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ON THE EXISTENCE OF PREMIXED LAMINAR FLAMES

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In this article, the existence of travelling wave solutions for premixed laminar flames in a model of slow, "constant density" combustion is studied. The model is governed by a simple system of an exothermic chemical reaction in a gas, via the reaction rate function, which is very natural, as we do not impose the assumption of its continuity. The existence of travelling waves is demonstrated and they are shown to be specific heteroclinic orbits of a three dimensional system of ordinary differential equations, connecting the unburned state points to a burned state point. The existence of these solutions is based on some general topological arguments in ordinary differential equations.

1. INTRODUCTION

One of the most important problems of combustion involves planar premixed flames, that is, one-dimensional deflagration waves. In a case of a single-step reaction involving one reactant, it reduces to a system of two reaction diffusion equations ([**10, 11**]). The existence of travelling wave solutions for this system has been established for both positive and zero ignition temperature ([**3, 4, 7, 11, 15, 16, 18, 19, 20**]). Here, we shall discuss the existence of travelling wave solutions in a model for slow, "constant density" combustion. Alternatively, the travelling wave solutions for this model may be derived from a more complicated system, assuming only a slow speed of propagation, and weak temperature and pressure variation ([**3, 4, 20**]). The model is a simple model of an exothermic chemical reaction in a gas and is as follows (for a background on the physical motivation and derivation of the following model, see Buckmaster and Ludford [**6**], Larrouturou [**10**], Wagner [**19, 20**] and Williams [**23**]):

$$\begin{cases} Y_t = (\nu(Y,T)Y_x)_x - DY\Phi(T), \\ T_t = (\lambda(Y,T)T_x)_x + qDY\Phi(T), \end{cases} \tag{1.1}$$

where T is the temperature and Y the mass fraction of the unburned gas. Note that the completely unburned state corresponds to $Y = Y_f$ and a totally burned state corresponds

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to $Y = 0$. Also the physically desirable values of the unknowns Y and T are non-negative. In fact, we consider $0 \leqslant Y \leqslant Y_f$ and $0 \leqslant T \leqslant T_b$ (see [**20**]). The parameters ν, λ, D and q are positive. The independent variables t and x are the time and space variables, respectively. Finally, we encounter the well known cold boundary difficulty, that is, the unburned state is not a stationary point of (1.1) since the "reaction rate function" $\Phi(T) \neq 0$, for $T > 0$. One resolution of the cold boundary difficulty can be based on activation energy asymptotic (see [**23**]). However, in our analysis we use the common mathematical idealisation of an ignition temperature, Φ is modified such that (see [**9, 13**])

$$\Phi(T) = \begin{cases} 0 & \text{for } T < T_i, \\ \Phi_1(T) & \text{for } T \geqslant T_i, \end{cases} \tag{1.2}$$

where $\Phi_1(T)$ is a smooth positive function and T_i the "ignition temperature" of the reaction. A typical example for $\Phi_1(T)$ is the Arrhenius law, that is $\Phi_1(T) = T^\gamma e^{-(A/T)}$ for some positive constants γ and A. Notice that $\Phi(T)$ is discontinuous at the point T_i. A careful discussion of this assumption and its consequences for detonation and deflagration wave (with one-step chemistry) can be found in [**10**] and [**13**].

The system (1.1), with $\Phi(T)$ in a very simple form, has received extensive mathematical treatment in recent years. Berestycki, Nicolaenko and Scheurer [**3**] proved the existence of a solution of (1.1). Also in [**4**] they considered the deflagration wave problem for a compressible reacting gas, with one reactant involved in a single-step chemical reaction. They showed how the one-dimensional travelling wave problem reduces to a system of two reaction-diffusion equations. Wagner [**20**] obtained a sufficient condition for the existence of travelling waves representing premixed laminar flames. In order to do this, he used a topological method in his article. The necessary condition has been given by Marion [**11**]. The existence of travelling wave solutions of (1.1) was established by Terman [**18**] in the case $T_i = 0$. Also stability and instability results for the travelling waves, where

$$\Phi(T) = \begin{cases} 0 & \text{for } T < 0, \\ Be^{-(E/T)} & \text{for } T \geqslant 0, \end{cases} \tag{1.3}$$

