PASSMAN-ZALESSKII RADICAL OF GROUP ALGEBRAS

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Recently Passman (attributing the origin of the idea to Zalesskii) has defined the following ideal in a ring, (**2**).

Definition. For a unitary ring R,

$$N^*R = \{\alpha \in R \mid \alpha S \text{ is nilpotent for all finitely generated subrings } S \text{ of } R\}.$$

For a group algebra KG over a field K of characteristic $p \neq 0$, he has proved the radical property:

$$N^*(K(G)/N^*K(G)) = 0.$$

We shall therefore call $N^*K(G)$, the Passman-Zalesskii Radical (PZ-Radical, in short) of KG.

If one defines

$$\wedge(G) = \{g \in G \mid |S: S \cap C_G(g)| < \infty \text{ for all finitely generated subgroups } S \text{ of } G\},$$

$$\wedge^+(G) = \{g \in \wedge(G) \mid |g| < \infty\}, \text{ and}$$

$$\wedge^p(G) = \langle g \in \wedge^+(G) \mid |g| = p^e \text{ for some } e\rangle,$$

then Passman has proved that,

$$N^*K(G) = JK\wedge^+(G).KG,$$

where J denotes the Jacobson Radical, (**2**).

Here we want to prove:

Theorem 1. $N^*K(G) = JK\wedge^p(G).KG.$
This will be proved, if we can prove:

Theorem 2. $JK\wedge^+(G) = JK\wedge^p(G).K\wedge^+(G).$
We firstly state the following result proved by Passman (**2**):

Lemma 1. *Let* $W \trianglelefteq G$, $W \subset \wedge^+(G)$. *Then,*

$$JK(W).KG \subseteq N^*K(G) \subseteq JK(G).$$

If we put $G = \wedge^+(G)$, $W = \wedge^p(G)$ in Lemma 1, we obtain,

Lemma 2. $JK(\wedge^p(G)) \subseteq JK(\wedge^+(G)).$
We shall also note the following result, though we shall not need it here:

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Proposition. *$JK(\wedge^+(G)) \subseteq \mathfrak{A}(\wedge^p(G))$, where $\mathfrak{A}(\wedge^p(G))$ is the augmentation ideal of* $\wedge^p(G)$ in $\wedge^+(G)$, (**3**).

Proof. $\wedge^+(G)/\wedge^p(G)$ is a locally finite group whose elements have order prime to p, (**2**). Therefore by Theorem 18.7 of (**1**), $K(\wedge^+(G)/\wedge^p(G))$ is semi-simple. But this algebra is isomorphic to $K(\wedge^+(G))/\mathfrak{A}(\wedge^p(G))$. Hence the result follows.

Proof of Theorem 2. In view of Lemma 2 above, we merely have to prove that $JK\wedge^+(G) \subseteq JK\wedge^p(G).K\wedge^+(G)$. Let $a \in JK\wedge^+(G)$. Then $T = \langle \text{Supp } a \rangle$ is a finite subroup of $\wedge^+(G)$, since the latter is locally finite. Hence

$$H = T.\wedge^p(G) \supseteq \text{Supp } a,\ |H\colon \wedge^p(G)| < \infty \text{ and } p \mid |H\colon \wedge^p(G)|$$

since $\wedge^+(G)/\wedge^p(G)$ is a locally finite group with no elements of order divisible by p. Then by Theorem 16.6 of (**1**),

$$JKH = JK\wedge^p(G).KH.$$

Now $a \in JK\wedge^+(G) \cap KH \subseteq JKH$ by Lemma 16.9 of (**1**). Hence

$$a \in JK\wedge^p(G).K\wedge^+(G).$$

This proves the theorem.

Alternatively the fact that

$$JK\wedge^+(G) \subseteq JK\wedge^p(G).K\wedge^+(G)$$

is also an immediate consequence of the corollary on page 55 of (**4**), and the fact that $\wedge^+(G)/\wedge^p(G)$ is locally finite with no element of order divisible by p. (The author wishes to thank the referee for pointing out the reference (**4**).)

Corollary. $\wedge^p(G) = 1$ *implies that*

(i) *$K\wedge^+(G)$ is semi-simple and* (ii) *the PZ-Radical of* $K(G) = 0$.

This compares with Theorem 18.7 of (**1**).

REFERENCES

(**1**) D. S. Passman, *Infinite Group-Rings* (Marcel Dekker Inc., New York, 1971).

(**2**) D. S. Passman, A New Radical for Group-Rings (to appear).

(**3**) I. Sinha, Augmentation-maps of Subgroups of a group, *Math. Z.* **94** (1966), 193-206.

(**4**) D. A. R. Wallace, Some Applications of Subnormality in groups in the study of Group-Algebras, *Math. Z.* **108** (1968), 53-62.

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