15
The Hamiltonian approach

We have been concentrating on the Euclidian path integral approach to lattice gauge theory. An alternative formulation, first advocated by Kogut and Susskind (1975), keeps a continuous time variable and only considers three-dimensional space as discrete. Working in the temporal gauge $A_0 = 0$, they define a Hamiltonian which is a function of the space components of the gauge field and a set of conjugate momenta. This formulation also permits a strong coupling expansion, which is now an application of quantum mechanical perturbation theory.

In this chapter we will derive the Kogut–Susskind Hamiltonian from the Wilson theory using the transfer matrix in direct analogy with the discussion in chapter 3. In this way we will see the equivalence of the two approaches. Which is preferable depends on taste and the particular question being asked. In the Wilson theory, space-time symmetry is more apparent, the particle spectrum is given by the singularity structure of Green's functions, and we have the simple analogy with statistical mechanics. In the Kogut–Susskind approach, we deal with a conventional quantum mechanical system with a well-defined Hamiltonian, the spectrum of the theory is directly the spectrum of this Hamiltonian, and phase transitions represent level crossings in the infinite volume limit.

As we wish to consider the continuous time limit of the Wilson theory, we introduce a different lattice spacing a_0 for the time direction. This gives the timelike plaquettes a different shape than the spacelike ones and the details of the argument in chapter 7 on the classical continuum limit must be slightly modified. The couplings on spacelike and timelike plaquettes are no longer equal in the action

$$S = -\beta_s \sum_{s} \operatorname{Tr} U_\square - \beta_t \sum_{t} \operatorname{Tr} U_\square . \tag{15.1}$$

Here the notation means that the first sum is over spacelike plaquettes only and the second over timelike ones. To obtain a proper classical limit we should take

$$\beta_s = 2na_0/(g_0^2\, a), \tag{15.2}$$

$$\beta_t = 2na/(g_0^2\, a_0), \tag{15.3}$$

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where a continues to denote the spacelike lattice spacing. As a_0 goes to zero with fixed a, β_s goes to zero and β_t goes to infinity.

The above argument is essentially classical. For the quantum theory we have seen that the bare charge is cutoff-scheme dependent. In particular, the spacelike and timelike couplings may correspond to different Λ parameters in the sense discussed in chapter 13. Indeed, if we do not allow for such a change in relative spacelike and timelike scales, the speed of light may need to be renormalized (Shigemitsu and Kogut, 1981; Hasenfratz and Hasenfratz, 1981). Therefore we introduce two bare couplings and their geometric mean

$$\beta_s = 2na_0/(g_s^2 a), \tag{15.4}$$

$$\beta_t = 2na/(g_t^2 a_0), \tag{15.5}$$

$$g_H^2 = g_s g_t. \tag{15.6}$$

The subscript on g_H stands for the Hamiltonian formulation. As with any bare couplings, these must all agree to lowest order

$$g_s^2 = g_t^2 + O(g_t^4) = g_H^2 + O(g_H^4). \tag{15.7}$$

Introducing a cutoff dependence into the couplings and taking a continuum limit at the asymptotically free fixed point, we conclude

$$g_s^2(a)/g_t^2(a) \underset{a\to 0}{\to} 1. \tag{15.8}$$

To proceed toward the Hamiltonian formulation, we now go to the temporal gauge. Fixing all timelike links to the identity, we see that a timelike plaquette represents a coupling between two spacelike links at subsequent times. Separating out time dependences, we relabel the sites with two indices, i and t, such that the first corresponds to the spatial coordinates and $a_0\, t$ represents the time. In this notation the unfixed links carry a time index and two space indices $U_{ij,\,t}$. The pure gauge theory action is now

$$S = -(2a/(g_t^2 a_0)) \sum_{\{ij\},\,t} \operatorname{Re}\operatorname{Tr}(U_{ij,\,t+1}^{-1} U_{ij,\,t}) - (2a_0/(g_s^2 a)) \sum_{\Box,\,t} \operatorname{Re}\operatorname{Tr}(U_{\Box,\,t}), \tag{15.9}$$

where the second sum is over all spacelike plaquettes and all times.