have been obtained by Clavin [**7**], Sivashinsky [**16**], and Roquejoffre and Terman [**15**]. Avrin [**1**] studied the equations with initial data that are bounded, uniformly continuous, and nonnegative but otherwise arbitrary. He established the existence of unique global strong solutions satisfying appropriate a priori estimates. With a positivity condition imposed on the initial data for the temperature, he showed that the concentration decays exponentially. Also in [**2**] he studied the qualitative behaviour of solutions to the initial-boundary value problem for the reaction-diffusion equations (1.1). In both cases where T and Y satisfy zero Neumann boundary conditions or fixed Dirichlet boundary conditions, extensive qualitative results have been given concerning complete asymptotic burning and eventual quenching. Weber, Mercer, Sidhu and Gray [**22**] considered (1.1) and assumed that the chemical reaction can be represented by the Arrhenius rate law. They applied a different non-dimensionalisation, and used the ratio of the activation energy to the heat of the reaction as a large parameter around which an asymptotic analysis was based. Mercer, Weber and Sidhu [**12**] then used this non-dimensionalisation to study the effects of heat loss on the routes to extinction of the combustion wave, given that the activation energy is large or that the heat of the reaction is small. Finally, Billingham and Mercer [**5**] investigated (1.1). They used the method of matched asymptotic expansions to obtain asymptotic approximations for the permanent form travelling wave solutions and their results were confirmed numerically.

Now, assume the reaction is exothermic and the reaction rate function $\Phi(T)$ is given by (1.2). Furthermore, consider that the liberated energy, q depends on T. The existence of premixed laminar flames is proved by the existence of travelling wave solutions. These waves are heteroclinic orbits between specific rest points of a three dimensional system of ordinary differential equations. The proof is carried out by using some general topological arguments in ordinary differential equations. Finally, the uniqueness of these waves is considered.

The rest of the paper is organised as follows. In Section 2, we introduce the hypotheses and the problem, then make some observations related to the problem. In Section 3 we shall show that travelling wave solutions for premixed laminar flames exist.

2. The hypotheses and the problem

A solution $\big(Y(x,t),T(x,t)\big)^T$ of (1.1) is called a travelling wave solution between two states $(Y_l,T_l)^T$ and $(Y_r,T_r)^T$, if there is a constant $s\in\mathbb{R}$, which is called the speed of combustion shock wave, satisfying "the jump and entropy conditions", moreover this solution depends only on the variable $\xi = x+st$ [**17**]. This means that a travelling wave solution of (1.1) has the following form:

$$\big(Y(x+st),T(x+st)\big)^T.$$

Thus in order to obtain travelling wave solutions, (1.1) reduces to the following system of equations:

$$\begin{cases} sY_\xi = (\nu Y_\xi)_\xi - DY\Phi(T), \\ sT_\xi = (\lambda T_\xi)_\xi + qDY\Phi(T). \end{cases} \tag{2.1}$$

Let $Z=\nu Y_\xi - sY$ and $Z_1 = \lambda T_\xi - sT$ be auxiliary variables, then we obtain:

$$\begin{cases} \nu Y_\xi = Z + sY, \\ \lambda T_\xi = Z_1 + sT, \\ Z_\xi = DY\Phi(T), \\ Z_{1\xi} = -qDY\Phi(T). \end{cases} \tag{2.2}$$

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From the two last equations of (2.2), we have:

$$qZ_\xi + Z_{1\xi} = 0. \tag{2.3}$$

The above equation can be integrated once to give:

$$qZ + Z_1 = c, \tag{2.4}$$

where c is a constant of integration. Since $Y_\xi(-\infty) = Y_\xi(+\infty) = T_\xi(-\infty) = T_\xi(+\infty) = 0$, it follows from (2.2) and (2.4), $c = -sT_b$ at the burned state and on the other hand $c = -qsY_f$ at the unburned state, so we must have

$$T_b = qY_f. \tag{2.5}$$

By using the relations (2.4) and (2.5), (2.2) reduces to a three dimensional system of ordinary differential equations as follows:

$$\begin{cases} \nu Y_\xi = Z + sY, \\ \lambda T_\xi = -sT_b - qZ + sT, \\ Z_\xi = DY\Phi(T). \end{cases} \tag{2.6}$$

