DECOMPOSITION OF REPRESENTATION ALGEBRAS

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Throughout this paper $\mathscr{G}$ is a finite group and $\mathscr{F}$ is a complete local principal ideal domain of characteristic p, where p divides $|\mathscr{G}|$. The notations of [5] are adopted; moreover we shall denote the isomorphism-class of an $\mathscr{F}\mathscr{G}$-representation module $\mathscr{M}$ by M, the class of $\mathscr{M}_{\mathscr{X}}$ by $M_{\mathscr{X}}$ and the class of $\mathscr{M}^{\mathscr{R}}$ by $M^{\mathscr{R}}$ for suitable groups $\mathscr{X}$ and $\mathscr{R}$.

Conlon [2] has shown that $A(\mathscr{G}) = A''_{\mathscr{X}}(\mathscr{G})$, where the direct sum is taken over the non-conjugate p-subgroups $\mathscr{D}$ of $\mathscr{G}$, and Green [4] has shown that $A''_{\mathscr{X}}(\mathscr{G}) \cong A''_{\mathscr{X}}(\mathscr{N})$, where $\mathscr{N}$ is the $\mathscr{G}$-normalizer of $\mathscr{X}$.

In this paper we shall assume that $\mathscr{H}$ is a normal p-subgroup of $\mathscr{G}$, and that $\mathscr{R}$ is a group satisfying

$$\mathscr{H} \leqq \mathscr{R} \leqq \mathscr{G}.$$

Various preliminaries appear in Section 1. Section 2 is devoted to defining new algebras $A_{\mathscr{R}}$ and $B^{\mathscr{G}}_{\mathscr{Y}}(\mathscr{H})$, and deriving a direct decomposition of $A_{\mathscr{R}}$ (Theorem 21). This result is used in Section 3 to get new direct decompositions of $A_{\mathscr{H}}(\mathscr{G})$ and $A'_{\mathscr{H}}(\mathscr{G})$. Finally two special cases are discussed in detail.

I would like to express my gratitude to the referee, who corrected a fault in my original proof of (21) and made many helpful suggestions concerning the organization of the paper.

1. Some maps

Suppose $\eta : \mathscr{U} \to \mathscr{V}$ is a homomorphism of groups. Then we define maps η^* and η_* as on p. 77 of [1]: if $\mathscr{M}$ is an $\mathscr{F}\mathscr{V}$-representation module then $\mathscr{M}\eta^*$ is an $\mathscr{F}\mathscr{U}$-representation module, and if $\mathscr{L}$ is an $\mathscr{F}\mathscr{U}$-representation module then $\mathscr{L}\eta_*$ is an $\mathscr{F}\mathscr{V}$-representation module. In particular if $\mathscr{U} \leqq \mathscr{V}$ we shall write $\theta(\mathscr{U}, \mathscr{V})$ for the natural embedding map; as noted in [1],

$$\mathscr{M}\theta(\mathscr{U}, \mathscr{V})^* = \mathscr{M}_{\mathscr{U}},$$
$$\mathscr{L}\theta(\mathscr{U}, \mathscr{V})_* = \mathscr{L}^{\mathscr{V}}.$$

We shall write $\psi_{\mathscr{R}}$ for the natural map from $\mathscr{R}$ to $\mathscr{R}/\mathscr{H}$.

We also write η^*, η_* for the linear maps

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$$\eta^* : A(\mathscr{V}) \to A(\mathscr{U})$$
$$\eta_* : A(\mathscr{U}) \to A(\mathscr{V})$$

obtained by defining $M\eta^*$ and $L\eta_*$ to be the isomorphism-classes of $\mathscr{M}\eta^*$ and $\mathscr{L}\eta_*$ respectively, and extending by linearity. Then η^* is an algebra homomorphism, and

$$M(L\eta_*) = [(M\eta^*)L]\eta_*.$$

In particular if $\theta = \theta(\mathscr{U}, \mathscr{V})$ and $\psi_{\mathscr{R}}$ are as above, then θ^* and θ_* are the r and t of Green [4], and $\psi^*_{\mathscr{R}} : A(\mathscr{R}/\mathscr{H}) \to A(\mathscr{R})$ is injective.

