}$ intersect in some nonzero subspace V, i.e. when $\mathcal{A}$ is not *essential*, then V already would have been a top element and so the top element $\hat{1} \neq V$ is an *extra* element on top of V in $L_{\mathcal{A}}$. The poset $L_{\mathcal{A}}$ is actually a lattice as its name indicates, with the join $V \vee W$ of two subspaces V, W given by their intersection $V \cap W$, and meet $V \wedge W$ given by the intersection of all subspaces in $\mathcal{A}$ that contain $V \cup W$. The *proper part* $\overline{L}_{\mathcal{A}}$ is the subposet $L_{\mathcal{A}} - \{\hat{0}, \hat{1}\}$. Abusing notation, we can think of any poset such as $\overline{L}_{\mathcal{A}}$ as a topological space by identifying it with the geometric realization of the *order complex* $\Delta(\overline{L}_{\mathcal{A}})$. Here $\Delta(P)$ is the simplicial complex having vertices corresponding to the elements of P and simplices corresponding to the chains (totally ordered subsets) in P.

Next we discuss Backelin's result. For an ideal I in $S = k[x_1, \ldots, x_n]$ which is homogeneous with respect to the standard $\mathbf{N}^n$-grading ($\deg(x_i) = 1$), the following invariant of $R = S/I$ was introduced by Backelin in [Ba2]:

$$\operatorname{rate}(R) := \sup\left\{\frac{a_i - 1}{i - 1} \,\middle|\, i \geqslant 2\right\}, \quad \text{where} \quad a_i := \max\{j \mid \operatorname{Tor}_i^T(k,k)_j \neq 0\}.$$

The rate of R measures the degree complexity of the infinite minimal free resolution of k over R, and plays a similar role to that played by *(Castelnuovo–Mumford) regularity* for finite graded resolutions. If I is a monomial ideal then Backelin stated that $\operatorname{rate}(S/I) \leqslant d - 1$, where d is the maximal degree of a minimal generator of I, cf. [ERT, Prop. 3]. This fact implies a vanishing theorem for the homology of $\overline{L}_{\mathcal{A}_I}$:

COROLLARY 5.1. *Let I be a monomial ideal in S and d be the maximal degree of a minimal generator of I. Let $\mathcal{B}_I$ be its canonical arrangement intersected with $\sum_i u_i = 0$ in $\mathbf{R}^n$. Then for any field k we have*

$$\tilde{\mathrm{H}}^i(\overline{L}_{\mathcal{B}_I}; k) = 0 \quad \textit{for} \quad i < \frac{n-1}{d-1} - 2.$$

The reason for considering the intersection of $\mathcal{A}_I$ with $\sum_i u_i = 0$ instead of $\mathcal{A}_I$ itself is that $\mathcal{A}_I$ is never essential, because the line $u_1 = \cdots = u_n$ is in the intersection of all its subspaces. This means that the proper part of its intersection lattice would be a cone and have no homology, so the vanishing property would be vacuously true.

Proof. By Backelin's result

$$\operatorname{Tor}_i^{S/I}(k,k)_j = 0 \quad \text{if} \quad j > (d-1)(i-1) + 1,$$

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where the subscript j refers to the usual **N**-grading by total polynomial degree on S/I and on $\mathrm{Tor}_*^{S/I}(k,k)$. Since $\mathrm{Tor}_i^{S/I}(k,k)_n$ contains $\mathrm{Tor}_i^{S/I}(k,k)_{x_1\cdots x_n}$ in our $\mathbf{N}^n$-graded notation, we conclude that

$$\mathrm{Tor}_i^{S/I}(k,k)_{x_1\cdots x_n} = 0 \quad \text{if} \quad n > (d-1)(i-1)+1.$$

Equation (2.1) from the proof of Theorem 1.3 allows us to rewrite this as

$$\tilde{\mathrm{H}}_{i-3}(\mathbf{S}^{n-2} \cap \mathcal{B}_I; k) = 0 \quad \text{if} \quad n > (d-1)(i-1)+1,$$
$$\tilde{\mathrm{H}}_i(\mathbf{S}^{n-2} \cap \mathcal{B}_I; k) = 0 \quad \text{if} \quad i < \frac{n-1}{d-1} - 2.$$

On the other hand, Corollary 2.5 of [ZZ] shows that $\tilde{\mathrm{H}}^i(\overline{L}_{\mathcal{B}_I}; k)$ is a direct summand in $\tilde{\mathrm{H}}_i(\mathbf{S}^{n-2} \cap \mathcal{B}_I; k)$, so the theorem follows. □

Inspired by Corollary 5.1, the next result generalizes it.

THEOREM 5.2. *Let* **F** *be any field,* $\mathcal{A}$ *an arrangement of linear subspaces in* $\mathbf{F}^m$*, and assume every maximal subspace in* $\mathcal{A}$ *has codimension at most* c*. Then*

$$\tilde{\mathrm{H}}^i(\overline{L}_{\mathcal{A}}; \mathbf{Z}) = 0 \quad \textit{for} \quad i < \frac{m}{c} - 2.$$

Proof. We can first reduce to the case where $L_{\mathcal{A}}$ is an *atomic* lattice, meaning that every element of L is the join of the elements below it which cover $\hat{0}$, or equivalently, every subspace in $\mathcal{A}$ is the intersection of the maximal subspaces in $\mathcal{A}$ containing it. To achieve this reduction, consider the *closure relation* on L defined by sending any subspace in L to the join of the elements covering $\hat{0}$ which lie below it. The closed sets $L' \subseteq L$ form a sublattice, and it is well known that the inclusion of the proper parts $\overline{L'} \hookrightarrow \overline{L}$ is a homotopy equivalence (see [BWa, Lem. 7.6]).