For simplicity, we replace Y, Z and D by $Y_f(1-X)$, $sY_f(W-1)$ and β^{-1}. In this way (2.6) becomes:

$$\begin{cases} \nu \dot{X} = s(X - W), \\ \lambda \dot{T} = s(T - T_b W), \\ \beta \dot{W} = \dfrac{1}{s}(1 - X)\Phi(T), \end{cases} \tag{2.7}$$

where "." means $d/(d\xi)$. In order for a solution $u(\xi) = \big(X(\xi), T(\xi), W(\xi)\big)^T$ to be a travelling wave solution from the state $u_l = (X_l, T_l, W_l)^T$ to the state $u_r = (X_r, T_r, W_r)^T$, satisfying the jump condition we must have $\lim_{\xi\to-\infty} u(\xi) = u_l$ and $\lim_{\xi\to+\infty} u(\xi) = u_r$. That is u_l and u_r must be two rest points of (2.7). Thus the existence of travelling wave solutions of (2.7) is proved in two steps. Firstly, we must find all of the rest points of (2.7) and secondly we must obtain all solutions of this system which connect a rest point corresponding to $0 \leqslant X < 1$ to a rest point corresponding to $X = 1$ as ξ increases from $-\infty$ to $+\infty$, whenever $\nu > 0, \lambda > 0, \beta > 0$. Now, for simplicity we let

$$\begin{cases} u = (X, T, W)^T, \\ G_1(u) = s(X - W), \\ G_2(u) = s(T - T_b W), \\ G_3(u) = \dfrac{1}{s}(1 - X)\Phi(T). \end{cases} \tag{2.8}$$

Then (2.7) can be written as:

$$A\dot{u} = G(u), \tag{2.9}$$

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where A=diag(ν, λ, β) and $G(u) = \big(G_1(u), G_2(u), G_3(u)\big)^T$.

At a rest point we must have:

$$\text{(2.10)} \qquad \begin{cases} s(X - W) = 0, \\ s(T - T_b W) = 0, \\ \dfrac{1}{s}(1 - X)\Phi(T) = 0, \end{cases}$$

The last equation of (2.10), at a rest point, will implies that $X = 1$ or $T < T_i$ (notice $\Phi(T) = 0$ for $T < T_i$ (T_i is the ignition temperature), and this set is contained in the region $0 \leqslant X < 1$).

CASE 1. $X = 1$. First equation of (2.10) implies $W = 1$ and then from the second equation of (2.10) we obtain $T = T_b$.

CASE 2. $\Phi(T) = 0$. In this case, at the rest point, we must have $T < T_i$. Then the first two equations of (2.10) give a set of rest points. To find this set, put $X = m$, $0 \leqslant m < 1$, in the last equation of (2.10) then the first of the two equations will imply that $W = m$ and consequently $T_m = mT_b$ for every $0 \leqslant m < 1$. Also from now on, we assume that the ignition temperature, T_i satisfies the inequalities:

$$\text{(2.11)} \qquad 0 < T_i < T_b.$$

By considering the above results, the rest points of (2.7) are:

$$\text{(2.12)} \qquad \begin{cases} u_1 = (1, T_b, 1)^T, \\ u_m = (m, T_m, m)^T, \ 0 \leqslant m < 1, T_m < T_i. \end{cases}$$

In the next section, we show that for a general discontinuous reaction rate function $\Phi(T)$, the travelling wave solutions for premixed laminar flames exist. In another words, there is a heteroclinic orbit of (2.10) which is running from u_l to u_m for some $0 \leqslant m < 1$.

3. Existence of travelling waves for premixed laminar flames

In this section we shall show that the travelling waves for premixed laminar flames exist. In order to do this, we make some observations related to the nature of the stable manifold of (2.7) at the rest point u_1. For this, the linearised system of (2.7) at the rest point u_1, can be written as

$$\dot{u} = M(u - u_1),$$

where

$$M = \begin{bmatrix} \dfrac{s}{\nu} & 0 & -\dfrac{s}{\nu} \\ 0 & \dfrac{s}{\lambda} & -\dfrac{T_b s}{\lambda} \\ -\dfrac{1}{\beta s}\Phi_1(T_b) & 0 & 0 \end{bmatrix}.$$

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Notice that the entries of the matrix are considered at the rest point u_1. Let $f(\eta)$ be the characteristic polynomial of this matrix. So we have:

$$f(\eta) = \Big(\frac{s}{\lambda} - \eta\Big)\Big(\eta^2 - \frac{s}{\nu}\eta - \frac{1}{\beta\nu}\Phi_1(T_b)\Big). \tag{3.1}$$

