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ARTICLE

Solving heterogeneous-belief asset pricing models with short-selling constraints and many agents†

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Abstract

Short-selling constraints are common in financial markets, while physical assets such as housing often lack markets for short-selling altogether. As a result, investment decisions are often restricted by such constraints. This paper studies asset prices in behavioral heterogeneous-belief models with short-selling constraints and arbitrarily many belief types. We provide conditions on beliefs such that short-selling constraints bind for different types, along with analytic expressions for price and demands that allow us to construct fast solution algorithms relevant for a wide range of models. An application studies how an alternative uptick rule, as in the United States, affects price dynamics and wealth distribution in a market with many belief types in *evolutionary competition*. In a numerical example, we highlight a scenario in which a modified version of the alternative uptick rule, triggered by smaller percentage falls in price, reduces both asset mispricing and wealth inequality relative to the current regulation. As extensions, we show how our method applies to multiple asset markets with short-selling constraints, additional heterogeneities, and price setting by a market maker.

Keywords: Asset pricing; heterogeneous beliefs; short-selling constraints; computational algorithm; evolutionary competition

1. Introduction

The practice of short-selling is common in financial markets but is also widely regulated. When investors go short, they borrow and immediately sell a financial asset before repurchasing and returning the asset to the lender, closing their position. Whereas a long position can be thought of as a bet that asset prices will increase, short-selling allows investors to bet on a fall in asset prices. It has been argued that such betting may increase volatility in financial markets. A common policy response among regulators has been to restrict short-selling; for example, during the 2008–2009 financial crisis many countries introduced short-selling bans following sharp declines in asset prices. Similar short-selling bans were reinstated in some European economies during the 2011–2012 sovereign debt crisis and the Covid-19 outbreak (see Siciliano and Ventoruzzo (2020)). It is therefore important that researchers be able to solve asset pricing models with short-selling constraints in an efficient manner.

In this paper, we show how to efficiently solve dynamic behavioral asset pricing models with short-selling constraints and arbitrarily many heterogeneous beliefs.[1] We are thinking here of discrete time, heterogeneous-belief asset pricing models such as Brock and Hommes (1998), LeBaron

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et al. (1999), and Westerhoff (2004); for instance, the population shares of different types may be endogenously determined by *evolutionary competition*. We also allow price beliefs to potentially depend on the current asset price (in a linear fashion), which is a generalization relative to some models in the literature. For the case of such dynamic models, we provide expressions for the market-clearing price and demands and show how these results can be used to construct fast solution algorithms, such that researchers may efficiently solve models with thousands of different agents or belief types. Our algorithm is supported by analytical results that do not seem to have been documented previously.

Our analysis is built around the behavioral heterogeneous-beliefs asset pricing model (see Brock and Hommes (1998)). We allow arbitrarily many belief types whose population shares may be exogenous or determined endogenously by *evolutionary competition*. The model with evolutionary competition has been studied in the many-types case by Brock et al. (2005), who allow short-selling by investors. We show that when investors face short-selling constraints, the market-clearing price and demands depend on *belief dispersion* across types. We also provide a fast solution algorithm that exploits these analytical results.

The difficulty in the many-types case stems from the demand functions being *piecewise-linear*, such that the market-clearing price depends on how many types are short-selling constrained. For a market with a large number of investor types, it is computationally intensive to solve for a price and demands. To overcome this problem, we exploit the fact that types who are short-selling constrained in a given period must be *more pessimistic* than those who are unconstrained, such that ordering types in terms of optimism reduces computational burden and solution time. As a result, it becomes feasible to simulate models with thousands of belief types for many periods while retaining solution accuracy.

We provide analytical results for a benchmark asset pricing model, followed by a *policy application* which studies an alternative uptick rule, as currently in place in the United States, in a model with a large number of belief types in *evolutionary competition*. The alternative uptick rule is a "circuit breaker" that bans short-selling if prices fall 10% or more in the previous trading period; surprisingly, there do not appear to be any previous assessments of this rule in the heterogeneous-beliefs literature. Our results indicate that an alternative uptick rule may attenuate (or prevent) falls in price, but we also find that such rules can hinder price discovery, increase price volatility and lead to explosive price paths. An alternative uptick rule can also have substantive distributional implications. In a numerical example, we find that a modified rule, triggered by price falls smaller than 10%, may reduce both mispricing and wealth inequality relative to the current regulation.

Following the policy application, we highlight some extensions of the benchmark model to which our algorithm can be applied with only minor amendments. Extensions include the case of multiple asset markets with short-selling constraints, heterogeneity in attention to the current price, heterogeneity in perceived return risk (or risk preferences), and the market-maker approach to price determination.

The closest papers in the literature are Anufriev and Tuinstra (2013) and Dercole and Radi (2020). Anufriev and Tuinstra (2013) add trading costs for short-selling into a two-type asset pricing model and find that this leads to additional (non-fundamental) steady states as beliefs are updated more aggressively; in a similar vein, but with the addition of a leverage constraint, see in't Veld (2016). By comparison, Dercole and Radi (2020) study the original "uptick rule" in the United States from 1938–2007, which banned short-selling at lower prices, and find that there is no clear-cut impact on price volatility. There is also a wider literature on non-smooth asset pricing models (see e.g. Tramontana et al. (2010)); short-selling constraints are a specific application that gives rise to such models.

The above papers all consider a small number of investor types and solve for prices and demands in specific cases. The present paper contributes to the literature by (i) solving for price and demands when there are arbitrarily many belief types with general price predictors who can be in evolutionary competition and (ii) providing efficient solution algorithms. Relative to Anufriev and Tuinstra (2013), we consider a full short-selling ban, which amounts to the special case of an infinite short-selling cost in their setting. Unlike their paper, however, we provide analytic solutions for price and demands in this case by giving conditions that determine the sets of short-selling constrained and unconstrained types in any given period. We make our results accessible by providing both analytic and numerical examples.

Our paper is part of a growing literature studying heterogeneous beliefs, asset prices, and the effectiveness of regulatory policies in financial markets (Westerhoff (2016)). In financial market models, it is known that differences in beliefs combined with short-selling constraints can lead to price bubbles (see e.g. Scheinkman and Xiong (2003)), but such regulations could also aid market stability as noted above. There has also been interest in the impact of short-selling restrictions in markets for *physical* investment assets like housing (see Shiller (2015), Fabozzi et al. (2020)) that are subject to boom and bust and may be cast in our framework.

Section 2 presents a baseline model, followed by analytical results in Section 3. Section 4 presents our policy application. Section 5 presents three extensions, and Section 6 concludes.

2. Model

Consider a finite set of myopic, risk-averse investor types $\mathcal{H} = \{h_1, \ldots, h_H\}$. At each date $t \in \mathbb{N}_+$, each type $h \in \mathcal{H}$ chooses a portfolio of a risky asset $z_{t,h}$ and a riskless bond with return $\tilde{r} > 0$ to maximize a mean-variance utility function over future wealth with risk aversion parameter $a > 0$. The risky asset has current price p_t, future price p_{t+1}, and pays stochastic dividends d_{t+1}, which are exogenous. Investors form subjective expectations of the future price and future dividends of the risky asset as described below. The underlying model follows Brock and Hommes (1998), except that the risky asset is in positive net supply $\overline{Z} > 0$ and short-selling is ruled out by constraints of the form $z_{t,h} \geq 0$ for all t and h.

2.1 Asset demand

We denote the subjective expectation of type h at date t by $\tilde{E}_{t,h}[.]$ and the subjective variance by $\tilde{V}_{t,h}[.]$. The portfolio choice of type $h \in \mathcal{H}$ at date t solves the problem:[2]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \tag{1}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+\tilde{r})\big(w_{t,h} - p_t z_{t,h}\big)$ is future wealth, $w_{t,h} - p_t z_{t,h}$ is holdings of the risk-free asset, and $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$ and weight $a/2 > 0$.

Given short-selling constraints, the date t demand of each investor type is:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}. \end{cases} \tag{2}$$

If the price p_t is small enough, then type h's short-selling constraint is slack and their demand for the risky asset decreases with the price; this is the standard demand function that arises in Brock and Hommes (1998), where short-selling constraints are absent. However, if the current price is high enough to make the expected excess return of type h *negative*, then the short-selling constraint will bind on type h and their position in the risky asset is zero.

Dividends follow an IID process $d_t = \overline{d} + \epsilon_t$, where $\overline{d} > 0$ and ϵ_t is zero-mean with fixed variance. We assume all investor types know the dividend process, such that $\tilde{E}_{t,h}\left[d_{t+1}\right] = \overline{d}$ for all t and h; there is no loss of generality as our solution method nests a generic specification of $\tilde{E}_{t,h}\left[d_{t+1}\right]$ at no extra cost.[3] From equation (2), we see that the short-selling constraint is more likely to bind on type h the more *pessimistic* their price expectation $\tilde{E}_{t,h}\left[p_{t+1}\right]$.

2.2 Price beliefs

We consider generic price beliefs which are *boundedly rational* and can depend linearly on the current price p_t via a common coefficient (we relax the latter assumption later on).

Assumption 1. *All price beliefs are of the form:*

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \bar{c}p_t + \tilde{f}_{t,h} \tag{3}$$

where $\bar{c} \in [0, 1 + \tilde{r})$ *and* $\tilde{f}_{t,h} \in \mathbb{R}$ *is a generic forecast that cannot depend on current price* p_t.

Assumption 1 allows a wide range of boundedly rational beliefs. The coefficient $\bar{c}$ allows linear dependence of price expectations on the current price; for example, investors may extrapolate one-for-one on top of the current price like the extrapolators in Barberis et al. (2018) or they may use the information of the current price with some weight (see, for instance, LeBaron et al. (1999), Westerhoff (2004)). We assume $\bar{c} \geq 0$ to allow the case of no dependence on the current price (i.e. $\bar{c} = 0$), and we assume $\bar{c} < 1 + \tilde{r}$ to ensure that individual demands are *decreasing* in the current price p_t; for reference, see (4) below. *Time-varying or heterogeneous values of* $\bar{c}$ are discussed as an extension in Section 5.2.

The generic forecast $\tilde{f}_{t,h}$ (which can differ across types and over time) permits a potentially non-linear response to *past* prices, such as non-linear trend-following rules. In addition, $\tilde{f}_{t,h}$ may contain type-specific "fixed effects," be subject to random disturbances, or be influenced by social networks as in Yang (2009) or Panchenko et al. (2013). Assumption 1 in Brock and Hommes (1998) is nested by (3) when $\bar{c} = 0$ and $\tilde{f}_{t,h} = E_t\left[p^*_{t+1}\right] + g_h(x_{t-1}, \ldots, x_{t-L_h})$, where $g_h : \mathbb{R}^{L_h} \to \mathbb{R}$ is a function that can differ across types, L_h is the lag of type h, and $x_t := p_t - p^*_t$ is the price deviation from the fundamental price p^*_t.