So assume that $L_{\mathcal{A}}$ is atomic, and let H be a maximal subspace in $\mathcal{A}$, i.e. an atom of $L_{\mathcal{A}}$. Our method is essentially a *deletion-contraction* induction on the number of subspaces in $\mathcal{A}$, in which we apply Mayer–Vietoris to the following decomposition $\overline{L} = X \cup Y$: $\overline{L} = \big(\overline{L} - \{H\}\big) \cup (\overline{L})_{\geqslant H}$, where $(\overline{L})_{\geqslant H}$ denotes the subposet of elements in $\overline{L}$ which lie weakly above H. Note that $\big(\overline{L} - \{H\}\big) \cap (\overline{L})_{\geqslant H} = (\overline{L})_{>H} \cong \overline{L}_{\mathcal{A}|_H}$, where $\overline{L}_{\mathcal{A}|_H}$ is the proper part of the intersection lattice for the arrangement of subspaces $\mathcal{A}|_H := \{V \cap H : V \in \mathcal{A}\}$, sitting inside the ambient space H. Also, we can define a closure relation on $\overline{L} - \{H\}$ which sends a subspace to the intersection of all subspaces of $\mathcal{A}$ other than H which contain it. Then the inclusion of the closed sets $\overline{L}_{\mathcal{A}-\{H\}} \hookrightarrow \overline{L}_{\mathcal{A}} - \{H\}$ induces a homotopy equivalence, where $\overline{L}_{\mathcal{A}-\{H\}}$ is the proper part of the intersection lattice for the arrangement $\mathcal{A} - \{H\}$. We conclude that part of the Mayer–Vietoris exact sequence looks like this:

$$\tilde{\mathrm{H}}_i(\overline{L}_{\mathcal{A}-\{H\}}; \mathbf{Z}) \oplus \tilde{\mathrm{H}}_i((\overline{L})_{\geqslant H}; \mathbf{Z}) \to \tilde{\mathrm{H}}_i(\overline{L}_{\mathcal{A}}; \mathbf{Z}) \to \tilde{\mathrm{H}}_{i-1}(\overline{L}_{\mathcal{A}|_H}; \mathbf{Z}).$$

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Since the poset $(\overline{L})_{\geqslant H}$ has a bottom element H, it is topologically a cone, and hence has no (reduced) homology. We can apply induction to $\mathcal{A} - \{H\}$ to conclude that $\tilde{\mathrm{H}}_i(\overline{L}_{\mathcal{A}-\{H\}}; \mathbf{Z})$ vanishes for $i < (m/c) - 2$. The codimensions (within H) of all the subspaces $V \cap H$ are again bounded by c since

$$\dim_{\mathbf{F}} H + \dim_{\mathbf{F}} V \leqslant \dim_{\mathbf{F}} V \vee H + \dim_{\mathbf{F}} V \wedge H$$

$$\dim_{\mathbf{F}} H - \dim_{\mathbf{F}} V \vee H \leqslant \dim_{\mathbf{F}} V \wedge H - \dim_{\mathbf{F}} V$$

$$\dim_{\mathbf{F}} H - \dim_{\mathbf{F}} V \cap H \leqslant m - \dim_{\mathbf{F}} V$$

$$\dim_{\mathbf{F}} H - \dim_{\mathbf{F}} V \cap H \leqslant c.$$

Thus, we can also apply induction to $\mathcal{A}|_H$. Note that since $\dim_{\mathbf{F}} H \geqslant m - c$, induction says that $\tilde{\mathrm{H}}_{i-1}(\overline{L}_{\mathcal{A}|_H}; \mathbf{Z})$ will vanish for $i - 1 < (m - c/c) - 2$, that is for $i < (m/c) - 2$. Thus the term $\tilde{\mathrm{H}}_i(\overline{L}_{\mathcal{A}}; \mathbf{Z})$ in the exact sequence is surrounded by terms, which vanish for $i < (m/c) - 2$, and the result follows. □

REMARKS 5.3

(1) To see that the vanishing theorem is sharp for every c, take arrangements of a maximal number of subspaces of codimension c which are pairwise orthogonal.
(2) The case of the theorem where $c = 1$ follows from a well-known result of Folkman [Fo] since in this instance $L_{\mathcal{A}}$ is known to be a *geometric lattice*.
(3) Using the formulas of Ziegler–Živaljević [ZZ] and Goresky–MacPherson [ZZ, Corol. 2.5] which express the homology of $\mathbf{R}^m - \mathcal{A}$ and $\mathbf{S}^{m-1} \cap \mathcal{A}$ in terms of the homology of the lower intervals in the intersection lattice L_A, one obtains other new and interesting vanishing theorems.
(4) It is known that every finite lattice L is isomorphic to $L_{\mathcal{A}}$ for some $\mathcal{A}$, so one can think of the theorem as an *embedding criterion* – it gives a lower bound for the codimension of the subspaces one will need to use in $\mathcal{A}$. The bound is based on the homology of $\overline{L}$ and the dimension of the ambient space. □

Abstracting the essential features from the proof of Theorem 5.2 we obtain the following more general result:

THEOREM 5.4. *Let L be a finite lattice with a function $r: L \to \mathbf{N}$ which is semimodular $r(x)+r(y) \leqslant r(x \vee y)+r(x \wedge y)$, and order-preserving, with $r(\hat{0}) = 0$, $r(\hat{1}) = m$ and $r(x) \leqslant c$ for all atoms x in L. Then $\tilde{\mathrm{H}}_i(\overline{L}; \mathbf{Z}) = 0$ for $i < (m/c) - 2$.*

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