The following results are easy to prove:

LEMMA 1. *If $\mathscr{L}$ is an $\mathscr{F}(\mathscr{R}/\mathscr{H})$-representation module, then*

$$\mathscr{L}\theta(\mathscr{R}/\mathscr{H}, \mathscr{G}/\mathscr{H})_*\psi^*_{\mathscr{G}} \cong \mathscr{L}\psi^*_{\mathscr{R}}\theta(\mathscr{R}, \mathscr{G})_*.$$

LEMMA 2. $$A_{\mathscr{R}/\mathscr{H}}(\mathscr{G}/\mathscr{H})\psi^*_{\mathscr{G}} \subseteq A_{\mathscr{R}}(\mathscr{G}).$$

If $\mathscr{I}$ is the subgroup with one element of a group $\mathscr{K}$, we shall write $P(\mathscr{K})$ for the projective ideal $A_{\mathscr{I}}(\mathscr{K})$. Then we define

$$P_{\mathscr{H}}(\mathscr{R}) = P(\mathscr{R}/\mathscr{H})\psi^*_{\mathscr{R}}$$
$$P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}) = P_{\mathscr{H}}(\mathscr{R})\theta(\mathscr{R}, \mathscr{G})_*$$

$P_{\mathscr{I}}(\mathscr{R})$ is therefore just $P(\mathscr{R})$; we shall write $P^{\mathscr{G}}(\mathscr{R})$ for $P^{\mathscr{G}}_{\mathscr{I}}(\mathscr{R})$. As a corollary to the definitions, we have

$$P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}) = P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H})\psi^*_{\mathscr{G}} \cong P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H}); \tag{3}$$

the equality comes from Lemma 1 and the isomorphism holds because $\psi^*_{\mathscr{G}}$ is injective.

THEOREM 4. *$P^{\mathscr{G}}(\mathscr{R})$ is an ideal of $A(\mathscr{G})$.*

PROOF. $P^{\mathscr{G}}(\mathscr{R})$ is spanned by the $P^{\mathscr{G}}$, where $\mathscr{P}$ ranges through the indecomposable projective $\mathscr{F}\mathscr{R}$-representation modules.

It is sufficient to show that, with such a $\mathscr{P}$ and with $\mathscr{Q}$ an $\mathscr{F}\mathscr{R}$-representation module

$$P^{\mathscr{G}}Q \in P^{\mathscr{G}}(\mathscr{R}).$$

By the Mackey formula (p. 324 of [3]),

$$P^{\mathscr{G}}Q = (PQ_{\mathscr{R}})^{\mathscr{G}}.$$

Also $P \in P(\mathscr{R})$ and $Q_{\mathscr{R}} \in A(\mathscr{R})$: since $P(\mathscr{R})$ is an ideal of $A(\mathscr{R})$ we know that

$$PQ_{\mathscr{R}} \in P(\mathscr{R})$$

whence

$$(PQ_{\mathscr{R}})^{\mathscr{G}} \in P^{\mathscr{G}}(\mathscr{R}),$$

giving the result.

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LEMMA 5. $\mathscr{F}(\mathscr{R}/\mathscr{H})\psi_{\mathscr{R}}^{*} \cong \mathscr{F}_{\mathscr{H}}\theta(\mathscr{H}, \mathscr{R})_{*}$.

An $\mathscr{F}\mathscr{R}$-representation module $\mathscr{Q}$ will be called *$\mathscr{H}$-trivial* if $Hq = q$ for all $H \in \mathscr{H}$ and $q \in \mathscr{Q}$; clearly $\mathscr{Q}$ is H-trivial if and only if there is an $\mathscr{F}(\mathscr{R}/\mathscr{H})$-module $\mathscr{P}$ satisfying

$$\mathscr{Q} \cong \mathscr{P}\psi_{\mathscr{R}}^{*}; \tag{6}$$

and if (5) holds then $\mathscr{P}$ is indecomposable if and only if $\mathscr{Q}$ is.