This shows that the eigenvalues of the matrix M are:

$$\begin{cases} \eta_1 = \dfrac{s}{\lambda}, \\ \eta_2 = \dfrac{1}{2}\Big(\dfrac{s}{\nu}\sqrt{(\dfrac{s}{\nu})^2 + \dfrac{4}{\beta\nu}\Phi_1(T_b)}\Big), \\ \eta_3 = \dfrac{1}{2}\Big(\dfrac{s}{\nu} - \sqrt{(\dfrac{s}{\nu})^2 + \dfrac{4}{\beta\nu}\Phi_1(T_b)}\Big). \end{cases} \tag{3.2}$$

From now on, we assume $s > 0$, then we shall have $\eta_1 > 0, \eta_2 > 0$ and $\eta_3 < 0$. Thus the following theorem is proved.

THEOREM 3.1. *Let η_k, $1 \leqslant k \leqslant 3$, be the eigenvalues of the matrix M at the rest point $u_1 = (1, T_b, 1)$ and $s > 0$. Then $\eta_1 > 0$, $\eta_2 > 0$ and $\eta_3 < 0$.*

Concerning the eigenvectors at this rest point we have the following theorem.

THEOREM 3.2. *Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the negative eigenvalue η_3, then either $y_1 > 0$, $y_2 > 0$ and $y_3 > 0$ or the reverse inequalities hold.*

PROOF: The eigenvector $(y_1, y_2, y_3)^T$ must satisfy the following equations at the rest point u_1.

$$\begin{cases} \Big[\dfrac{s}{\nu} - \eta_3\Big]y_1 - \dfrac{s}{\nu}y_3 = 0, \\ \Big(\dfrac{s}{\lambda} - \eta_3\Big)y_2 - \dfrac{sT_b}{\lambda}y_3 = 0, \\ \Big(-\dfrac{1}{\beta s}\Phi_1(T_b)\Big)y_1 - \eta_3 y_3 = 0. \end{cases} \tag{3.3}$$

From the last equation of (3.3) we have $\operatorname{sgn} y_1 = \operatorname{sgn} y_3$. By considering the second equation of (3.3) we have $\operatorname{sgn} y_2 = \operatorname{sgn} y_3$. Thus $\operatorname{sgn} y_1 = \operatorname{sgn} y_2 = \operatorname{sgn} y_3$. ∎

In order to show the existence of travelling waves we define

$$D = \big\{u \in \mathbb{R}^3 : G_1(u) > 0,\ G_2(u) > 0,\ 0 < X < 1,\ 0 < W < 1,\ T \leqslant T_b\big\}.$$

Notice that the rest point u_1 is located on ∂D. Moreover, by Theorem 3.1, the stable manifold at u_1 is one dimensional. Concerning this manifold we have the following lemma.

LEMMA 3.1. *Let D be as above. Then the stable manifold at u_1 intersects D on a curve.*

PROOF: As we have shown before, the linearised system of (2.7) at the rest point u_1 has the following form:

$$(3.4)\qquad \begin{cases} \dot{X} = \frac{s}{\nu}(X-1) - \frac{s}{\nu}(W-1) = \frac{s}{\nu}(X-W) =: G_{1l}(u), \\ \dot{T} = \frac{s}{\lambda}(T-T_b) - \frac{T_b s}{\lambda}(W-1) = \frac{s}{\lambda}(T-T_bW) =: G_{2l}(u), \\ \dot{W} = -\frac{1}{s\beta}(X-1)\Phi_1(T_b) = \frac{1}{s\beta}(1-X)\Phi_1(T_b) =: G_{3l}(u), \end{cases}$$

or briefly,

$$\dot{u} = M\big(u-u_1\big) = \big(G_{1l}(u), G_{2l}(u), G_{3l}(u)\big)^T.$$

Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the negative eigenvalue $\eta_3 = \Big(s - \sqrt{s^2 + \frac{4\nu}{\beta}\Phi_1(T_b)}\Big)\Big/2\nu$. Now consider the solution:

$$u(\xi) = \big(X(\xi), T(\xi), W(\xi)\big)^T = (y_1, y_2, y_3)^T e^{\eta_3\xi} + u_1,$$

of the linear system (3.4). Then $u(\xi) \in D_s$ where

$$D_s = \big\{u \in \mathbb{R}^3 : G_{1l}(u) > 0,\ G_{2l}(u) > 0,\ G_{3l}(u) > 0\big\}.$$

