EXISTENCE THEOREM FOR GROUPS

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(Received 18th June 1963)

1. Introduction

Given a partial automorphism of a group G, i.e. an isomorphic mapping μ of a subgroup A of G onto a second subgroup B of G, it is known (**2**, Theorem I) that there always exists a group H containing G and an inner automorphism of H which extends μ; i.e. there exists an element t of H, such that the transform by t of any element of A is its image under μ.

Also it is known (**2**, Theorem II) that this result generalises to the case in which G possesses any number, finite or infinite, of partial automorphisms. These can be simultaneously extended to inner automorphisms of one and the same supergroup.

In (**1**) conditions are derived which are sufficient for extending two partial automorphisms of a given group to commutative automorphisms (inner ones) of an extension supergroup.

In this paper we consider a given group G and a set of partial automorphisms $\mu(\sigma)$ of G, where σ ranges over an index set Σ and derive conditions which are sufficient for $\mu(\sigma)$ to be all extendable to inner automorphisms t_σ of one and the same supergroup $G^* \supseteq G$ such that for a fixed $\alpha \in \Sigma$, t_α commutes with every t_σ. An obviously necessary condition for this is that $\mu(\alpha)$ commutes with every $\mu(\sigma)$, $\sigma \in \Sigma$.

The sufficient conditions obtained, though not necessary, are wide enough to give Corollaries (1) and (2) as special cases.

2. First step of the construction

Let A_σ and B_σ, where σ ranges over an index set Σ, be subgroups of a given group G, and assume that for every $\sigma \in \Sigma$, μ_σ is an isomorphic mapping of A_σ onto B_σ. Let α be a fixed element of Σ, such that

$$(A_\alpha \cap A_\theta)\mu_\alpha = B_\alpha \cap A_\theta, \quad \text{.......(1)}$$

$$(A_\alpha \cap B_\theta)\mu_\alpha = B_\alpha \cap B_\theta, \quad \text{.......(2)}$$

$$(A_\alpha \cap A_\theta)\mu_\theta = A_\alpha \cap B_\theta, \quad \text{.......(3)}$$

$$(B_\alpha \cap A_\theta)\mu_\theta = B_\alpha \cap B_\theta, \quad \text{.......(4)}$$

for all $\theta(\neq \alpha) \in \Sigma$,

$$g\mu_\alpha\mu_\theta = g\mu_\theta\mu_\alpha \quad \text{.......(5)}$$

for all $\theta \in \Sigma$ and whenever $g\mu_\alpha$, $g\mu_\theta$, $(g\mu_\alpha)\mu_\theta$ and $(g\mu_\theta)\mu_\alpha$ are defined.

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Take a sequence of groups

$$\ldots, G^{(-1)}, G^{(0)}, G^{(1)}, G^{(2)}, \ldots$$

which are copies of the group G, i.e. each group $G^{(i)}$, $i = 0, \pm 1, \pm 2, \ldots$ is isomorphic to G under a fixed isomorphic mapping γ^i:

$$G^{(i)} = G\gamma^i.$$

Lemma. *$G^{(i)}$ contains, for every $\sigma \in \Sigma$, subgroups $A_\sigma^{(i)}$ and $B_\sigma^{(i)}$ which are isomorphic.*

Proof. Each group $G^{(i)}$ contains, for every $\sigma \in \Sigma$, subgroups $A_\sigma^{(i)}$ and $B_\sigma^{(i)}$ which are images of A_σ, B_σ respectively under the mapping γ^i. Let μ_σ^i be the mapping defined as follows. If

$$a_\sigma^{(i)} = a_\sigma\gamma^i, \quad a_\sigma \in A_\sigma,$$

$$b_\sigma^{(i)} = b_\sigma\gamma^i, \quad b_\sigma \in B_\sigma$$

and if

$$a_\sigma\mu_\sigma = b_\sigma,$$

then we put

$$a_\sigma^{(i)}\mu_\sigma^i = b_\sigma^{(i)}.$$

This is the mapping,

$$\mu_\sigma^i = (\gamma^i)^{-1}\mu_\sigma\gamma^i.$$

μ_σ^i is an isomorphism of A_σ^i onto B_σ^i.

For any two integers i, j such that $i<j$, we define by induction a sequence of groups $P^{i,j}$ as follows.

We first form the free product of $G^{(i)}$ and $G^{(i+1)}$ amalgamating $B_\alpha^{(i)} \subseteq G^{(i)}$ with $A_\alpha^{(i+1)} \subseteq G^{(i+1)}$ by putting $b_\alpha^{(i)} = a_\alpha^{(i+1)}$ whenever

$$b_\alpha^{(i)} = b_\alpha\gamma^i, \quad b_\alpha \in B_\alpha,$$

$$a_\alpha^{(i+1)} = a_\alpha\gamma^{i+1}, \quad a_\alpha \in A_\alpha$$

and

$$a_\alpha\mu_\alpha = b_\alpha.$$

Thus the isomorphism underlying the amalgamation is $(\gamma^i)^{-1}\mu_\alpha\gamma^{i+1}$. This mapping is defined on $B_\alpha^{(i)}$; moreover it is the identical mapping on $B_\alpha^{(i)}$.