Suppose $\mathscr{P}$ and $\mathscr{Q}$ are indecomposable modules satisfying (6); write $\mathscr{K}$ for the vertex of $\mathscr{Q}$. Then $\mathscr{K}$ contains $\mathscr{H}$, and $\mathscr{K}/\mathscr{H}$ is a vertex of $\mathscr{R}$. So we have

LEMMA 7. *$P_{\mathscr{H}}(\mathscr{R})$ has a basis (as a $\mathscr{C}$-space) consisting of the classes Q of the indecomposable $\mathscr{H}$-trivial $\mathscr{F}\mathscr{R}$-representation modules $\mathscr{Q}$ which are $\mathscr{H}$-projective. Hence $P_{\mathscr{H}}^{\mathscr{G}}(\mathscr{R})$ is spanned (as a $\mathscr{C}$-space) by the classes $Q\theta(\mathscr{R}, \mathscr{G})_{*}$ of the $\mathscr{F}\mathscr{G}$-representation modules induced from these $\mathscr{Q}$; these induced modules are clearly also $\mathscr{H}$-trivial and $\mathscr{H}$-projective.*

If X is any element of $\mathscr{G}$, write

$$\gamma_{X,\mathscr{R}} : \mathscr{R}^{X} \to \mathscr{R}$$

for the group isomorphism $R^{X} \to R\,(R \in \mathscr{R})$. For any $\mathscr{F}\mathscr{R}$-representation module $\mathscr{Q}$, $\mathscr{Q}\gamma_{X,\mathscr{R}}^{*}$ is the conjugate module $\mathscr{Q}^{X}$, and the induced map

$$\gamma_{X,\mathscr{R}}^{*} : A(\mathscr{R}) \to A(\mathscr{R}^{X})$$

is a $\mathscr{C}$-algebra isomorphism. The following properties are easy to check.

(8) $\gamma_{X,\mathscr{G}}^{*}$ *is the identity map on* $A(\mathscr{G})$;

(9) $\gamma_{X\mathscr{H},\mathscr{R}/\mathscr{H}}^{*}\psi_{\mathscr{R}}^{*} = \psi_{\mathscr{R}}^{*}\gamma_{X,\mathscr{R}}^{*}$;

(10) $\gamma_{X,\mathscr{R}}^{*}\theta(\mathscr{R}^{X}, \mathscr{G})_{*} = \theta(\mathscr{R}, \mathscr{G})_{*}\gamma_{X,\mathscr{G}}^{*} = \theta(\mathscr{R}, \mathscr{G})_{*}$;

(11) *If $\mathscr{Q}$ is an indecomposable $\mathscr{F}\mathscr{R}$-module with vertex $\mathscr{K}$ then $\mathscr{Q}\gamma_{X,\mathscr{R}}^{*}$ is an indecomposable $\mathscr{F}\mathscr{R}^{X}$-module with vertex $\mathscr{K}^{X}$.*

LEMMA 12. *If $X \in \mathscr{G}$, then $P_{\mathscr{H}}^{\mathscr{G}}(\mathscr{R}) = P_{\mathscr{H}}^{\mathscr{G}}(\mathscr{R}^{X})$.*

PROOF. Using (11) with $\mathscr{R}$ replaced by $\mathscr{R}/\mathscr{H}$ we get

$$P(\mathscr{R}^{X}/\mathscr{H}) = P(\mathscr{R}/\mathscr{H})\gamma_{X\mathscr{H},\mathscr{R}/\mathscr{H}}^{*}$$

so, by definition

$$\begin{aligned} P_{\mathscr{H}}^{\mathscr{G}}(\mathscr{R}^{X}) &= P(\mathscr{R}/\mathscr{H})\gamma_{X\mathscr{H},\mathscr{R}/\mathscr{H}}^{*}\psi_{\mathscr{R}^{X}}^{*}\theta(\mathscr{R}^{X}, \mathscr{G})_{*} \\ &= P(\mathscr{R}/\mathscr{H})\psi_{\mathscr{R}}^{*}\gamma_{X,\mathscr{R}}^{*}\theta(\mathscr{R}^{X}, \mathscr{G})_{*} && \text{by (9)} \\ &= P(\mathscr{R}/\mathscr{H})\psi_{\mathscr{R}}^{*}\theta(\mathscr{R}, \mathscr{G})_{*} && \text{by (10)} \\ &= P_{\mathscr{H}}^{\mathscr{G}}(\mathscr{R}). \end{aligned}$$