For convenience, let $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \bar{c}$. Given $\tilde{E}_{t,h}\left[d_{t+1}\right] = \bar{d}$, we have $f_{t,h} = \tilde{f}_{t,h} + \bar{d} - a\sigma^2\overline{Z}$ and the demands in (2) can be written as

$$z_{t,h} = \begin{cases} \frac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{4}$$

Writing demands in terms of $f_{t,h}$ is convenient because the latter does *not* depend on the current price p_t and allows us to add the term $a\sigma^2\overline{Z}$ in the numerator of the demand function (see top line of (4)), which simplifies the algebra of the price solution; see Proposition 1. Writing demands this way is also consistent with a "deviation from fundamentals" representation; see Brock and Hommes (1998) and several other papers in the related literature.[4]

2.3 Population shares

We allow the population shares $n_{t,h}$ of investor types to be endogenous and time-varying, but we rule out any dependence on the current price p_t (see Assumption 2).

Assumption 2. *We consider population shares of the form* $n_{t,h} = \hat{n}_h(\boldsymbol{n}_{t-1}, \boldsymbol{u}_{t-1})$ *where* $\hat{n}_h$ *is a real function such that* $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, $n_{t,h} \in (0, 1)$ $\forall t, h$, *where* $\boldsymbol{n}_{t-1}$ $(\boldsymbol{u}_{t-1})$ *is the vector of past population shares (resp. past fitness levels). In particular, we rule out dependence of* $n_{t,h}$ *on the current asset price* p_t *(though not dependence on lagged prices* p_{t-1}, p_{t-2} *etc.).*

Assumption 2 is quite general. For instance, population shares may be *endogenously* determined by evolutionary competition as in Brock and Hommes (1998). Following Brock and Hommes (1997), a popular approach is a discrete-choice logistic model $n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} \exp(\beta U_{t,h})}$, where the intensity of choice $\beta \in [0, \infty)$ determines how fast agents switch to better-performing predictors. Various fitness measures $U_{t,h}$ are used in the literature, including realized profits net of a predictor cost (Brock and Hommes (1998)) and forecast accuracy (e.g. Ap Gwilym (2010)), which may potentially be included in a risk-adjusted measure of profits (De Grauwe and Grimaldi (2006)). While Assumption 2 rules out the "extreme" population shares of 0 or 1, it is straightforward (though analytically cumbersome) to relax this assumption, and we show how this can be done in Section 5 of the Supplementary Appendix.[5]

The presence of past fitness levels $\mathbf{u}_{t-1}$ in the function $\hat{n}_h$ allows evolutionary competition mechanisms such as the logit specification above, while the inclusion of past population shares $\mathbf{n}_{t-1}$ allows for asynchronous updating (see e.g. Hommes (2013), Ch. 5). It would not impose any extra burden to allow the function $\hat{n}_h$ to be time-varying or to include additional endogenous variables in the vector $\mathbf{u}_{t-1}$; however, as emphasized in Assumption 2, we rule out dependence of population shares on the *current* price p_t (or any future values).

The special case of fixed population shares $n_{t,h} = 1/H$ is relevant for agent-based or social network models where types are *individuals*, while exogenous time-varying population shares may be used to study herding in beliefs, as in the models of Kirman (1991, 1993).

3. Solving the model

The asset market clears when $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ subject to (4) and Assumptions 1 and 2. Given positive outside supply $\overline{Z} > 0$, there exists a unique market-clearing price p_t (see Anufriev and Tuinstra (2013), Proposition 2.1). We now characterize the price and demands.

3.1 Benchmark results

Proposition 1. *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\boldsymbol{n}_{t-1}, \boldsymbol{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types) at date t. Then, the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}\right) \leq a\sigma^2\overline{Z}$, then no type is short-selling constrained at date t $\left(\mathcal{B}_t^* = \mathcal{H}, \mathcal{S}_t^* = \emptyset\right)$ and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{5}$$

with demands $z_{t,h} = \left(a\sigma^2\right)^{-1}\left(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t\right) \geq 0 \; \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}\right) > a\sigma^2\overline{Z}$, at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}, \mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}\right) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}\left(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}\right)$, and the price and demands are given by

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{6}$$

and $z_{t,h} = \left(a\sigma^2\right)^{-1}\left(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t\right) \geq 0 \; \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \; \forall h \in \mathcal{S}_t^*$.

Proof. See the Appendix. □

Proposition 1 gives the market-clearing price and demands for an arbitrarily large set of belief types whose population shares may be endogenously determined. Since the results apply at any date $t \in \mathbb{N}_+$, we can find a solution recursively for $t = 1, 2, \ldots$, starting from period 1. If *no* types are short-selling constrained, the asset price depends on *all beliefs* as in (5); however, when short-selling constraints *bind*, only the beliefs of the unconstrained types matter for price determination; see (6). Intuitively, the market price depends on the demands (hence beliefs) of "buyers" who participate in the market by taking a long position.

An important difference relative to Proposition 2.1 in Anufriev and Tuinstra (2013) (which shows the existence of a unique market-clearing price) is that Proposition 1 gives conditions that determine the sets of constrained and unconstrained types (see parts (i) and (ii)), such that we have an explicit solution for the price for an arbitrary number of belief types.[6] As noted in Section 1, the short-selling ban considered here amounts to the special case of an infinite short-selling tax in Anufriev and Tuinstra (2013). However, since a full ban on short-selling is a benchmark case, it is useful to have an explicit solution. These analytical results allow us to build a fast computational algorithm as discussed below.

Part (i) of Proposition 1 gives a simple condition on beliefs that checks whether short-selling constraints are slack for all types. If so, the price is given by p_t^* in (5), which is the standard solution absent short-selling constraints; see for example Brock and Hommes (1998). Short-selling constraints will bind on at least one type if *belief dispersion* relative to the most pessimistic type, $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\})$, is large enough; see Proposition 1 Part (ii). In this case, at least one type—and at most $H-1$ types—will be short-selling constrained at the market-clearing price. Intuitively, those types with relatively low valuations of the asset (low $f_{t,h}$) will want to short sell if the price is "biased" upwards relative to their valuation by more optimistic types with non-trivial population shares, and hence, at the market-clearing price, the "pessimists" will be short-selling constrained. In this case, the sets of unconstrained and short-selling constrained types $\mathcal{B}_t^*$, $\mathcal{S}_t^*$ are determined by "cutoff" conditions that depend on the extent of belief dispersion within the group (see directly above (6)).

Finally, note that when one or more types are short-selling constrained, the market-clearing price p_t is higher than the (hypothetical) price p_t^* if short-selling constraints were absent—that is short-selling constraints raise the asset price, as argued by Miller (1977).

We now give an application of these results using a simple two-type example.

Example 1. *Consider two types h_1, h_2 with generic beliefs f_{t,h_1}, f_{t,h_2} that satisfy Assumption 1 and have the form $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$, as in (4), for $h = h_1, h_2$. The population shares are given by a discrete-choice logistic model: $n_{t,h} = \frac{exp\left(\beta U_{t-1,h}\right)}{\sum_{h\in\{h_1,h_2\}} exp\left(\beta U_{t-1,h}\right)}$, where $\beta \in [0,\infty)$ is the intensity of choice, $U_{t,h} = R_t z_{t-1,h}$ is the profit of type h at date t, $R_t = p_t + d_t - (1+\tilde{r})p_{t-1}$ is the realized return, and $z_{t-1,h}$ is the demand of type h at date $t-1$; see (4).*

By Proposition 1, if $\sum_{h\in\{h_1,h_2\}} n_{t,h}(f_{t,h} - \min\{f_{t,h_1}, f_{t,h_2}\}) \le a\sigma^2\overline{Z}$ neither type is short-selling constrained, and $p_t = \sum_{h\in\{h_1,h_2\}} n_{t,h}f_{t,h}/(1+r)$ by (5), where $r = \tilde{r} - \overline{c}$. If the above condition is not met, then either $f_{t,h_1} - f_{t,h_2} > a\sigma^2\overline{Z}/n_{t,h_1}$ (if h_1 is more optimistic) or $f_{t,h_2} - f_{t,h_1} > a\sigma^2\overline{Z}/n_{t,h_2}$ (if h_2 is more optimistic). In the former case, $\mathcal{B}_t^* = \{h_1\}$, $\mathcal{S}_t^* = \{h_2\}$, and by (6) the market-clearing price is $p_t = \left[(1+r)n_{t,h_1}\right]^{-1}\left(n_{t,h_1}f_{t,h_1} - (1-n_{t,h_1})a\sigma^2\overline{Z}\right)$, with demands $z_{t,h_1} = \overline{Z}/n_{t,h_1}$, $z_{t,h_2} = 0$. In the latter case, $\mathcal{B}_t^* = \{h_2\}$, $\mathcal{S}_t^* = \{h_1\}$, so $p_t = \left[(1+r)n_{t,h_2}\right]^{-1}\left(n_{t,h_2}f_{t,h_2} - \left(1-n_{t,h_2}\right)a\sigma^2\overline{Z}\right)$ and $z_{t,h_1} = 0$, $z_{t,h_2} = \overline{Z}/n_{t,h_2}$.

Suppose that beliefs follow the two-type Brock and Hommes (1998) model, where $\overline{c} = 0$ such that $r = \tilde{r}$. Type h_1 is a fundamentalist with $\tilde{E}_{t,h_1}\left[p_{t+1}\right] = \overline{p}$, where $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$ is the fundamental price, and h_2 is a 1-lag chartist: $\tilde{E}_{t,h_2}\left[p_{t+1}\right] = \overline{p} + \overline{g}(p_{t-1} - \overline{p})$, where $\overline{g} > 0$. Note that these beliefs imply that $f_{t,h_1} = (1+r)\overline{p}$ and $f_{t,h_2} = (1+r)\overline{p} + \overline{g}(p_{t-1} - \overline{p})$; see (4). Assuming $p_{t-1} > \overline{p}$, the chartist is more optimistic at date t, and hence by Proposition 1:

$$p_t = \begin{cases} \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p})}{1+r} & \text{if } \overline{g}(p_{t-1} - \overline{p}) \le a\sigma^2\overline{Z}/n_{t,h_2} \\[2ex] \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p}) - \left(1-n_{t,h_2}\right)a\sigma^2\overline{Z}}{n_{t,h_2}(1+r)} & \text{if } \overline{g}(p_{t-1} - \overline{p}) > a\sigma^2\overline{Z}/n_{t,h_2} \end{cases} \tag{7}$$

which replicates a known result in the literature.[7]

The above example is particularly simple: if belief dispersion is large enough that some type is constrained, then ranking the two types by optimism $f_{t,h}$ immediately determines the set of short-selling constrained types ($\mathcal{S}_t^*$) and the set of unconstrained types ($\mathcal{B}_t^*$). In a general setting with many types, however, there are many candidates for the sets $\mathcal{B}_t^*$, $\mathcal{S}_t^*$, and this number increases *exponentially* as the number of types H is increased. In fact, including the case where short-selling constraints are slack for all types, there are $2^H - 1$ candidates for $\mathcal{B}_t^*$, $\mathcal{S}_t^*$.[8] As a result, the task of finding the price is *computationally intensive* when there are a large number of types H, as seems plausible in many real-world asset markets.