To see this notice that:

$$\big(G_{1l}(u), G_{2l}(u), G_{3l}(u)\big)^T = M(u-u_1) = MYe^{\eta_3\xi} = \eta_3 Y e^{\eta_3\xi} = (\eta_3 y_1, \eta_3 y_2, \eta_3 y_3)^T e^{\eta_3\xi},$$

where M is the coefficient matrix of (3.4). By Theorem 3.2, we may assume that $y_1 < 0, y_2 < 0$ and $y_3 < 0$. Since $\eta_3 < 0$, it follows from the last equality that $\big(G_{1l}(u), G_{2l}(u), G_{3l}(u)\big) \in D_s$. This means that the stable manifold of (3.4), at the rest point u_1, which is the line:

$$M_s = \big\{u \in \mathbb{R}^3 :\ \ u - u_1 = (y_1, y_2, y_3)^T c, c \in \mathbb{R}\big\},$$

lies in D_s for $c > 0$ and lies in D for $c > 0$ and small. Thus the stable manifold of (2.7) at the rest point u_1 intersects D on a curve. ∎

Now consider the following system of ordinary differential equations

$$(3.5)\qquad \begin{cases} \nu\dot{X} = G_1(u), \\ \lambda\dot{T} = G_2(u), \\ \beta\dot{W} = (1-X)\frac{1}{s}\Phi_1(T) =: \widetilde{G}_3(u). \end{cases}$$

where $G_k(u)$, $k = 1, 2$, are defined by (2.8) and $\Phi_1(T)$ by (1.2). Notice that the above system is, mathematically, well defined for all $X > 0, T > 0$ and $W > 0$, moreover it is the same with (2.7) for $0 < X < 1$ and $T > T_i$. This system leads us to the proof of the existence of travelling waves for premixed laminar flames. In order to do this, we shall prove the next Lemma which is similar to [8, Lemma 3.4].

Lemma 3.2. *Let D be as above. Then there is a unique orbit of (3.5) which lies in D, its ω-limit set is u_1, and this orbit intersects the set $\Delta = \{u \in \overline{D} : G_1(u) > 0, G_2(u) > 0$ and $W = 0\}$. Along this orbit $X(\xi), T(\xi)$ and $W(\xi)$ are increasing.*

Proof: The proof is organised in the following steps.

Step 1. The system (3.5) is gradient-like with respect to $h(u) = W$ in D and is locally Lipschitz in a neighbourhood of $\overline{D}$.

Step 2. Notice that D is homeomorphic to the cylinder $\{V \in \mathbb{R}^3 : v_1^2 + v_2^2 < 1,\ 0 < v_3 < 1\}$ and so $\{u \in \overline{D} : h(u) = c\}$ corresponds to the set $\{V \in \mathbb{R}^3 : v_1^2 + v_2^2 < 1,\ v_3 = c\}$ for $c \in [0, 1]$ under the homeomorphism.

Step 3. The rest point u_1 is the only rest point of (3.3) which is located in the set $\{u \in \overline{D} : h(u) = 1\}$.

Step 4. The flow goes out of D on $\{u \in \partial D : 0 < h(u) < 1\}$. To see this, let $u_0 \in \{u \in \partial D : 0 < W < 1\}$. Then $G_1(u_0) = 0$ or $G_2(u_0) = 0$ or $X = 0$ or $X = 1$ or $T = T_b$. Now, suppose $G_1(u_0) = 0$. If we differentiate $G_1(u)$ along the orbits of (3.5) we obtain:

$$\frac{dG_1(u)}{d\xi}_{|G_1(u_0)=0} = -\frac{s}{\nu}\widetilde{G}_3(u_0) < 0.$$

Thus the flow goes out of $\overline{D}$ on $G_1(u_0) = 0$. Let $G_2(u_0) = 0$ and differentiate $G_2(u)$ along the orbits to obtain:

$$\frac{dG_2(u)}{d\xi}_{|G_2(u_0)=0} = -\frac{sT_b}{\lambda}\widetilde{G}_3(u_0).$$