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LEMMA 13. *If $\mathscr{R}$ and $\mathscr{S}$ are subgroups of $\mathscr{G}$, both containing $\mathscr{H}$, then*

$$P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R})P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{S}) \subseteqq \sum_{X \in \mathscr{G}} P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}^X \cap \mathscr{S}).$$

PROOF. Suppose $\mathscr{Q}$ is an $\mathscr{F}\mathscr{R}$-module and $\mathscr{T}$ an $\mathscr{F}\mathscr{S}$-module. Then, from Mackey's 'tensor product' theorem ([3], p. 325),

$$\begin{aligned}(14) \qquad & \mathscr{Q}\theta(\mathscr{R}, \mathscr{G})_* \otimes \mathscr{T}\theta(\mathscr{S}, \mathscr{G})_* \\ & \cong \bigoplus_X \left((\mathscr{R}^X)_{\mathscr{R}^X \cap \mathscr{S}} \otimes (\mathscr{T})_{\mathscr{R}^X \cap \mathscr{S}}\right)\theta(\mathscr{R}^X \cap \mathscr{S}, \mathscr{G})_*\end{aligned}$$

where X runs through a complete set of $(\mathscr{R}, \mathscr{S})$-double coset representatives in $\mathscr{G}$. In particular, if $\mathscr{Q}$ and $\mathscr{T}$ are both $\mathscr{H}$-projective and $\mathscr{H}$-trivial, then each tensor product appearing on the right in (14) will also be an $\mathscr{H}$-projective, $\mathscr{H}$-trivial $\mathscr{F}(\mathscr{R}^X \cap \mathscr{S})$-module.

Lemma 13 now comes directly from Lemma 7.

LEMMA 15. *If $\mathscr{R}' \leqq_{\mathscr{G}} \mathscr{R}$, then $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}') \subseteqq P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R})$.*

PROOF. From Lemma 12 we can assume $\mathscr{R}' \leqq \mathscr{R}$. If $\mathscr{Q}$ is an indecomposable projective $\mathscr{F}(\mathscr{R}'/\mathscr{H})$-representation module, $\mathscr{Q}^{\mathscr{R}/\mathscr{H}}$ is projective, so

$$Q^{\mathscr{R}/\mathscr{H}} \in P(\mathscr{R}/\mathscr{H})$$

whence

$$Q^{\mathscr{R}/\mathscr{H}}\theta(\mathscr{R}/\mathscr{H}, \mathscr{G}/\mathscr{H})_* \in P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H}).$$

The left hand side is $Q^{\mathscr{G}/\mathscr{H}}$, which is $Q\theta(\mathscr{R}'/\mathscr{H}, \mathscr{G}/\mathscr{H})_*$; Q could have been any basis element for $P(\mathscr{R}'/\mathscr{H})$, so

$$P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}'/\mathscr{H}) \subseteqq P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H}),$$

and from (3)

$$P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}') \subseteqq P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}).$$

2. The algebras $A_{\mathscr{R}}$ and $B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$.

If $\mathscr{Q}$ is an indecomposable projective $\mathscr{F}(\mathscr{R}/\mathscr{H})$-representation module then $\mathscr{Q}\theta(\mathscr{R}/\mathscr{H}, \mathscr{G}/\mathscr{H})_*$ is projective, and belongs to $P(\mathscr{G}/\mathscr{H})$. So $P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H})$ is a subalgebra of $P(\mathscr{G}/\mathscr{H})$; by Theorem 4 it must be an ideal. Since $P(\mathscr{G}/\mathscr{H})$ is a finite direct sum,

$$P(\mathscr{G}/\mathscr{H}) \cong \oplus\, C,$$

we must have

$$P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H}) \cong \oplus\, \mathscr{C}$$

for some finite number of summands. $P^{\mathscr{G}/\mathscr{H}}(\mathscr{R}/\mathscr{H})$ is non-empty, so it has an identity element; by (3), $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R})$ must also have an identity element, which we will write as $I_{\mathscr{R}}$.