In analogy with chapter 3 we wish to find a Hilbert space and an operator T such that

$$Z = \int (\mathrm{d}U)\,\mathrm{e}^{-S} = \operatorname{Tr} T^N, \tag{15.10}$$

where N is the number of discrete times and we have imposed periodic boundary conditions. From the logarithm of T we will obtain the Hamiltonian. The first term in eq. (15.9) will generate the kinetic energy and the second, the potential.

The space in which T operates is a direct product of spaces of square-integrable functions over the gauge group. A state $|\psi\rangle$ in this space is specified by a wave function $\psi(U)$ which is a function of link variables U_{ij} which are group elements associated with each bond of a spacelike lattice. The inner product in this space is

$$\langle\psi'|\psi\rangle = \int (\mathrm{d}U)\,\psi'^{+}(U)\,\psi(U). \tag{15.11}$$

For simplicity we use the same notation as for the path integral, but in eq. (15.11) only spacelike variables enter. We can expand the states of this space in the non-normalizable basis $\{|U\rangle\}$, where a state in this set is determined by a group element U_{ij} on each spacelike bond. These satisfy a condition that the reversed links are not independent

$$U_{ij} = U_{ji}^{-1}. \tag{15.12}$$

The overlap of states in this basis is

$$\langle U'|U\rangle = \prod_{\{ij\}} \delta(U'_{ij}, U_{ij}), \tag{15.13}$$

where the delta function over the group was introduced in chapter 9, eq. (9.14). The completeness statement is

$$1 = \int (\mathrm{d}U)\,|U\rangle\,\langle U|. \tag{15.14}$$

The general state takes the form

$$|\psi\rangle = \int (\mathrm{d}U)\,|U\rangle\,\psi(U). \tag{15.15}$$

Working in this Hilbert space, one may write down by inspection the matrix elements of an operator satisfying eq. (15.10)

$$\begin{aligned}\langle U'|T|U\rangle &= \exp\left((2a/(g_t^2\,a_0))\sum_{\{ij\}} \mathrm{Re\,Tr}\,(U'^{-1}_{ij}U_{ij})\right)\\ &\quad\times \exp\left((2a_0/(g_s^2\,a))\sum_{\square} \mathrm{Re\,Tr}\,(U_{\square})\right)\end{aligned} \tag{15.16}$$

Just as we expressed T for quantum mechanics in terms of the operators $\hat{p}$ and $\hat{x}$, we would like to write this T in terms of some simple operators in the present Hilbert space. We begin by defining a set of matrix valued operators $\hat{U}_{ij}$ and unitary operators $R_{ij}(g)$, where g is an element of the gauge group

$$\hat{U}_{ij}|U\rangle = U_{ij}|U\rangle, \tag{15.17}$$

$$\left.\begin{aligned} R_{ij}(g)\,|U\rangle &= |U'\rangle,\\ U'_{ij} &= gU_{ij}\end{aligned}\right\} \tag{15.18}$$

and R_{ij} does not alter any other links. The operators $\hat{U}$ clearly are the analog of the coordinate $\hat{x}$ in ordinary quantum mechanics. The operators

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$R_{ij}(g)$ satisfy the group representation property

$$R_{ij}(g)\,R_{ij}(g') = R_{ij}(gg'). \tag{15.19}$$

They translate the variables U and thus are related to the canonical momentum, in a sense which will be made more precise shortly. In terms of these quantities, T takes the form

$$T = \Big(\prod_{\{ij\}}\Big(\int \mathrm{d}g\, R_{ij}(g) \quad \exp\left((2a/(g_t^2\, a_0))\,\mathrm{Re\,Tr}\, g\right)\Big) \times \exp\left((2a_0/(g_s^2\, a))\sum_{\Box} \mathrm{Re\,Tr}\,\hat{U}_{\Box}\right), \tag{15.20}$$

where $\hat{U}_{\Box}$ is the product of the $\hat{U}_{ij}$ around the corresponding plaquette.