Hence, the flow goes out of $\overline{D}$ on $G_2(u_0) = 0$. If we differentiate T along the orbits we get:

$$\frac{dT}{d\xi}_{|T=T_b} = \frac{s}{\nu}(T_b - T_bW) = \frac{s}{\nu}T_b(1 - W) > 0.$$

Hence, the flow goes out of $\overline{D}$ on $T = T_b$, and also if we differentiate X along the orbits to obtain:

$$\frac{dX}{d\xi}_{|X=0} = \frac{s}{\nu}(0 - W) = -\frac{s}{\nu}W < 0,$$
$$\frac{dX}{d\xi}_{|X=1} = \frac{s}{\nu}(1 - W) > 0.$$

Therefore the flow goes out of $\overline{D}$ on $X = 0$ and $X = 1$.

Step 5. The stable manifold of (3.5) at the rest point u_1 intersects D in a nonempty set.

Proof of the next step is similar to the proof of [**14**, Theorem 3.1].

Step 6. Consider Steps 1-5, then there is a point $p \in \{u \in \partial D : h(u) = 0\}$ such that $\lim\limits_{\xi \to \infty} p.\xi = u_1$.

PROOF OF STEP 6: Let q be a point on the stable manifold of (3.5) at the rest point u_1 which is located in D. Then $\lim_{\xi\to\infty} q.\xi = u_1$. Moreover, the orbit $q.\xi$ is defined for $\xi < 0$ as long as this orbit lies in D. Since $h(q) > 0$, this orbit cannot approach to the surface $\{u \in \partial D : h(u) = 1\}$ as ξ decreases from 0. On the other hand, by Step 4, this orbit cannot approach to the surface $\{u \in \partial D : 0 < h(u) < 1\}$ either, as ξ decreases. Finally, this orbit cannot stay in D for all $\xi < 0$, otherwise there must be a rest point of (3.5) in $\{u \in \overline{D} : 0 \leqslant h(q) \leqslant h(u) < 1\}$. But step 3 shows that such a rest point is disallowed to exist. Therefore there is a $\xi_0 < 0$ such that $q.\xi_0 \in \{u \in \partial D : h(u) = 0\}$. Let $p = q.\xi_0$. Then $p.\xi$ is the desired orbit.

Thus by the step 6, there must be an orbit of (3.5) lying in D, initiating at a point on the surface $W = 0$ and running to the point u_1 as $\xi \to +\infty$. Finally, (3.5) and the set D shows that along these orbits $X(\xi), T(\xi)$ and $W(\xi)$ are increasing. ∎

Let $\widetilde{u}(\xi), \xi \in [\xi_0, \infty)$ be the orbit which is given by the above lemma. Then $\widetilde{u}(\xi_0) \in \{u \in \overline{D} : W = 0\}$ and $\lim_{\xi\to\infty} \widetilde{u}(\xi) = u_1$. Concerning the orbit $\widetilde{u}(\xi)$ we have the following lemma.

LEMMA 3.3. *Let $\widetilde{u}(\xi)$ be as above and ν, λ and β be positive. If $\beta \gg \max(\lambda, \nu)$, then the orbit $\widetilde{u}(\xi)$ meets the hypersurface $T = T_i$ for some $\widetilde{\xi} \in (\xi_0, \infty)$ with $0 < \widetilde{X}(\widetilde{\xi}) < 1$ and $0 < \widetilde{W}(\widetilde{\xi}) < 1$.*

PROOF: Let $(X_i, T_i, W_i)^T$ be the unique solution of the equation

$$X - W = 0,\ T - T_b W = 0 \text{ and } T = T_i.$$

Therefore $W_i = T_i/T_b$ and $X_i = T_i/T_b$. Then from $0 < T_i < T_b$ we have $0 < X_i < 1, 0 < W_i < 1$ and $\{u \in \overline{D} : G_1(u) = 0, G_2(u) = 0, 0 < W < W_i\} \subset \{u \in \overline{D} : T \leqslant T_i\}$. Let $0 < X_0 < X_i, 0 < W_0 < W_i, D_0 = \{u \in D : 0 < W < W_0, T > T_i\}$ and $\delta = \min_{u\in\overline{D}_0} [G_2(u) + G_1(u)]$. Then $\delta > 0$ (because if $\delta = 0$, then $G_2(u) = G_1(u) = 0$ this means $X = W$ and $T = T_b W$. Thus $T = T_b X$. If $T > T_i$ then $T_b X > T_i$ and then $X > T_i/T_b = X_i$ and this is a contradiction to the requirement that $0 < X < X_0 < X_i$).