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Now define

$$A_{\mathscr{R}} = A_{\mathscr{H}}(\mathscr{G}) I_{\mathscr{R}}.$$

From Lemma 12 we have, for any $X \in \mathscr{G}$,

(16) $$I_{\mathscr{R}} = I_{\mathscr{R}^X},$$

(17) $$A_{\mathscr{R}} = A_{\mathscr{R}^X}.$$

LEMMA 18. *If* $\mathscr{R}' \leqq_{\mathscr{G}} \mathscr{R}$ *then* $A_{\mathscr{R}'} | A_{\mathscr{R}}$.

PROOF. From Lemma 15 we see $I_{\mathscr{R}'} \in P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R})$, and $I_{\mathscr{R}'}$ is idempotent. So there is an orthogonal decomposition

$$I_{\mathscr{R}} = I_{\mathscr{R}'} + (I_{\mathscr{R}} - I_{\mathscr{R}'})$$

which yields

$$A_{\mathscr{R}} = A_{\mathscr{R}'} \oplus A_{\mathscr{H}}(\mathscr{G})(I_{\mathscr{R}} - I_{\mathscr{R}'}).$$

From Lemma 13 we also get

LEMMA 19. *If* $\mathscr{R}$ *and* $\mathscr{S}$ *are subgroups of* $\mathscr{G}$, *both containing* $\mathscr{H}$, *then*

$$A_{\mathscr{R}} A_{\mathscr{S}} \subseteqq \sum_{X \in \mathscr{G}} A_{\mathscr{R}^X \cap \mathscr{S}}$$

For convenience, write $\pi(\mathscr{R})$ to denote some complete set of groups $\mathscr{S}$ which are distinct to within $\mathscr{G}$-conjugacy and satisfy

$$\mathscr{H} \leqq \mathscr{S} \leqq \mathscr{R},$$

and $\pi'(\mathscr{R})$ to denote $\pi(\mathscr{R}) \setminus \{\mathscr{R}\}$.

We define $A'_{\mathscr{R}}$ to be $\sum A_{\mathscr{R}'}$, where $\sum$ means algebra sum over $\mathscr{R}' \in \pi'(\mathscr{R})$. From Lemma 18, $A'_{\mathscr{R}}$ is a finite sum of direct summands of $A_{\mathscr{R}}$, so

$$A'_{\mathscr{R}} | A_{\mathscr{R}};$$

consequently there is an algebra $B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H})$ defined by

$$A_{\mathscr{R}} = A'_{\mathscr{R}} \oplus B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H})$$

which satisfies

$$B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H}) \cong A_{\mathscr{R}} / A_{\mathscr{R}'}.$$

In particular

(20) $$B^{\mathscr{G}}_{\mathscr{H}}(\mathscr{H}) = A_{\mathscr{H}}.$$

THEOREM 21. $A_{\mathscr{R}} = \oplus B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$, *where* $\oplus$ *is algebra direct sum over* $\mathscr{S} \in \pi(\mathscr{R})$.

PROOF. We proceed by induction on $\mathscr{R}$. From (20) the theorem holds when $\mathscr{R}$ is replaced by $\mathscr{H}$. Suppose that whenever $\mathscr{H} \leqq \mathscr{K} < \mathscr{R}$,

$$A_{\mathscr{K}} = \bigoplus_{\mathscr{S} \in \pi(\mathscr{K})} B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H});$$

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This yields

$$A'_{\mathscr{R}} = \sum_{\mathscr{X} \in \pi'(\mathscr{R})} \bigoplus_{\mathscr{S} \in \pi(\mathscr{X})} B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$$
$$= \sum_{\mathscr{S} \in \pi'(\mathscr{R})} B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H});$$

using the definition of $B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H})$,

$$A_{\mathscr{R}} = \sum_{\mathscr{S} \in \pi(\mathscr{R})} B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H}). \tag{22}$$

To prove that the sum in (22) is direct, we must show that for any $\mathscr{X} \in \pi(\mathscr{R})$, $B^{\mathscr{G}}_{\mathscr{X}}(\mathscr{H})$ has zero intersection with $\sum B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$, the sum being over $\mathscr{S} \in \pi(\mathscr{R})\diagdown\{\mathscr{X}\}$. Now as each $B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$ is an ideal direct summand of $A_{\mathscr{S}}$, it is (by (18)) an ideal direct summand of $A_{\mathscr{R}}$, and has an identity element; from this it is easy to see that all we must prove is that

$$B = B^{\mathscr{G}}_{\mathscr{X}}(\mathscr{H}) \cap B^{\mathscr{G}}_{\mathscr{Y}}(\mathscr{H}) \neq 0 \tag{23}$$

is impossible for $\mathscr{X}, \mathscr{Y} \in \pi(\mathscr{R})$ unless $\mathscr{X} = \mathscr{Y}$.