To overcome this problem, we now present a version of Proposition 1 that reduces the number of candidates that need to be checked and is useful for computational purposes. Similar to Anufriev and Tuinstra (2013), we use the fact that types who are short-selling constrained in a given period t must be *more pessimistic* than those who were unconstrained; see (4). As a result, ranking types in terms of optimism can speed up discovery of the set of short-selling constrained types $\mathcal{S}_t^*$. We have already seen a relevant case in Example 1: knowing that the chartist type was more optimistic than the fundamental type allowed us to narrow down to 2 cases for the price rather than 3 ($= 2^2 - 1$) if beliefs were left unordered. We now show how this principle can be applied in a general setting with many belief types.

Consider the function $\tilde{h}_t : \mathcal{H} \to \tilde{\mathcal{H}}_t$, where $\tilde{\mathcal{H}}_t := \{1, \ldots, \tilde{H}_t\}$ is an adjusted set of types with the property that the most optimistic type(s) in $\mathcal{H}$ get label $\tilde{H}_t$, the next most optimistic type(s) gets label $\tilde{H}_t - 1$, and so on, down to the least optimistic type(s) in $\mathcal{H}$ with label 1. Types with equal optimism get the *same* label, so $\tilde{H}_t \leq H$, which implies that $|\tilde{\mathcal{H}}_t| \leq |\mathcal{H}|$. In the case of ties in terms of optimism, the period t population share of the "group" is the sum of the population shares of the individual types. We first present a corollary based on the adjusted set of types $\tilde{\mathcal{H}}_t$, before presenting an efficient solution algorithm.

Corollary 1. *Let $\tilde{\mathcal{H}}_t = \{1, \ldots, \tilde{H}_t\}$ be the set defined above, such that beliefs are ordered as $\tilde{E}_{t,1}[p_{t+1}] < \tilde{E}_{t,2}[p_{t+1}] < \ldots < \tilde{E}_{t,\tilde{H}_t}[p_{t+1}]$, or equivalently $f_{t,1} < f_{t,2} < \ldots < f_{t,\tilde{H}_t}$. Let $disp_{t,k} := \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,k})$, where $k \in \{1, \ldots, \tilde{H}_t - 1\}$. Then, we have the following:*

$$p_t = \begin{cases} \dfrac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{1+r} := p_t^* & \text{if } disp_{t,1} \leq a\sigma^2 \overline{Z} \\ \dfrac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2 \overline{Z}}{(1-n_{t,1})(1+r)} := p_t^{(1)} & \text{if } disp_{t,2} \leq a\sigma^2 \overline{Z} < disp_{t,1} \\ \dfrac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1}+n_{t,2}) a\sigma^2 \overline{Z}}{(1-n_{t,1}-n_{t,2})(1+r)} := p_t^{(2)} & \text{if } disp_{t,3} \leq a\sigma^2 \overline{Z} < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - \left(\sum_{h=1}^{\tilde{H}_t - 1} n_{t,h}\right) a\sigma^2 \overline{Z}}{\left(1 - \sum_{h=1}^{\tilde{H}_t - 1} n_{t,h}\right)(1+r)} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t - 1} > a\sigma^2 \overline{Z} \end{cases} \tag{8}$$

where $p_t^{(k^*)}$ is the price if types $1, \ldots, k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*, \;\; p_t^{(0)} := p_t^*. \tag{9}$$

Proof. It follows from Proposition 1. See the Appendix. □

Corollary 1 streamlines the task of finding the market-clearing price. In Proposition 1, where beliefs are unordered, there are $2^H - 1$ cases to check, as compared to $\tilde{H}_t \leq H$ when belief types are ordered as in Corollary 1. Clearly, this amounts to a substantial reduction in computational burden in models with a large number of types H. For example, with only 15 distinct beliefs (types) at date t, there are $2^{15}-1=32,767$ candidates for the sets $\mathcal{B}_t^*, \mathcal{S}_t^*$ when types are not ordered by optimism. However, if we order types from least to most optimistic and construct the set $\tilde{\mathcal{H}}_t=\{1,, \ldots, 15\}$, then there are only 15 candidates for the sets and the market-clearing price, corresponding to Corollary 1 when $\tilde{H}_t=15$. Ranking types in terms of optimism has previously been suggested by Anufriev and Tuinstra (2013); the main difference with our algorithm is that having analytical results as in Corollary 1 facilitates fast computation of solutions with many types; see Table 1 in Section 4.2.1.

The final part of Corollary 1 is important. It tells us that the market price when one or more short-selling constraints are binding is higher than the (hypothetical) price p_t^* if short-selling constraint are absent and that $p_t^*<p_t^{(1)}$ and $p_t^{(1)}<p_t^{(2)}<\ldots<p_t^{(k^*-1)}<p_t^{(k^*)}$, that is the price is smaller the fewer short-selling constraints are assumed to be binding. These properties are useful because we can use the unconstrained solution p_t^* to obtain a *lower bound* $\underline{k}$ for the actual number of types k^* who are short-selling constrained, by counting the number of negative (unconstrained) demands at price p_t^*. This gives our algorithm a good start. In a similar way, counting the number of negative (unconstrained) demands at prices $p_t^{(k)}$, for $k<k^*$, will give an improved estimate of k^* when it lies above the lower bound $\underline{k}$.

We now present a computational algorithm which efficiently finds the number of short-selling constrained types k^* and hence the market-clearing price and demands.

3.2 Computational algorithm

Our computational algorithm is simple and exploits the analytical results in Corollary 1. The main steps are as follows:

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering beliefs as $f_{t,1}<f_{t,2}<\ldots<f_{t,\tilde{H}_t}$ and find the associated population shares $n_{t,h}$ of types $h=1,\ldots,\tilde{H}_t$.
2. Compute $disp_{t,1}=\sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h}-f_{t,1})$. If $disp_{t,1}\le a\sigma^2\overline{Z}$, accept $p_t=p_t^*$ as the date t price, compute demands and move to period $t+1$. Otherwise, move to Step 3.
3. Set $p_t^{guess}=p_t^*$ and find the largest k such that $z_{t,k}^{guess}=\frac{f_{t,k}+a\sigma^2\overline{Z}-(1+r)p_t^{guess}}{a\sigma^2}<0$, and denote this value $\underline{k}$. Starting from $k=\underline{k}$, check if $disp_{t,k+1}\le a\sigma^2\overline{Z}<disp_{t,k}$; if not, try $k=k_{prev}+1$ until a k^* is found such that $disp_{t,k^*+1}\le a\sigma^2\overline{Z}<disp_{t,k^*}$.
4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t=p_t^{(k^*)}:=\frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}f_{t,h}-\left[\sum_{h=1}^{k^*} n_{t,h}\right]a\sigma^2\overline{Z}}{(1+r)\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}}$, compute demands and move to period $t+1$.

The above algorithm is efficient for two reasons. First, if the condition in Step 2 is met, no computation time is wasted checking cases where one or more types have binding short-selling constraints. Second, if the condition in Step 2 is not met, using the unconstrained solution p_t^* as a guess immediately gives a lower bound $\underline{k}$ on the number of short-selling constrained types k^*. Note that $\underline{k}$ is a lower bound for k^* since $p_t^{(k)}>p_t^*$ for all $k\le k^*$ (see Corollary 1); that is, binding short-selling constraints *raise* price relative to the counterfactual scenario of no constraints. Therefore, if types $1,\ldots,k$ would like to short-sell at price p_t^*, they *must* also be short-selling constrained at price $p_t^{(k^*)}>p_t^*$ (see (4)), implying that $k^*\ge\underline{k}$.

In practice, the speed of the computational algorithm can be improved with an iterative procedure. Specifically, rather than increasing k in steps of 1 as in Step 3 (whenever $\underline{k}$ is not a solution), the algorithm will "jump" closer to the true number of constrained types k^* if we repeatedly replace p_t^{guess} with an updated price $p_t^{(k)}$ based on the current guess k in Step 3 and generate an updated value of k, say k', that equals the number of negative (unconstrained) demands at this price. In other words, we exploit the property that $p_t^{(1)} < \ldots < p_t^{(k^*-1)} < p_t^{(k^*)}$ (Corollary 1) to find k^* faster. For a large number of types such as several thousand or more, there is a considerable speed-up with 5-10 iterations of this procedure.[9] Computation speed is thus an important advantage of our algorithm that uses the results in Corollary 1.

3.3 Conditional short-selling constraints

Let us briefly consider *conditional* short-selling constraints ahead of our policy application which studies such a rule. Let $g(p_{t-1}, \ldots, p_{t-K})$ be the "trigger" for the short-selling constraint, with K being the longest lag in the price that is considered. If $g(p_{t-1}, \ldots, p_{t-K}) \leq 0$, short-selling is banned at date t; if $g(p_{t-1}, \ldots, p_{t-K}) > 0$, the short-selling constraint is lifted; for example the US alternative uptick rule has $g^{AUR}(p_{t-1}, p_{t-2}) = p_{t-1} - (1-\kappa)p_{t-2}$, where $\kappa = 0.1$, such that short selling is banned if the price falls 10% or more in the previous trading period.

Let $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},\ldots,p_{t-K})\leq 0\}}$ be an indicator variable equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1}, \ldots, p_{t-K}) \leq 0$) and equal to 0 otherwise. With this formulation, investors may take negative positions in periods where the indicator variable is zero (short-selling constraints absent) but are restricted to non-negative positions in periods where the indicator variable is 1 (short-selling ban); for further details, see Section 1.2 of the Supplementary Appendix. The demand of type $h \in \mathcal{H}$ is thus given by

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}-(1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}}{1+\tilde{r}} \textit{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}}{1+\tilde{r}} \textit{ and } \mathbb{1}_t = 1. \end{cases} \tag{10}$$

The only difference in the demand schedules relative to (2) is the compound "if-or" and "if-and" statements that include the value of the indicator variable. As a result, it is straightforward to amend Proposition 1 and Corollary 1, as explained in the following remark.

Remark 1. *In the above model with a conditional short-selling constraint, the market-clearing price and demands follow Proposition 1, except that in part (i) the "if..." statement is replaced by "if... or* $\mathbb{1}_t = 0$*," and in part (ii) the "if..." statement is replaced by "if... and* $\mathbb{1}_t = 1$*." A proposition and proof are provided in Section 3.1 of the Supplementary Appendix.*

4. Application: Alternative uptick rule

We now consider an application with an alternative uptick rule, as currently in place in the United States. Under the rule, short-selling is banned next period following price falls of 10% or more. This contrasts with the original uptick rule in place from 1938 to 2007, which banned short-selling of shares following any fall in price (regardless of the magnitude). We work with a version of the Brock and Hommes (1998) model with a large number of types and an alternative uptick rule; this case does not seem to have been studied in the literature.