Now suppose the orbit $\widetilde{u}(\xi)$, $\xi \in [\xi_0, +\infty)$, does not meet the set $\{u \in D : T = T_i, 0 < X < 1 \text{ and } 0 < W < 1\}$. Then from $G_2(u) > 0$ in D and $\lim_{\xi\to\infty} \widetilde{T}(\xi) > T_i$, we have $\widetilde{T}(\xi) > T_i$ for all $\xi \in [\xi_0, +\infty)$.

Let ξ_1 be the solution of the equation $\widetilde{W}(\xi) = W_0$, where $\widetilde{W}(\xi)$ is the third component of $\widetilde{u}(\xi)$. Since $\dot{\widetilde{W}} > 0$, thus $\widetilde{W}$ is increasing along the orbit $\widetilde{u}(\xi)$. Since $\widetilde{W}(\xi_0) = 0, \widetilde{W}(\xi_1) = W_0$, then $\xi_1 > \xi_0$ and for all $\xi \in (\xi_0, \xi_1), 0 < \widetilde{W}(\xi) < W_0$, this means that $\widetilde{u}(\xi)$ remains in D_0 for $\xi \in (\xi_0, \xi_1)$.

Now along the orbit $\widetilde{u}(\xi)$ in D_0 we must have:

$$\begin{aligned}\frac{d}{dW}[T+X] &= \frac{1}{(dW/d\xi)}\Big[\frac{dT}{d\xi}+\frac{dX}{d\xi}\Big]\\ &= \frac{\beta}{(1-X)\Phi_1(T)}\Big[\frac{G_2(u)}{\lambda}+\frac{G_1(u)}{\nu}\Big]\\ &\geqslant \frac{\beta\delta\eta}{\max(\lambda,\nu)}>0,\end{aligned}$$

where $1/\eta=\max_{u\in\overline{D}} G_3(u)$. Therefore

$$\begin{aligned}\widetilde{T}(\xi)+\widetilde{X}(\xi)|_{\xi_0}^{\infty} &= \int_{\xi_0}^{\infty}\Big[\frac{1}{\lambda}G_2(\widetilde{u}(\xi))+\frac{1}{\nu}G_1(\widetilde{u}(\xi))\Big]d\xi\\ &\geqslant \int_{\xi_0}^{\xi_1}\Big[\frac{1}{\lambda}G_2(\widetilde{u}(\xi))+\frac{1}{\nu}G_1(\widetilde{u}(\xi))\Big]d\xi\\ &= \int_0^{W_0}\frac{\beta}{(1-X)\Phi_1(T)}\Big[\frac{G_2(u)}{\lambda}+\frac{G_1(u)}{\nu}\Big]dW\\ &\geqslant \frac{\beta\delta\eta W_0}{\max(\lambda,\nu)}.\end{aligned} \tag{3.6}$$

On the other hand $\widetilde{T}(\infty)+\widetilde{X}(\infty)-\widetilde{T}(\xi_0)-\widetilde{X}(\xi_0)=T_b+1-\widetilde{T}(\xi_0)-\widetilde{X}(\xi_0)$. Since $G_2(u)>0$ and $G_1(u)>0$ in D, thus $T-T_bW\geqslant 0$ and $X-W\geqslant 0$, respectively. This shows that $T+X\geqslant T_bW+W$ or $\widetilde{T}(\xi_0)+\widetilde{X}(\xi_0)\geqslant T_b\widetilde{W}(\xi_0)+\widetilde{W}(\xi_0)$ or $-\widetilde{T}(\xi_0)-\widetilde{X}(\xi_0)\leqslant -T_b\widetilde{W}(\xi_0)-\widetilde{W}(\xi_0)$. Finally, we have:

$$(T_b+1)(1-\widetilde{W}(\xi_0))\geqslant T_b+1-\widetilde{T}(\xi_0)-\widetilde{X}(\xi_0).$$