Let

$$P^{i,i+1} = \{G^{(i)} * G^{(i+1)};\ B_\alpha^{(i)} = A_\alpha^{(i+1)}\},$$

and inductively

$$P^{i,j} = \{P^{i,j-1} * G^{(j)};\ B_\alpha^{(j-1)} = A_\alpha^{(j)}\}.$$

We then form the union

$$P^* = \bigcup_{n=1}^{\infty} P^{-n,+n}$$

and define the mapping μ_α^* as follows. For any $x \in P^*$, i.e. $x \in G^{(i)}$ for some suitable i, let $x(\gamma^i)^{-1} = g \in G$, $g\gamma^{i+1} = y$; then we put

$$x\mu_\alpha^* = y,$$

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which means that on $G^{(i)}$, μ_α^* is the mapping $(\gamma^i)^{-1}\gamma^{i+1}$.

Now we have the following result.

Theorem 1. *The group G is embedded in the supergroup* $P^* \supseteq G$ *which possesses an automorphism* μ_α^* *extending* μ_α *and for every* $\theta(\neq \alpha) \in \Sigma$, P^* *contains the subgroups*

$$A_\theta^* = \{\dots, A_\theta^{(-1)}, A_\theta^{(0)}, A_\theta^{(1)}, \dots\}$$

and

$$B_\theta^* = \{\dots, B_\theta^{(-1)}, B_\theta^{(0)}, B_\theta^{(1)}, \dots\}$$

which are isomorphic under a mapping $\bar{\mu}_\theta$ *such that for any* $x \in A_\theta^*$,

$$x\bar{\mu}_\theta\mu_\alpha^* = x\mu_\alpha^*\bar{\mu}_\theta. \qquad \dots\dots(6)$$

The proof of this theorem follows the same lines as that of lemmas 5-9 in (**1**). What is true there for μ holds here for μ_α and what is true there for ν holds here for every μ_θ, $\theta(\neq \alpha) \in \Sigma$.

3. Second step of the construction

Now we form the group

$$\tilde{P} = \{P^*, t_\alpha\}$$

generated by P^* and an element t_α and define

$$t_\alpha^{-1}p^*t_\alpha = p^*\mu_\alpha^*$$

for all $p^* \in P^*$.

Thus t_α induces an inner automorphism of P and equation (6) gives

$$t_\alpha^{-1}(x\bar{\mu}_\theta)t_\alpha = (t_\alpha^{-1}xt_\alpha)\bar{\mu}_\theta$$

which shows that the inner automorphism induced by t_α commutes with every $\bar{\mu}_\theta$, $\theta(\neq \alpha) \in \Sigma$.

For every $\theta(\neq \alpha) \in \Sigma$, define

$$t_\alpha\bar{\mu}_\theta = t_\alpha, \qquad \dots\dots(7)$$

thus $\bar{\mu}_\theta$ becomes an isomorphism of

$$\tilde{A}_\theta = \{A_\theta^*, t_\alpha\}$$

onto the group

$$\tilde{B}_\theta = \{B_\theta^*, t_\alpha\}$$

which also commutes with the inner automorphism induced by t_α.

Applying Theorem II, (**2**) we can embed G in a group G^* containing a group T freely generated by a set of elements t_θ, $\theta(\neq \alpha) \in \Sigma$, such that for any $\theta(\neq \alpha)$ in Σ the transform by t_θ of an element in A_θ is its image $\bar{\mu}_\theta$, i.e.

$$t_\theta^{-1}\tilde{a}_\theta t_\theta = \tilde{a}_\theta\bar{\mu}_\theta \qquad \dots\dots(8)$$

for any $\tilde{a}_\theta \in \tilde{A}_\theta$.

This means that t_θ induces an inner automorphism of G^* which extends $\bar{\mu}_\theta$ and thus extends μ_θ also.

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Putting $\tilde{a}_\theta = t_\alpha$ in (8) and making use of (7) we get

$$t_\theta^{-1} t_\alpha t_\theta = t_\alpha \bar{\mu}_\theta = t_\alpha;$$

thus t_α commutes with every t_θ. This completes the proof of the following theorem.

Theorem 2. *For the existence of an extension group G^* containing G and inner automorphisms t_σ of G^* extending μ_σ for every $\sigma \in \Sigma$ such that t_α commutes with every t_σ it is sufficient that relations* (2.1)-(2.5) *hold.*

4. Special cases

From Theorem 2, the following are immediate consequences.

Corollary 1. *With the previous notation, it is sufficient for the existence of an extension group with the required property that together with* (2.5) *the relations*

$$A_\alpha \cap A_\theta = A_\theta \cap B_\alpha = A_\alpha \cap B_\theta = B_\alpha \cap B_\theta = \{e\},$$

hold for all $\theta(\neq \alpha) \in \Sigma$; where e denotes the unit element of the group.

For then the conditions of Theorem 2 will be trivially satisfied.

Corollary 2. *Again, with the same notation, if A_σ coincides with B_σ for every $\sigma \in \Sigma$, i.e. if μ_σ maps A_σ onto itself then for the required extension to be effected, it is sufficient together with* (2.5) *that*

$$(A_\alpha \cap A_\sigma)\mu_\alpha = (A_\alpha \cap A_\sigma)\mu_\sigma = A_\alpha \cap A_\sigma$$

for every σ in Σ.

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https://doi.org/10.1017/S0013091500025578 Published online by Cambridge University Press