Since the alternative uptick rule bans short-selling following price falls of 10% or more, but *not* otherwise, it is a *conditional* short-selling constraint; see Section 3.3. Accordingly, the indicator variable has the form $\mathbb{1}_t := \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}}$ for $\kappa = 0.1$, and the solution is described by Remark 1. Demands of types $h \in \mathcal{H}$ are thus given by a version of (10):

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}-(1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}}{1+\tilde{r}} \text{ or } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}} = 0 \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+\bar{d}}{1+\tilde{r}} \text{ and } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}} = 1 \end{cases} \tag{11}$$

where we have assumed IID dividends $d_t = \bar{d} + \epsilon_t$ with $\tilde{E}_{t,h}[d_{t+1}] = \bar{d}\ \forall t, h$.

Equation (11) shows that short-selling is banned in period t only if $p_{t-1} \leq (1-\kappa)p_{t-2}$. Following Brock and Hommes (1998), we consider linear predictors of the form:

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{p} + b_h + g_h(p_{t-1} - \overline{p}), \qquad b_h \in \mathbb{R}, \quad g_h \geq 0. \tag{12}$$

Equation (13) is a standard specification in the literature. The intercept term consists of the fundamental price $\overline{p}$ plus "bias" b_h in the price forecast of type h, whereas g_h is trend-following parameter of type h. Type h is a pure fundamentalist investor if $b_h = g_h = 0$, while larger values of g_h or $|b_h|$ imply, respectively, stronger trend-following and stronger forecast bias.

The fundamental price $\overline{p}$ is the unique fundamental solution under common rational expectations; see Brock and Hommes (1998). Given that the risky asset is in positive net supply $\overline{Z} > 0$, the fundamental price is $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$, where $r := \tilde{r}$ is the interest rate on the riskless asset. Writing the predictor (12) in price deviations $x_t := p_t - \overline{p}$ gives:

$$\hat{E}_{t,h}\left[x_{t+1}\right] = b_h + g_h x_{t-1}, \qquad \text{where } \hat{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] - \overline{p}. \tag{13}$$

Given the indicator $\mathbb{1}_t = \mathbb{1}_{\{x_{t-1}+\kappa\overline{p} \leq (1-\kappa)x_{t-2}\}}$, the demands in (11) can be written as:

$$z_{t,h} = \begin{cases} \frac{\hat{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \vee \mathbb{1}_{\{x_{t-1}+\kappa\overline{p} \leq (1-\kappa)x_{t-2}\}} = 0 \\ 0 & \text{if } x_t > \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \wedge \mathbb{1}_{\{x_{t-1}+\kappa\overline{p} \leq (1-\kappa)x_{t-2}\}} = 1. \end{cases} \tag{14}$$

Fitness $U_{t,h}$ is a linear function of past profits net of predictor costs $C_h \geq 0$. Profits at date t are given by scaling demand $z_{t-1,h}$ by the realized excess return $R_t := p_t + d_t - (1+r)p_{t-1} = x_t - (1+r)x_{t-1} + a\sigma^2\overline{Z} + \epsilon_t$, where ϵ_t is the IID dividend shock, and we abstract from memory of past performance. For all $t \geq 1$ fitness and population shares are given by

$$U_{t,h} = R_t z_{t-1,h} - C_h, \quad n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h \in \mathcal{H}} \exp(\beta U_{t,h})}, \qquad \text{where } \beta \in [0, \infty). \tag{15}$$

The fitness levels $U_{t,h}$ determine the population shares $n_{t+1,h}$ of each type via a discrete-choice logistic model with intensity of choice β. The intensity of choice determines how fast agents switch to better-performing predictors. In the special case, $\beta = 0$ no switching occurs; increasing β implies more switching to relatively profitable predictors. Following Brock and Hommes (1997, 1998), this *evolutionary competition* mechanism has been widely studied.

We use the same parameters as in Section 3.1 of Anufriev and Tuinstra (2013): $\overline{Z} = 0.1$, $a\sigma^2 = 1$, $r = 0.1$, and we set $\overline{d} = 0.6$, giving a fundamental price $\overline{p} = \frac{\overline{d} - a\sigma^2\overline{Z}}{r} = 5$. In their model there are two types: a fundamentalist type with $\hat{E}_{t,f}\left[x_{t+1}\right] = 0$ and cost $C = 1$, and a chartist type with $\hat{E}_{t,c}\left[x_{t+1}\right] = \overline{g}x_{t-1}$, where $\overline{g} = 1.2$, and cost 0. We consider a large number of types $H = 1,000$, with predictors described by (13), population shares $n_{t,h}$ given by (15), and predictor costs C_h depending on the "closeness" of beliefs to a pure fundamentalist.

4.1 Benchmark exercise

We first perform a sanity check by giving 500 types a pure fundamental predictor (at cost $C = 1$) and the remaining 500 types the same chartist predictor $\overline{g} = 1.2$ at zero cost. In this case, there are two "groups" in the population whose aggregate population shares are determined endogenously based on past fitness. As a result, we should replicate the numerical bifurcation diagram in Anufriev and Tuinstra (2013, Figure 5) for the case of a *two-type* model with $d_t = \overline{d}$ and no short-selling constraint; see Fig. 1 below.

For sufficiently low values of the intensity of choice β, the fundamental steady state $x = 0$ is the unique price attractor. Intuitively, we are in the case $(1+r) < \overline{g} < 2(1+r)$ and positive outside supply in Fig. 1, for which (Anufriev and Tuinstra (2013, Proposition 3.1)) shows that the

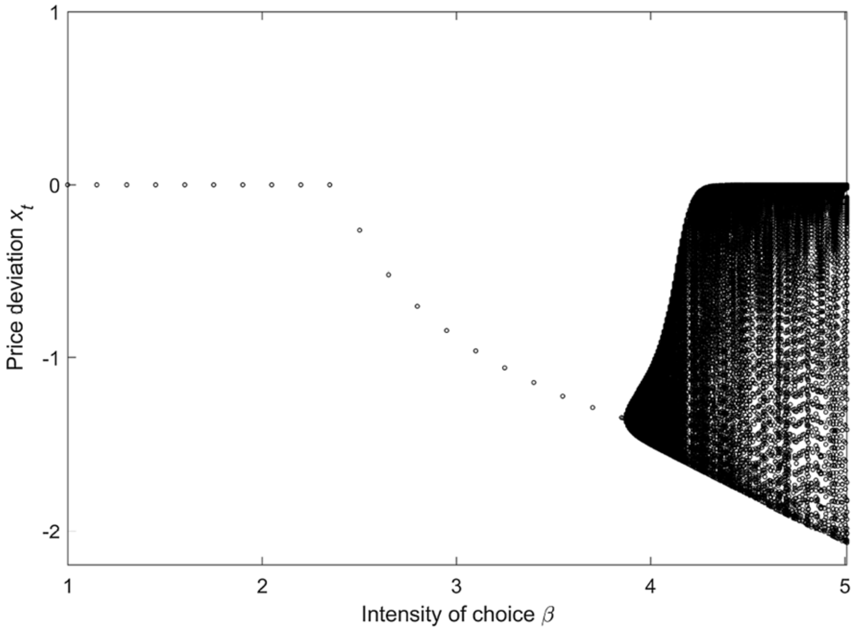

Figure 1. Bifurcation diagram in the absence of short-selling constraints. For each β, we plot 300 points following a transitory of 3000 periods from given initial values $x_0 \in (-4, 0)$.

fundamental steady state is globally stable for sufficiently small values of the intensity of choice β. Once a critical value of β is exceeded, there exist two non-fundamental steady states in addition to the fundamental steady state, which is locally stable. As β is increased further, however, the fundamental steady state becomes unstable, while the non-fundamental steady states are locally stable if β is not too large.

Given negative initial price, only the non-fundamental steady state with $x < 0$ is an attractor for the price dynamics at intermediate values of β; this amounts to the lower "fork" seen for β between (approx.) 2.4 and 3.8 in Fig. 1. Increasing β further causes the non-fundamental steady states to lose their stability through a Neimark-Sacker bifurcation, leading to an invariant closed curve and (quasi-)periodic dynamics. The results in Fig. 1 are consistent with those in Anufriev and Tuinstra (2013) for the same parameter values. Note that while we obtained the above diagram using $H = 1000$ types rather than two, we effectively have a two-type model since the groups consist of homogeneous investor types.

4.2 Simulated time series: Four scenarios

We now introduce heterogeneity proper by having many different types. We consider four different scenarios where the initial price x_0 is held fixed and only the intensity of choice β or the degree of heterogeneity (in g_h, b_h and C_h) are changed.[10] We first present simulated price series in four scenarios (without any noise) and then provide some results on computation speed and accuracy with stochastic dividends. We then consider some distributional implications of an alternative uptick rule by simulating the wealth distribution.

4.2.1 Four price simulations

The simulated price series in the four scenarios (S1–S4) are presented in Fig. 2. All four time series are started from the same initial price $x_0 = 3$, and we assume deterministic dividends $d_t = \overline{d} = 0.6$

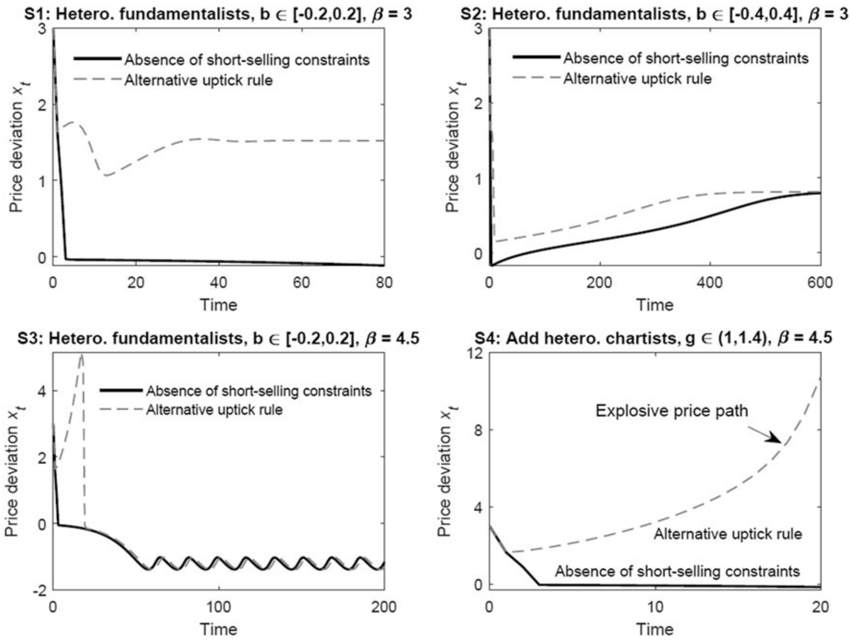

Figure 2. Simulated price series in four scenarios from an initial value $x_0 = 3$.

for all t in order to focus on the underlying dynamics. The four scenarios correspond to heterogeneity among the 500 fundamental types (with $g_h = 0$) due to bias b_h which is linearly spaced on the interval $[-0.2, 0.2]$ and predictor costs $C_h = 1 - |b_h|$ for such types (S1); the same setting as S1 except that heterogeneity is increased such that $b_h \in [-0.4, 0.4]$ (S2); the same setting as S1 except that the intensity of choice is increased from $\beta = 3$ to $\beta = 4.5$ (S3); and the same setting as S3 except that chartists are also heterogeneous with g_h drawn from a uniform distribution on the interval $(1, 1.4)$.