Since $\widetilde{W}(\xi_0)=0$, thus the above result and (3.6) will imply:

$$T_b+1\geqslant\frac{\beta\delta\eta W_0}{\max(\lambda,\nu)}.$$

Notice that this inequality is impossible for $\beta\gg\max(\lambda,\nu)$ (this inequality makes sense, see [**21**, p. 1046] or [**24**]). Thus the proof is complete. ∎

From now on we assume that $\beta\gg\max(\lambda,\nu)$, or the orbit $\widetilde{u}(\xi)$ meets the surface $T=T_i$ at the point $\widetilde{u}_i=(\widetilde{X}_i,\widetilde{T}_i,\widetilde{W}_i)^T$, where $0<\widetilde{X}_i<1$ and $0<\widetilde{W}_i<1$. We call the point $\widetilde{u}_i$ the ignition point. Notice that this point is unique.

Now we have our main theorem as follows.

THEOREM 3.3. *Suppose that* (2.7) *admits the rest points* u_1 *and* u_m, *for some* $0\leqslant m<1$. *Then for given* $\lambda,\nu,\beta>0$ *with* $\beta\gg\max(\lambda,\nu)$, *there is a unique orbit of* (2.7) *which is running from* u_m *to* u_1, *for some* $0\leqslant m<1$.

PROOF: In the region $T<T_i$, the last equation of (2.7) becomes $\dot{W}=0$. Thus, in this region, along the orbits of this system $W(\xi)$ is constant. Here we let $W(\xi)=\widetilde{W}_i$,

where $\widetilde{W}_i$ is the third component of $\widetilde{u}_i$, the ignition point. On the surface $W = \widetilde{W}_i$, (2.7) reduces to the following two dimensional system of equations, in the region $T \leqslant T_i$:

$$\text{(3.7)} \qquad \begin{cases} \nu \dot{X} = s[X - \widetilde{W}_i], \\ \lambda \dot{T} = s[T - T_b \widetilde{W}_i]. \end{cases}$$

The solution of this system is as follow:

$$\text{(3.8)} \qquad \begin{cases} X(\xi) = \widetilde{W}_i + e^{(s/\nu)\xi}, \\ T(\xi) = T_b \widetilde{W}_i + e^{(s/\lambda)\xi}, \end{cases}$$

Now consider the region

$$D' = \left\{ (X,T) \in \mathbb{R}^2 : X > \widetilde{W}_i,\ T > T_b \widetilde{W}_i \text{ and } T + X < T_i + \widetilde{X}_i \right\}.$$

Therefore each orbit of (3.7) initiating at a point on $\partial D' \cap \left\{ (X,T) : T + X = T_i + \widetilde{X}_i \right\}$ lies in $\overline{D'}$ for $\xi < 0$ and goes to $(\widetilde{W}_i, T_b \widetilde{W}_i)$ as ξ tends to $-\infty$. Note that along this orbit $X(\xi)$ and $T(\xi)$ are increasing.

Now, consider again the ignition point $\widetilde{u}_i = (\widetilde{X}_i, T_i, \widetilde{W}_i)$. By the above argument, there is a unique orbit of (2.7), say

$$\approx{u}\,(\xi) = \left(\approx{X}\,(\xi), \approx{T}\,(\xi), \approx{W}\,(\xi)\right), \quad -\infty < \xi < \widetilde{\xi},$$

with

$$\begin{aligned} &\approx{u}\,(\widetilde{\xi}) = (\widetilde{X}_i, T_i, \widetilde{W}_i), \\ &\approx{W}\,(\xi) = \widetilde{W}_i, \quad \text{for} \quad \xi \leqslant \widetilde{\xi}, \end{aligned}$$

and

$$\lim_{\xi \to -\infty} \approx{u}\,(\xi) = (\widetilde{W}_i, T_b \widetilde{W}_i, \widetilde{W}_i).$$

Along this orbit $X(\xi)$ and $T(\xi)$ are increasing and $W(\xi)$ is constant. This orbit lies in D, the domain which is used in the proof of Lemma 3.1. Now define

$$u(\xi) = \begin{cases} \widetilde{u}(\xi) & \xi \geqslant \widetilde{\xi}, \\ \approx{u}\,(\xi) & \xi < \widetilde{\xi}. \end{cases}$$

Then $u(\xi)$ is a complete orbit of (2.7) lying in D and is running from u_m to u_1 for some $0 \leqslant m < 1$. This completes the proof. $\square$

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