We now wish to consider the limit as a_0 goes to zero. As a_0 becomes small, the integrals in eq. (15.20) become dominated by group elements near the identity. We parametrize the elements as in chapter 6

$$g = \mathrm{e}^{\mathrm{i}\omega^{\alpha}\lambda^{\alpha}} = \mathrm{e}^{\mathrm{i}\omega\cdot\lambda}, \tag{15.21}$$

where

$$\mathrm{Tr}\,(\lambda^{\alpha}\lambda^{\beta}) = \tfrac{1}{2}\delta^{\alpha\beta}, \tag{15.22}$$

$$[\lambda^{\alpha}, \lambda^{\beta}] = \mathrm{i} f^{\alpha\beta\gamma}\lambda^{\gamma}. \tag{15.23}$$

The invariant group measure takes the form

$$\mathrm{d}g = J(\omega)\prod_{\alpha}\mathrm{d}\omega^{\alpha}. \tag{15.24}$$

The only properties of the Jacobean function J that we will need are that in a neighborhood of the identity it is regular and non-vanishing and that

$$J(\omega) = J(-\omega), \tag{15.25}$$

which follows because $\mathrm{d}g = \mathrm{d}g^{-1}$.

As with any representation of the group, the operator $R_{ij}(g)$ can be written in terms of a set of generators for that representation

$$R_{ij}(g) = \exp\,(\mathrm{i}\omega^{\alpha} l_{ij}^{\alpha}) = \exp\,(\mathrm{i}\omega\cdot l_{ij}). \tag{15.26}$$

In our Hilbert space the l_{ij}^{α} are Hermitian operators satisfying

$$[l_{ij}^{\alpha}, l_{ij}^{\beta}] = \mathrm{i} f^{\alpha\beta\gamma} l_{ij}^{\gamma}, \tag{15.27}$$

$$[l_{ij}^{\alpha}, \hat{U}_{ij}] = -\lambda^{\alpha}\hat{U}_{ij}, \tag{15.28}$$

$$[l_{ij}^{\alpha}, \hat{U}_{ji}] = \hat{U}_{ji}\,\lambda^{\alpha}, \tag{15.29}$$

$$[l_{ij}^{2}, l_{ij}^{\alpha}] = 0 = [l_{ij}^{2}, R_{ij}(g)]. \tag{15.30}$$

The operators corresponding to different links all commute. In eq. (15.30) we have introduced the quadratic Casimir operator for the group

$$l_{ij}^{2} = \sum_{\alpha} l_{ij}^{\alpha}\, l_{ij}^{\alpha}. \tag{15.31}$$

These operators may be all represented by differential operators in the group parameters. For example, with the group $U(1) = \{\exp(i\theta)\}$, we have a single generator $\lambda = 2^{-\frac{1}{2}}$ and

$$l_{ij} = 2^{-\frac{1}{2}}\mathrm{d}/\mathrm{d}\theta_{ij}. \tag{15.32}$$

To consider a link in the reversed direction, first note that eq. (15.12) carries over to the operators

$$\hat{U}_{ij} = \hat{U}^{\dagger}_{ji}. \tag{15.33}$$

The connection between l_{ij} and l_{ji} follows from

$$\begin{aligned} R_{ji}(g)|U_{ij}\rangle &= |U_{ij}g^{-1}\rangle \\ &= |(U_{ij}g^{-1}U_{ij}^{-1})\,U_{ij}\rangle \\ &= R_{ij}(U_{ij}g^{-1}U_{ij}^{-1})|U_{ij}\rangle. \end{aligned} \tag{15.34}$$

This implies for the generators

$$l^{\alpha}_{ji}|U\rangle = -G(U_{ij})^{\alpha\beta}l^{\beta}_{ij}|U\rangle, \tag{15.35}$$

where $G(g)^{\alpha\beta}$ denotes the adjoint representation of the group

$$g^{-1}\lambda^{\alpha}g = G(g)^{\alpha\beta}\lambda^{\beta}. \tag{15.36}$$

As this is a real orthogonal representation, we have

$$l^2_{ij} = l^2_{ji}. \tag{15.37}$$

Thus the quadratic Casimir does not depend on the direction chosen for the link.