The price paths in the above scenarios are quite different, even though the additional heterogeneities are small (see Fig. 2). In Scenario 1 (top left), we see that if short-selling constraints are absent, the price quickly falls and then slowly converges on a non-fundamental steady state $x < 0$ (black line). Under an alternative uptick rule, by comparison, the initial drop in price is halted because the short-selling constraint binds; the price then oscillates around this higher value before converging on a non-fundamental steady state with $x > 0$. Thus, the alternative uptick rule gives quite a different long-run outcome. In Scenario 2, only bias differences among fundamentalists are increased, but this ensures that price converges on the same non-fundamental steady state in both cases (Fig. 2, top right). Thus, in contrast to Scenario 1, long-run price implications of the alternative uptick rule are absent in this case. This difference in results seems to be related to differences in performance across types when price is initially falling in the two scenarios; intuitively, greater heterogeneity implies more belief dispersion, but also larger differences in performance and hence population shares.

In Scenario 3, the intensity of choice β is set at 4.5 rather than at 3, and this is the only difference relative to Scenario 1. In this case, there are permanent price oscillations in both cases (bottom left); however, the short-run price dynamics under an alternative uptick rule are quite different, with an initial price spike after the short-selling constraint first binds, since the short-selling constraint binds on many types simultaneously. Lastly, in Scenario 4, (bottom right) heterogeneity in chartists is added on top of Scenario 3. In this case, the reversal in price under an alternative

Table 1. Computation times and accuracy in Scenario 3: $T = 500$ periods

No. of types	Regime	Time (s)	Bind freq.	max ($Error_t$)
$H = 1000$	No short-sell constraints	0.02	–	2.4e-14
	Alt. uptick rule: $\kappa = 0.1$	0.03	1/500	3.2e-14
	Orig. uptick rule: $\kappa = 0$	0.05	34/500	4.8e-14
$H = 10,000$	No short-sell constraints	0.17	–	2.3e-13
	Alt. uptick: $\kappa = 0.1$	0.18	1/500	6.3e-13
	Orig. uptick: $\kappa = 0$	0.25	43/500	3.0e-13
$H = 50,000$	No short-sell constraints	0.82	–	1.2e-12
	Alt. uptick: $\kappa = 0.1$	0.84	1/500	2.3e-12
	Orig. uptick: $\kappa = 0$	0.94	36/500	1.8e-12

Notes: $\max(Error_t) := \max\{Error_1, \ldots, Error_T\}$, where we define the date t simulation error as $Error_t = \left|\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}\right|$. Demands $z_{t,h}$ depend on the computed market-clearing price.

uptick rule is reinforced by trend-following into a permanent price "bubble" where the asset price diverges to $+\infty$. By comparison, when short-selling constraints are absent the price converges on a non-fundamental steady state $x < 0$, and hence the explosive price dynamics can be attributed to the short-selling regulation.

Computation speed and accuracy. Table 1 reports simulation times for Scenario 3 when the simulation length is 500 periods and the number of types is increased from $H = 1000$ to $H = 10,000$ and $H = 50,000$. Dividend shocks $d_t = \overline{d} + \epsilon_t$ are stochastic, and we consider the cases $\kappa = 0.1$ and $\kappa = 0$ (original uptick rule) because short-selling constraints bind more frequently in the latter case. We also include a measure of accuracy based on the distance between demand and supply at the computed price, that is $Error_t := \left|\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}\right|$; in particular, we report the largest absolute deviation recorded across all 500 periods.[11]

The results in Table 1 show that the solution algorithm is fast and accurate. The final column confirms that excess demand is essentially zero in all simulations, and the accuracy here is similar to when short-selling constraints are absent (top rows), in which case the standard analytical solution $x_t = (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}\hat{E}_{t,h}\left[x_{t+1}\right]$ is used to compute the price and the simulation error. Simulation times are below one second in all cases, increase with the number of types H, and are higher under the original uptick rule (where $\kappa = 0$), since this causes the short-selling constraint to bind in a much larger number of periods, as shown in the fourth column.[12] Even when short-selling constraints bind more frequently, computation times do not increase much and accuracy of the solution is preserved.

4.2.2 Distributional implications

We now consider some distributional effects of an alternative uptick rule. Recall that wealth of type h evolves as $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+r)\big(w_{t,h} - p_t z_{t,h}\big)$, such that an alternative uptick rule will affect wealth distribution through its impact on price and demands $z_{t,h}$. Note that if the short-selling constraint binds on type h at date t, then $z_{t,h} = 0$ and hence wealth evolves as $w_{t+1,h} = (1+r)w_{t,h}$. By being out of the market in period t, type h foregoes potential returns but also avoids potential losses. Thus, the implications of an alternative uptick rule for wealth distribution will depend on the distribution of returns and losses.

We stick with the same four scenarios as in Fig. 2 but we now focus on a measure of wealth inequality across investor types. In particular, we plot the Gini coefficient of the wealth

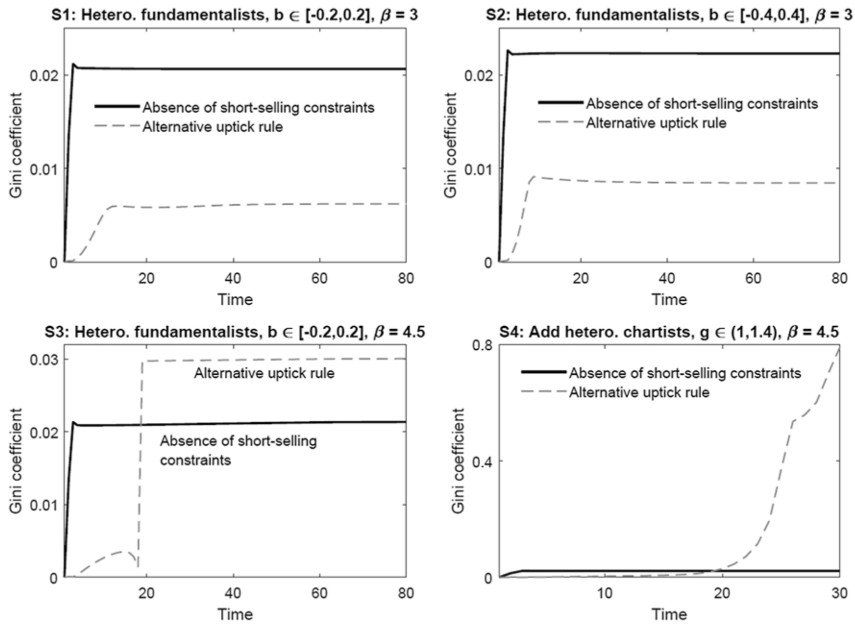

Figure 3. Simulated Gini coefficient of wealth in Scenarios 1 to 4.

distribution across types at each date t of our simulations. We assume all investor types have equal initial wealth, which we set at 50. The results are shown in Fig. 3.

An alternative uptick rule has mixed effects on wealth inequality. In Scenario 1 (Fig. 3, top left), the Gini coefficient initially increases and then settles, but there is a smaller increase in inequality with an alternative uptick rule since price does not fall sharply for several periods (Fig. 2, top left), which benefits more fundamental types. Such redistribution is smaller and more gradual under the alternative uptick rule because the fall in asset prices is smaller and, since price stabilizes, inequality remains lower in the long run. We see similar results in Scenario 2 (top right) where price also falls less under the alternative uptick rule.

In Scenario 3, wealth inequality is initially muted under an alternative uptick rule because price rises rather than falls (Fig. 2, bottom left). However, this initial period is followed by a severe drop in price, such that more fundamental types outperform more chartist types, and wealth inequality increases before stabilizing (see Fig. 3, bottom left). As a result, wealth inequality across types is initially lower under an alternative uptick rule but ends up higher in the long run. Finally, in Scenario 4 (Fig. 3, bottom right), wealth inequality across types is *initially* lower under an alternative uptick rule, as the initial period of falling prices is ended as in Scenario 1 (Fig. 2, bottom right). However, since price then explodes, strong chartist types make large profits and fundamental types losses, such that wealth inequality across types increases dramatically, with a Gini coefficient of around 0.8 by period 30.[13]

To better understand the wealth dynamics in Scenario 4, Fig. 4 plots the wealth distribution across types in periods $t = 3$, $t = 6$ and $t = 24$ under both unrestricted short-selling (top panel) and an alternative uptick rule (bottom panel). We see that wealth inequalities appear rather quickly under unrestricted short-selling, but not under an alternative uptick rule, where the initial fall in price is halted. However, as time increases, the price bubble under the alternative uptick rule soon leads to much greater inequality than if short-selling constraints are absent, and by period 24 an extremely large number of types have wealth levels that are a small fraction of the highest wealth type. These results are consistent with the rapid and sustained increase in the Gini coefficient observed in Fig. 3.

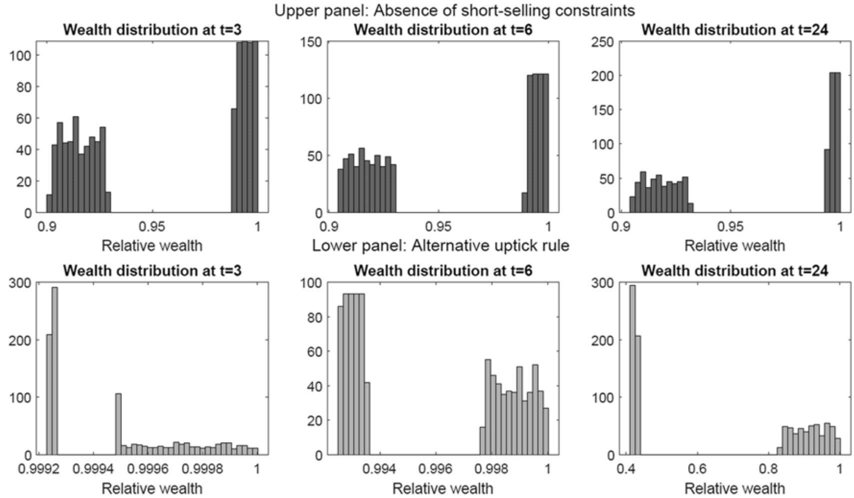

Figure 4. Simulated wealth distribution across types in Scenario 4.