Suppose $\mathscr{X}$ and $\mathscr{Y}$ are members of $\pi(\mathscr{R})$ which satisfy (23). Then the identity element E of B is non-zero; and

$$E = E_{\mathscr{X}} E_{\mathscr{Y}},$$

where $E_{\mathscr{X}}$ and $E_{\mathscr{Y}}$ are the identity elements of $B^{\mathscr{G}}_{\mathscr{X}}(\mathscr{H})$ and $B^{\mathscr{G}}_{\mathscr{Y}}(\mathscr{H})$ respectively. Hence

$$E \in A_{\mathscr{X}} A_{\mathscr{Y}} \subseteqq \sum_{X \in \mathscr{G}} A_{\mathscr{X}^X \cap \mathscr{Y}}$$

(using Lemma 19). If $\mathscr{Y} \not\leqq_{\mathscr{G}} \mathscr{X}$, then each $\mathscr{X}^X \cap \mathscr{Y}$ is a *proper* subgroup of $\mathscr{Y}$, so $E \in A'_{\mathscr{Y}}$; but this means

$$E = EE_{\mathscr{Y}} \in A'_{\mathscr{Y}} B^{\mathscr{G}}_{\mathscr{Y}}(\mathscr{H}) = 0$$

which is impossible. So $\mathscr{X} \geqq_{\mathscr{G}} \mathscr{Y}$, and similarly $\mathscr{Y} \geqq_{\mathscr{G}} \mathscr{X}$; therefore $\mathscr{X}$ and $\mathscr{Y}$ are $\mathscr{G}$-congugate and (by the definition of $\pi(\mathscr{R})$) $\mathscr{X} = \mathscr{Y}$.

This shows that the sum in (22) is direct, so we have the theorem.

3. The decomposition of $A_{\mathscr{H}}(\mathscr{G})$

From theorem 3.17 of [2], $I_{\mathscr{G}}$ is the identity element of $A_{\mathscr{H}}(\mathscr{G})$; so

$$A_{\mathscr{G}} = A_{\mathscr{H}}(\mathscr{G}); \tag{24}$$

applying this to Theorem 21 we have

THEOREM 25. $$A_{\mathscr{H}}(\mathscr{G}) = \bigoplus_{\mathscr{S} \in \pi(\mathscr{G})} B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H}).$$

To compare our decomposition (25) with Conlon's decomposition

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(26) $$A_{\mathscr{H}}(\mathscr{G}) \bigoplus_{\mathscr{K} \in \alpha(\mathscr{H})} A''_{\mathscr{K}}(\mathscr{G})$$

(where $\alpha(\mathscr{H})$ is a complete set of non-$\mathscr{G}$-conjugate subgroups of $\mathscr{H}$) in [2], it is convenient to observe that any such decomposition determines, and is determined by, an orthogonal idempotent decomposition of $I_{\mathscr{G}}$; (25) comes from

$$I_{\mathscr{G}} = \sum_{\mathscr{S} \in \pi(\mathscr{G})} E_{\mathscr{S}},$$

whereas (26) could be written

$$I_{\mathscr{G}} = \sum_{\mathscr{K} \in \alpha(\mathscr{H})} F_{\mathscr{K}}.$$

Then we obtain a refinement

$$I_{\mathscr{G}} = \sum_{\mathscr{S}, \mathscr{K}} E_{\mathscr{S}} F_{\mathscr{K}}$$

$$A_{\mathscr{H}}(\mathscr{G}) = \bigoplus_{\mathscr{S}, \mathscr{K}} A_{\mathscr{H}}(\mathscr{G}) E_{\mathscr{S}} F_{\mathscr{K}},$$

which could also be written as

(27) $$A_{\mathscr{H}}(\mathscr{G}) = \bigoplus_{\mathscr{S}, \mathscr{K}} \{A''_{\mathscr{K}}(\mathscr{G}) \cap B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})\}.$$