With this bit of group theory in hand, we return to the transfer matrix and insert eqs (15.21), (15.24) and (15.26) into eq. (15.20)

$$\begin{aligned} T = (\prod_{\{ij\}}(\int(\prod_{\alpha}\mathrm{d}\omega^{\alpha})\,J(\omega)\exp(\mathrm{i}l_{ij}\cdot\omega)\exp((2a/(g_t^2\,a_0))\,\mathrm{Tr}\cos(\omega\,.\,\lambda))) \\ \times\exp((2a_0/(g_s^2\,a))\sum_{\square}\mathrm{Re}\,\mathrm{Tr}\,U_{\square}). \end{aligned} \tag{15.38}$$

When a_0 goes to zero, the integral over ω is dominated by ω near the maximum of $\mathrm{Tr}\cos(\omega\cdot\lambda)$. For a unitary group this maximum always occurs near $\omega = 0$. We expand about this point

$$\mathrm{Tr}\cos(\omega\cdot\lambda) = n - \tfrac{1}{4}\omega^2 + O(\omega^4). \tag{15.39}$$

Inserting this into eq. (15.38), we do the Gaussian ω integrals to obtain the result

$$T = K\exp(-a_0 H + O(a_0^2)), \tag{15.40}$$

where K is an irrelevant constant factor and

$$H = (g_t/g_s)((g_H^2/(2a))\sum_{\{ij\}} l^2_{ij} + (2/(g_H^2\,a))\sum_{\square}\mathrm{Re}\,\mathrm{Tr}\,\hat{U}_{\square}). \tag{15.41}$$

This is the Kogut–Susskind Hamiltonian.

The two terms in eq. (15.41) have a direct interpretation in analogy to the usual continuum gauge theory Hamiltonian. The second term is a sum over spacelike plaquettes and represents the lattice form of the magnetic

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field squared. The first term involves the canonical momenta and represents the electric field squared. Indeed, the operator l_{ij} corresponds directly to the flux of electric field passing through link ij.

In eq. (15.41) we have removed a factor of g_t/g_s so that the remainder of the Hamiltonian only depends on the mean, g_H. Note that by virtue of eq. (15.8), this prefactor approaches unity in the continuous space limit. Thus for spectrum calculations in the continuum, we can ignore this factor. The coupling g_H has its own associated Λ parameter, defined in analogy with eq. (13.19). As indicated there, the relationship of this parameter with any other scheme can be determined perturbatively. Hasenfratz and Hasenfratz (1981) have calculated

$$\Lambda_H/\Lambda_0 \begin{cases} = 0.84, n = 2 \\ = 0.91, n = 3. \end{cases} \tag{15.42}$$

The above Hamiltonian possesses a large amount of symmetry due to the remaining gauge freedom of the theory. As we have only specified the temporal gauge, we can still do time-independent gauge transformations. An operator that performs such a transformation at space site i is

$$J_i(g) = \prod_{\{ij\} \supset i} R_{ij}(g), \tag{15.43}$$

where the product extends over all bonds emanating from site i. This is a symmetry operator which commutes with the Hamiltonian. All physical states should be singlets under this operation in the sense that

$$J_i(g) | \psi \rangle = | \psi \rangle. \tag{15.44}$$

In terms of the generators l_{ij}, this amounts to

$$\sum_{\{ij\} \supset i} l_{ij}^{\alpha} | \psi \rangle = 0. \tag{15.45}$$

This equation says that the net electric flux out of any site is zero. Thus we have a discrete version of Gauss's law. Alternatively we could study external sources by allowing some sites to be other than a gauge singlet. Note that the counting of degrees of freedom parallels continuum treatments. The temporal gauge has removed timelike links as variables. Gauss's law removes one variable per group generator on each site. Thus the final theory has two degrees of freedom for each gauge boson, as expected from the possible polarizations in the continuum theory.

A strong coupling series is easily formulated for this Hamiltonian. When g_H is large, the electric term dominates. The kinetic part of the Hamiltonian is diagonalized by placing all links into singlet states with $l_{ij}^2 = 0$. The natural basis of states for the strong coupling expansion is in terms of definite representations of the gauge group on each link. The potential or magnetic term in the Hamiltonian then acts as a perturbation which excites links into intermediate states involving higher representations. The first correction involves the excitation of the links around a single plaquette into the fundamental representation. For further details we refer the reader to the review by Kogut (1979).

Here we have only considered pure gauge fields. The Hamiltonian is easily extended to include fermionic or other matter fields. With fermions one again has the doubling problem alluded to in chapter 5 except that one factor of two is saved because time is continuous.

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