4.3 Policy exercise: Varying κ

Having studied price dynamics and wealth distribution, we now ask whether "fine-tuning" the policy parameter κ in the alternative uptick rule could lead to a better mix between wealth inequality and mispricing from a policy perspective. Recall that $\kappa \geq 0$ represents the minimum percentage fall in price in period $t-1$ that will trigger a ban on short-selling in period t; and $\kappa = 0.1$ under the current US short-selling regulation.

We consider an *ad hoc* loss function which says policymakers dislike mispricing (deviations from the fundamental price) and wealth inequality measured by the Gini coefficient:

$$L_\kappa = \sum_{t=1}^{T} |x_t| + \lambda \sum_{t=1}^{T} Gini_t \tag{16}$$

where $\lambda > 0$ is the relative weight on wealth inequality.

The loss function (16) treats positive and negative price deviations equally, and we assume policymakers care about the sum of mispricings and the sum of Gini coefficients over a horizon of T periods. The loss is denoted L_κ because it depends on the parameter κ in the alternative uptick rule. Recall that $\kappa = 0.1$ corresponds to the current alternative uptick rule in the United States, while $\kappa = 0$ corresponds to the original uptick rule that was followed until 2007. We restrict our analysis to values that satisfy $\kappa \in [0, 0.1]$, that is the original uptick rule, the alternative uptick rule, and intermediate options in between these two cases.

In Fig. 5, we plot mispricing, wealth inequality, and the loss function L_κ for 50 different values of κ equally spaced in the interval [0,0.1]. We focus on Scenario 4, in which both heterogeneous fundamentalists and chartists are present, and we set $\beta = 3.5$ and $T = 40$. Mispricing and inequality (as in (16)) are normalized by dividing by their values when short-selling constraints are absent in all periods; hence the y-axis values tell us whether *for a given κ value* the rule lowers mispricing and inequality. Note that if mispricing (inequality) is smaller than 1, then for λ is sufficiently small (large) the loss in (16) will be lower with an uptick rule than under unfettered short-selling. Glancing at Fig. 5, we see that the short-selling rule reduces inequality but raises mispricing relative to unrestricted short-selling.

As κ is increased (so that larger drops in price are needed to trigger a short-selling ban), the impact on mispricing and inequality is quite different (left and middle panel). Increasing κ from 0 (original uptick rule) to a small positive value lowers both mispricing and wealth inequality;

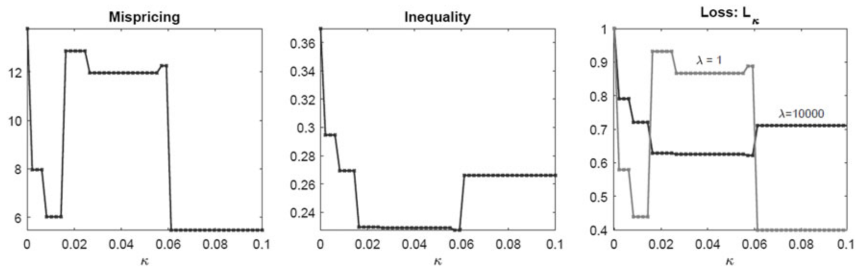

Figure 5. Mispricing, wealth inequality, and the loss L_κ in Scenario 4 when $\beta = 3.5$. Mispricing and inequality are ratios to the values when short-selling is unrestricted; we normalize $\max(Loss)$ to 1. The parameter κ takes on 50 values linearly spaced on $[0, 0.1]$. The plot of the loss is shown in the final panel for two different λ values, $\lambda = 1$ and $\lambda = 10, 0000$.

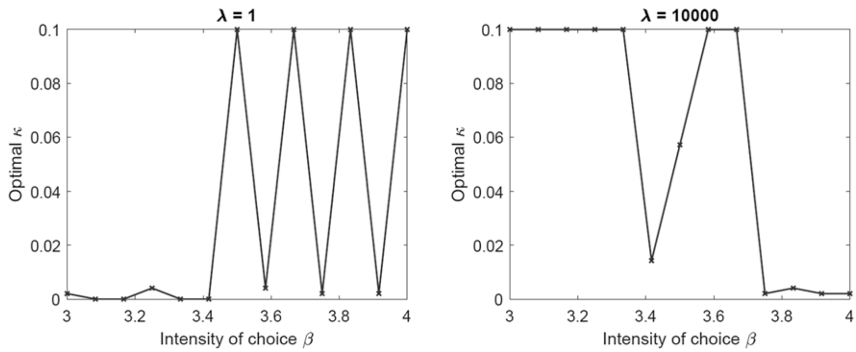

Figure 6. Optimal κ in Scenario 4 for various β and λ. The relative weight on inequality is set at either $\lambda = 1$ (left panel) or $\lambda = 10, 000$ (right panel). Results are based on 50 values of κ linearly spaced on the interval $[0, 0.1]$ and 13 values of β linearly spaced on $[3, 4]$.

however, increasing κ further leads to a *tradeoff*: mispricing increases while inequality goes on falling; then, at the highest values of κ, inequality is increased whereas mispricing is minimized. Because inequality is minimized at intermediate values of κ (close to 0.06) while mispricing is minimized at the highest values of κ (including $\kappa = 0.1$), the loss is sensitive to the value of λ, the relative weight on wealth inequality.[14] In particular, for relatively low values such as $\lambda = 1$ the loss is minimized by the alternative uptick rule ($\kappa = 0.1$), whereas for sufficiently high weight λ on inequality the loss is minimized at intermediate κ (≈ 0.06) because inequality is also (Fig. 5, middle and right panel). As a first pass, the results in Fig. 5 suggest policymakers might improve on the current alternative uptick rule by choosing a value of κ which is positive but smaller than 10%.

We now consider robustness. We start by computing the optimal (i.e. loss-minimizing) value of κ, among our 50 values linearly spaced on the interval $[0, 0.1]$, for two different values of λ when the intensity of choice β is increased; see Fig. 6. Here we follow the rule that if the loss-minimizing value of κ is unique, we report that value; otherwise, we report as optimal the highest (lowest) value of κ in the set of loss-minimizing values when $\kappa = 0.1$ is (is not) included in the set. This rule means that $\kappa = 0.1$ is reported as optimal whenever we find the current alternative uptick rule cannot be improved upon.

From Fig. 6, we see that the optimal value of κ is sensitive even to small changes in β once the intensity of choice is large enough. The conclusion that intermediate values of κ can minimize the loss (16) is robust: $\kappa \in (0, 0.1)$ is optimal in around one-half of the cases in each panel. For a relatively low weight on inequality (left panel), the optimal κ is either 0 (original uptick rule), 0.1 (alternative uptick rule), or slightly above 0. By comparison, with a large relative weight on inequality the original uptick rule is never optimal, and intermediate κ close to 2% and 6% is

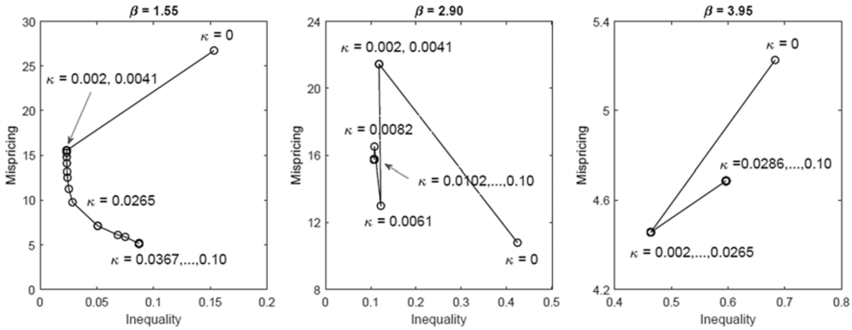

Figure 7. Plots of mispricing versus inequality: various β. Each panel plots the relationship between mispricing and wealth inequality for a given β as the policy parameter κ is varied. Mispricing and inequality are ratios to the values when short-selling is unrestricted. The parameter κ takes on 50 values linearly spaced on the interval [0, 0.1].

optimal in some cases (right panel). In short, a high weight on inequality seems to justify moving away from the polar cases of the original and the alternative uptick rules, such that a short-selling ban is triggered by price falls somewhat smaller than 10%.

To illustrate how κ influences the mix between mispricing and wealth inequality, we now present plots of this relationship as κ is varied. We do this for three different values of the intensity of choice β to highlight some different cases; see Fig. 7. In the first case (left panel) where $\beta = 1.55$, the original uptick rule is clearly dominated since setting $\kappa > 0$ lowers both mispricing and inequality. Once $\kappa > 0$ we see a clear *tradeoff*: increasing κ lowers mispricing but raises inequality, such that the optimal value of κ will get smaller as λ is increased. In the second case ($\beta = 2.90$, middle panel), there is no clear relationship between mispricing and inequality. The original uptick rule ($\kappa = 0$) minimizes mispricing in this case, but inequality is reduced by setting $\kappa > 0$. Setting κ close to zero (0.006) raises inequality slightly but reduces mispricing by around one-fifth compared to higher values of κ (0.008–0.1), so would be optimal for policymakers who prefer to balance mispricing and inequality. Finally, in the right panel of Fig. 7 we see a case where intermediate values of κ (0.002–0.0265) are optimal *irrespective* of the value of λ because such a policy *minimizes mispricing and inequality*, such that both the original uptick rule and the alternative uptick rule are dominated. While this is just a single numerical example, it shows that policies which lie *between* the original uptick rule ($\kappa = 0$) and the current alternative uptick rule ($\kappa = 0.1$) may improve price discovery *and* reduce wealth inequality across types.

In summary, our results suggest it might be possible to improve on the current alternative uptick rule by having short-selling bans triggered by smaller falls in price. While we should be cautious about drawing general conclusions from a small number of policy experiments, these results clearly raise questions about whether current regulation could be improved.

5. Extensions

Having studied a benchmark model and a policy application, we now consider some extensions to the model. Any minor extensions or generalizations, such as individual investors in a social network; physical investment assets like housing or time-varying expected dividends; and short-selling up to a limit, are discussed in Section 2 of the Supplementary Appendix.

5.1 Multiple assets subject to short-selling constraints

We first consider *multiple* risky assets in positive net supply and subject to unconditional short-selling constraints. Let $z^m_{t,h}$ be the date t demand of type h for asset $m \in \{1, \ldots, M\}$, where $M \geq 2$.