Suppose $\mathscr{D}$ is any group. Then (see [2] and [4]) there are $\mathscr{C}$-algebra isomorphisms

$$A(\mathscr{D}) \cong A_{\mathscr{H}}(\mathscr{D}) \cong A_{\mathscr{H}}(\mathscr{G}),$$

where $\mathscr{H}$ is the Sylow p-subgroup of $\mathscr{D}$ and $\mathscr{G}$ is the $\mathscr{D}$-normalizer of $\mathscr{H}$. Therefore (27) gives a decomposition of $A(\mathscr{D})$ in the general case, to within isomorphism.

4. Special cases of $B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$

We shall consider the structure of $B^{\mathscr{G}}_{\mathscr{S}}(\mathscr{H})$ in two special cases.

First consider $\mathscr{S} = \mathscr{H}$. $P(\mathscr{H}|\mathscr{H})$ consists of the $\mathscr{C}$-multiples of F, where F is the isomorphism-class of the module $\mathscr{F}$, so $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{H})$ consists of the $\mathscr{C}$-multiples of $F_{\mathscr{H}}\theta(\mathscr{H}, \mathscr{G})_*$; and since $F^2 = F$ and $I_{\mathscr{H}}$ is to be idempotent,

$$I_{\mathscr{H}} = [\mathscr{G} : \mathscr{H}]^{-1}(F_{\mathscr{H}})^{\mathscr{G}}.$$

A calculation yields

(28) *$A_{\mathscr{H}}$ is spanned by the module-classes of the form $N^{\mathscr{G}}$, where N is a basis element of $A(\mathscr{H})$.*

Moreover $I_{\mathscr{H}}$ is the idempotent I of proposition 3 of [1], and by that result $A_{\mathscr{H}}$ is isomorphic to $A(\mathscr{H})$ if every indecomposable $\mathscr{F}\mathscr{H}$-representation module is $\mathscr{G}$-stable.

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The second special case occurs when $\mathscr{R}$ is a subnormal p-extension of another group $\mathscr{S}$, where $\mathscr{H} \leqq \mathscr{S}$. If $\mathscr{P}$ is a projective indecomposable $\mathscr{F}(\mathscr{S}/\mathscr{H})$-representation module then $\mathscr{P}^{\mathscr{R}|\mathscr{H}}$ is indecomposable. So any element of $P(\mathscr{R}/\mathscr{H})$ can be written

$$L\theta(\mathscr{S}/\mathscr{H}, \mathscr{R}/\mathscr{H})_*$$

for some $L \in P(\mathscr{S}/\mathscr{H})$, and the typical element of $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R})$ is

$$\begin{aligned} L\theta(\mathscr{S}/\mathscr{H}, \mathscr{R}/\mathscr{H})_* \theta(\mathscr{R}/\mathscr{H}, \mathscr{G}/\mathscr{H})_* \psi^*_{\mathscr{G}} \\ = L\theta(\mathscr{S}/\mathscr{H}, \mathscr{G}/\mathscr{H})_* \psi^*_{\mathscr{G}} \\ \in P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{S}). \end{aligned}$$

This means $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}) \subseteqq P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{S})$; but the reverse inclusion also holds, so $P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{R}) = P^{\mathscr{G}}_{\mathscr{H}}(\mathscr{S})$, whence $I_{\mathscr{R}} = I_{\mathscr{S}}$ and $A_{\mathscr{R}} = A_{\mathscr{S}}$. Since $A_{\mathscr{S}} \subseteqq A'_{\mathscr{R}}$, we have $A_{\mathscr{R}} = A'_{\mathscr{R}}$, so

$$B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H}) = 0. \tag{29}$$

In particular suppose $\mathscr{R}$ is a p-group. Then

THEOREM 30. *If $\mathscr{R}$ is a p-group properly containing $\mathscr{H}$, $B^{\mathscr{G}}_{\mathscr{R}}(\mathscr{H}) = 0$.*

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