Following Westerhoff (2004), type h's demand for asset m depends not only on the expected excess return on asset m but also on the relative attractiveness of that asset. In particular, a fraction $w_t^m \in (0,1)$ of each investor type participates in a given market m, with this fraction determined by comparison with all other markets (see below). Differently from Westerhoff, we add unconditional short-selling constraints, such that $z_{t,h}^m \geq 0$ for all m. Each asset market has IID dividends $d_t^m = \overline{d}^m + \epsilon_t^m$, and we assume $\tilde{E}_{t,h}[d_{t+1}^m] = \overline{d}^m > 0\ \forall m$.

Analogous to (2), the demand of type $h \in \mathcal{H}$ in market $m \in \{1, \ldots, M\}$ is

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{\tilde{E}_{t,h}[p_{t+1}^m] + \overline{d}^m - (1+\tilde{r})p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{\tilde{E}_{t,h}[p_{t+1}^m] + \overline{d}^m}{1+\tilde{r}} \\ 0 & \text{if } p_t^m > \frac{\tilde{E}_{t,h}[p_{t+1}^m] + \overline{d}^m}{1+\tilde{r}} \end{cases} \tag{17}$$

where p_t^m is price in market m and σ_m^2 is the subjective return variance (assumed constant).

The demand function (17) has the same form as in the benchmark case (see (2)), except for the scaling by the share w_t^m that participates in the market. As in Westerhoff (2004), we assume the participation shares w_t^m depend on relative attractiveness of each market, A_t^m:

$$w_{t+1}^m = \frac{\exp(\beta A_t^m)}{\sum_{m=1}^M \exp\left(\beta A_t^m\right)}, \qquad A_t^m = f\left(\left[p_t^m - \overline{p}^m\right]\right) \tag{18}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(0) = 0$ and $\beta \in [0, \infty)$ is the intensity of choice.

Equation (18) shows that the participation shares w_t^m are determined by *evolutionary competition*, with attractiveness, A_t^m, of market m depending on the deviation of the market price from the fundamental price $\overline{p}^m$.[15] Given equations (17)–(18), the fundamental price in market m is $\overline{p}^m = \frac{\overline{d}^m - a\sigma_m^2 M\overline{Z}_m}{\tilde{r}}$, where $\overline{Z}_m > 0$ is the fixed supply of asset m per investor.[16]

Belief types in each market m follow a market-specific version of Assumption 1 (see (3)):

$$\tilde{E}_{t,h}\left[p_{t+1}^m\right] = \overline{c}^m p_t^m + \tilde{f}_{t,h}^m, \qquad \overline{c}^m \in [0, 1+\tilde{r}) \tag{19}$$

where $\tilde{f}_{t,h}^m$ is a generic price forecast of type h in market m that does not depend on p_t^m.

Let $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m / w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$ (see (3)–(4)). Then, by (17),

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r^m)p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m}. \end{cases} \tag{20}$$

We assume the population shares in each market m are determined by Assumption 2. Market-clearing in each market is thus given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m / w_t^m. \tag{21}$$

With the change in variables in (21), the market-clearing condition has the same form as in the benchmark model, except for a scaling of supply by $1/w_t^m$. Hence, we have the following.

Remark 2. *In the above model with M risky assets subject to short-selling constraints, the expressions for the market-clearing prices p_t^m and demands $z_{t,h}^m$ in each market $m \in \{1, \ldots, M\}$ are given by Proposition 1, except that $p_t, f_{t,h}, r, \overline{Z}$ must be replaced by $p_t^m, f_{t,h}^m, r^m, \overline{Z}_m/w_t^m$, and the demands $z_{t,h}$ are replaced by market-specific demands $z_{t,h}^m$ in (20). A reworked version of Proposition 1 is provided in Section 3.2 of the Supplementary Appendix.*

5.2 Additional heterogeneity

We now consider additional heterogeneity relative to the baseline model through price beliefs with different weights on the current price and different perceptions of return risk. We confine technical details to the Supplementary Appendix and provide an illustrative example.

5.2.1 Heterogeneous responses to current price

Assumption 1 allows a common response of beliefs to price via the term $\bar{c}p_t$ in (3), where $\bar{c} \in [0, 1 + \tilde{r})$. We now allow *heterogeneity across types*.[17] In this case, price beliefs are

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \bar{c}_h p_t + \tilde{f}_{t,h} \tag{22}$$

where $\bar{c}_h \in [0, 1 + \tilde{r})$. Defining $f_{t,h} := \tilde{f}_{t,h} + \bar{d} - a\sigma^2\bar{Z}$ and $r_h := \tilde{r} - \bar{c}_h$, demands are now

$$z_{t,h} = \begin{cases} \frac{f_{t,h} + a\sigma^2\bar{Z} - (1+r_h)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h}+a\sigma^2\bar{Z}}{1+r_h} \\ 0 & \text{if } p_t > \frac{f_{t,h}+a\sigma^2\bar{Z}}{1+r_h} \end{cases} \tag{23}$$

where the only difference relative to (4) is that r is now *type-specific*.

Equations (22)–(23) show that optimism is no longer determined solely by $f_{t,h}$; however, we can distinguish least and most optimistic types by looking at the term $\frac{f_{t,h}+a\sigma^2\bar{Z}}{1+r_h}$, since a given type h will short-sell only if this term is sufficiently small. For example, a type with large $f_{t,h}$—say a chartist type in a bull market—need not value the asset more highly than a fundamental type who ignores the trend completely, if the latter puts much more weight on the *current* price when forming their forecast of the future price; see (22).

Given the demand function (23), the sets of unconstrained and short-selling constrained types $\mathcal{B}_t^*, \mathcal{S}_t^*$ depend on $\min_{h\in\mathcal{B}_t^*}\left\{\frac{f_{t,h}+a\sigma^2\bar{Z}}{1+r_h}\right\}$ and $\max_{h\in\mathcal{S}_t^*}\left\{\frac{f_{t,h}+a\sigma^2\bar{Z}}{1+r_h}\right\}$, rather than $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}$ and $\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ as in the benchmark case (see Proposition 1, part (ii)). Nevertheless, it is simple to amend Proposition 1, Corollary 1, and the computational algorithm. We provide these results in Section 4.1 of the Supplementary Appendix, along with a simple numerical example which shows that the solution subject to short-selling constraints is fast, accurate, and characterized by persistent price cycles that would otherwise be absent.[18]

5.2.2 Heterogeneity in variances

Now consider heterogeneity in *subjective risk perceptions* (or, equivalently, heterogeneity in the risk aversion parameter a), with price beliefs given by (3) as in the benchmark model. In this case, the terms $a\sigma^2$ in the demand function (2) become type-specific, that is $a\sigma_h^2$, and it is convenient to define $\tilde{a}_h := \left(a\sigma_h^2\right)^{-1}$, $f_{t,h} := \tilde{f}_{t,h} + \bar{d} - \bar{Z}/\tilde{a}_h$, and $r := \tilde{r} - \bar{c}$ as before. The demand of each type $h \in \mathcal{H}$ is now given by

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} + \bar{Z}/\tilde{a}_h - (1+r)p_t) & \text{if } p_t \leq \frac{f_{t,h}+\bar{Z}/\tilde{a}_h}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h}+\bar{Z}/\tilde{a}_h}{1+r}. \end{cases} \tag{24}$$

In this case, types can be ranked in terms of optimism by looking at the predetermined forecast component $\tilde{f}_{t,h}$ as in the baseline model; see (24) and the definition of $f_{t,h}$ above. Nevertheless, the market-clearing price *is* affected by heterogeneous variances; intuitively, this is because asset demand by type h equals their expected excess return scaled by a risk correction, so higher risk perceptions (or risk aversion) will lower demand *ceteris paribus*.

Working through the algebra reveals that the short-selling conditions and market price now depend on the *risk-weighted* population shares $\tilde{a}_h n_{t,h}$, as shown in amended versions of Proposition 1 and Corollary 1 in Section 4.2 of the Supplementary Appendix (a computational algorithm and a numerical example with many types are also provided). Since it is difficult to provide intuition in a general setting, we now give a brief two-type example, with evolutionary competition between types, that builds directly on Example 1.

5.2.3 A simple example

Let us re-solve Example 1 for the case of heterogeneous variances.

Example 2. *Reconsider Example 1 with two types: fundamentalist h_1 with $\tilde{E}_{t,h_1}\left[p_{t+1}\right]=\overline{p}$, where $\overline{p}$ is the fundamental price, and chartist h_2 with $\tilde{E}_{t,h_2}\left[p_{t+1}\right]=\overline{p}+\overline{g}(p_{t-1}-\overline{p})$, where $\overline{g}>0$.*[19] *The population shares are given by $n_{t,h}=\frac{exp\left(\beta U_{t-1,h}\right)}{\sum_{h\in\{h_1,h_2\}}exp(\beta U_{t-1,h})}$, where $\beta\in[0,\infty)$, $U_{t,h}=R_t z_{t-1,h}$, $R_t=p_t+d_t-(1+\tilde{r})p_{t-1}$, $r:=\tilde{r}$, and $z_{t-1,h}$ is the demand at date $t-1$. We assume as in Example 1 that $p_{t-1}>\overline{p}$, so the chartist is more optimistic at date t. We also assume types differ in their subjective variances: $\sigma^2_{h_1}\neq\sigma^2_{h_2}$ such that $\tilde{a}_{h_1}\neq\tilde{a}_{h_2}$.*

By comparison to equation (7), we now have the following price solution:

$$p_t=\begin{cases}\overline{p}+\left[\dfrac{\tilde{a}_{h_2}n_{t,h_2}}{\tilde{a}_{h_1}n_{t,h_1}+\tilde{a}_{h_2}n_{t,h_2}}\right]\dfrac{\overline{g}(p_{t-1}-\overline{p})}{1+r} & \text{if } \overline{g}(p_{t-1}-\overline{p})\leq\left[coef_t\right]\overline{Z}\\[2ex] \overline{p}+\dfrac{n_{t,h_2}\overline{g}(p_{t-1}-\overline{p})-\left[(1-n_{t,h_2})/\tilde{a}_{h_2}\right]\overline{Z}}{n_{t,h_2}(1+r)} & \text{if } \overline{g}(p_{t-1}-\overline{p})>\left[coef_t\right]\overline{Z}\end{cases}\tag{25}$$

where $coef_t:=\dfrac{\tilde{a}_{h_1}n_{t,h_1}+\tilde{a}_{h_2}n_{t,h_2}}{\tilde{a}_{h_1}\tilde{a}_{h_2}n_{t,h_2}}$ and terms in square brackets differ relative to Example 1.

Equation (25) shows how price differs relative to Example 1; in particular, heterogeneity in subjective variances means that the *risk-weighted* population shares of both fundamentalists and chartists influence the price. Further, relative to Example 1 the "if" conditions under which short-selling occurs now differ, as the latter conditions are intimately related to market-clearing. In a similar vein to the above example, a reworked version of Example 1 can easily be formulated for the case of heterogeneous forecast weights on the current price (Section 5.2.1 above) using the results in Section 4.1 of the Supplementary Appendix.

5.3 Market-maker plus short-selling constraints

As a final extension, we let price be determined by a market maker in response to demand, rather than by market-clearing; see, for example, Beja and Goldman (1980), Chiarella (1992), Farmer and Joshi (2002) and Westerhoff (2003). As is standard in the literature, we consider price impact functions which are linear in excess demand. We allow the price set by the market maker to increase with *current* and *past* excess demand as follows:[20]

$$p_t=p_{t-1}+\mu[\lambda(Z_t-\overline{Z})+(1-\lambda)(Z_{t-1}-\overline{Z})]\tag{26}$$

where $\mu>0$, $\lambda\in(0,1]$ and $Z_t:=\sum_{h\in\mathcal{H}}n_{t,h}z_{t,h}$ is aggregate demand per investor at date t, such that $Z_t-\overline{Z}$ can be interpreted as (average) excess demand per investor type.

When $\lambda\in(0,1)$, past demand matters for the current price, whereas if $\lambda=1$ only current demand Z_t matters. We stick with the beliefs $\tilde{E}_{t,h}\left[p_{t+1}\right]=\overline{c}p_t+\tilde{f}_{t,h}$ in Assumption 1 and consider two different specifications of asset demands. First, for the case of baseline demands, equation (4) holds as before. In this case, we can easily obtain amended versions of Proposition 1 and

Corollary 1 since we just need to solve for the price p_t using the price equation (26) rather than by imposing market-clearing. These analytical results are similar to the baseline case and are provided in Section 3.3 of the Supplementary Appendix.

Now consider an alternative demand specification from the market-maker literature. A common demand specification is $\tilde{a}_h(\tilde{E}_{t,h}\left[p_{t+1}\right] - p_t)$, where $\tilde{a}_h > 0$ is a type-specific coefficient; see for example Westerhoff (2004). With a short-selling constraint $z_{t,h} \geq 0$, demands are adjusted to

$$z_{t,h} = \begin{cases} \tilde{a}_h\left(\tilde{E}_{t,h}\left[p_{t+1}\right] - p_t\right) & \text{if } p_t \leq \tilde{E}_{t,h}\left[p_{t+1}\right] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}\left[p_{t+1}\right]. \end{cases} \tag{27}$$

The key difference relative to the baseline specification of demand, (2), is that each type's asset demand is scaled by the *type-specific* coefficient $\tilde{a}_h$. Note that more pessimistic types—that is those with lower expectations $\tilde{E}_{t,h}\left[p_{t+1}\right]$—are more likely to be short-selling constrained at a given price p_t set by the market maker. The analytical results for this case are lengthy and quite technical, so we provide these in the Supplementary Appendix; in particular, we give amended versions of Proposition 1, Corollary 1, and the Computational Algorithm.

6. Conclusion

This paper has studied dynamic behavioral asset pricing models with short-selling constraints and many investor types with heterogeneous beliefs. Our results provide analytical expressions for price and demands, along with conditions on beliefs such that short-selling constraints bind on different types, allowing us to construct computationally efficient solution algorithms. The analysis is built around a Brock and Hommes (1998) model with short-selling constraints and generic price predictors; we also presented extensions for the cases of multiple risky assets; additional heterogeneities; and pricing by a market maker.

The utility of these results was shown via examples and a numerical application that studied an alternative uptick rule, as currently in place in the United States, in a market with a large number of belief types in *evolutionary competition*. The results highlight the complicated relationship between the design of short-selling regulations and their implications for asset mispricing and wealth distribution. In particular, an alternative uptick rule may attenuate (or prevent) falls in price, but we also found that such rules can hinder price discovery, increase price volatility, and lead to explosive price paths. An alternative uptick rule can also have substantive distributional (wealth) implications, and we showed a scenario in which a modified alternative uptick rule, which bans short-selling following smaller falls in price, reduces both mispricing and wealth inequality relative to the current regulation.

There are several promising avenues for future research. First, it would be of interest to investigate whether adding short-selling constraints in models with many belief types improves the ability of such models to reproduce empirical stylized facts, especially during times of market turmoil, when such constraints are more likely to be present. In a similar vein, it may be feasible to estimate such models in order to evaluate the relative empirical contribution of adding short-selling constraints. Second, from a policy perspective, there has been interest in whether short-selling restrictions lead to mispricing and might cause or exacerbate price bubbles, both in financial markets and housing markets (Shiller (2015), Fabozzi et al. (2020)). It would be of interest to investigate this further alongside the distributional implications of short-selling restrictions in models with many investor types or agents.

Finally, from a technical perspective, there are some modeling specifications of interest which are not covered by the results presented in this paper. For instance, one could confront a large number of investor types with additional restrictions such as a short-selling tax (rather than full ban) or leverage constraints (Anufriev and Tuinstra (2013), in't Veld (2016)); the elimination of investors who hit zero or negative wealth; or margin calls that prevent a short position being maintained in future periods. These approaches might have important implications not just for the price effects of short-selling restrictions, but also their distributional implications that have received little attention so far.

Supplementary material. To view supplementary material for this article, please visit https://doi.org/10.1017/S1365100523000639

Notes

1 Short-selling constraints appear to have first been studied, in a static model, by Miller (1977).
2 We assume (as is standard) that $\tilde{E}_{t,h}[y_t] = y_t$ and $\tilde{V}_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2 \tilde{V}_{t,h}[y_{t+1}]$.
3 See the definition of $f_{t,h}$ in (4), which potentially allows $\tilde{E}_{t,h}\left[d_{t+1}\right]$ to vary over time and across types.
4 Given our assumptions, the fundamental price is $p_t^* = \overline{p} := (\overline{d} - a\sigma^2\overline{Z})/\tilde{r}$, so $f_{t,h} = \tilde{f}_{t,h} + \tilde{r}\overline{p}$ (see (4)). Thus, $f_{t,h} - (1+r)p_t = \hat{E}_{t,h}\left[x_{t+1}\right] - (1+r)x_t$, where $r = \tilde{r} - \overline{c}$, $x_t := p_t - \overline{p}$, and $\hat{E}_{t,h}\left[x_{t+1}\right] := \tilde{f}_{t,h} - (1-\overline{c})\overline{p}$. Demands follow (4), with x_t replacing p_t and $\hat{E}_{t,h}\left[x_{t+1}\right]$ replacing $f_{t,h}$. For an applied example, see Section 4.
5 Allowing $n_{t,h} \in [0, 1]$ means the set of types relevant for price determination (i.e. market-clearing) can be time-varying, which reduces analytical tractability of the results relative to Proposition 1.
6 After their Proposition 2.1, Anufriev and Tuinstra (2013, p. 1529) note that their expressions should not be confused with an explicit solution: "Note however that x_t is still implicitly defined by (10) since the right-hand side also depends upon x_t through the definition of the sets $P(x_t)$, $Z(x_t)$ and $N(x_t)$. Below we will derive the market equilibrium price x_t explicitly for some special cases" (they study a two-type example).
7 In particular, given our assumption that $p_{t-1} > \overline{p}$, equation (7) is consistent with Proposition 2.2 in Anufriev and Tuinstra (2013) when short-selling is prohibitively costly.
8 The number of candidate sets equals the size of the power set of $\mathcal{H}$ with a correction of "minus 1." The correction arises because the asset market cannot clear if $\mathcal{B}_t^*$ is an empty set (i.e. if no type holds the asset).
9 Note that this procedure will not overshoot k^* because the guessed price will remain below the market-clearing price. If $k' = k_{prev}$ or if k^* is reached, the iterations are terminated early using a "break" command.
10 All other parameters are the same as in the previous section, so $a\sigma^2 = 1$, $\overline{d} = 0.6$, $r = 0.1$ and $\overline{Z} = 0.1$.
11 Dividend shocks ϵ_t were drawn at date 0 from a truncated-normal distribution with mean zero, standard deviation $\sigma_d = 0.01$ and support $[-\overline{d}, \overline{d}]$. Simulations were run in MATLAB 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20 GHz processor and 8 GB of RAM.
12 Recall that $\kappa = 0.1$ means short-selling constraint is banned in period t only if price fell by 10% or more in the previous period. For $\kappa = 0$, short-selling constraint is banned following any previous fall in price.
13 The "kink" in period 26 arises because we assume that types that hit negative wealth have it reset to zero, and period 26 is the first period in which this rule is triggered.
14 Note that mispricing, inequality, and the loss can remain unchanged as κ is increased since falls in price may exceed the threshold set by κ and this can remain true even when κ increases.
15 Westerhoff (2004) sets $f\left(\left[p_t^m - \overline{p}^m\right]\right) = \ln\left[\left(1 + c\left[p_t^m - \overline{p}^m\right]^2\right)^{-1}\right]$, where $c > 0$, such that attractiveness declines with distance from the fundamental price due to the risk of being caught in a bubble that collapses.
16 If all investors are fundamentalists, then $A_t^m = f(0) = 0$ for all m, such that $w_t^m = 1/M$ for all m. Using this result in conjunction with the demands (17), common rational expectations $E_t[p_{t+1}^m]$ and market-clearing leads to the equation $p_t^m = (1+\tilde{r})^{-1}\left[E_t\left[p_{t+1}^m\right] + \overline{d}^m - a\sigma_m^2 M\overline{Z}_m\right]$, which can be solved forwards to give $\overline{p}^m$.
17 Allowing time-variation in $\overline{c}$ is straightforward. Here we take values as fixed and focus on heterogeneity.
18 See Figure 1 and Table 1 in Section 4.1 of the Supplementary Appendix.
19 Recall $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$. This price is derived for all agents fundamentalists with common variance σ^2.
20 Allowing price to be a non-linear function of *past* excess demand $Z_{t-1} - \overline{Z}$ does not pose any difficulty as this variable is predetermined at date t; however, we use linearity in *current* excess demand to solve for p_t.

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Appendix

Proof of Proposition 1.

A unique market-clearing price exists by Proposition 2.1 in Anufriev and Tuinstra (2013).

Case 1: Short-selling constraint is slack for all h ∈ $\mathcal{H}$.

Let us guess that $z_{t,h} = \left(a\sigma^2\right)^{-1}\left(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t\right) \geq 0 \quad \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $p_t = p_t^* := (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0 \quad \forall h \in \mathcal{H}$, which amounts to $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} \leq \min_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z}$. Given $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, the above inequality simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}\right) \leq a\sigma^2\overline{Z}$, as stated in Proposition 1.

Case 2: Short-selling constraint slack for all h ∈ $\mathcal{B}_t^*$ and binds for all h ∈ $\mathcal{H} \setminus \mathcal{B}_t^*$.

Let us guess that $z_{t,h} = \left(a\sigma^2\right)^{-1}\left(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t\right) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of constrained types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$, and hence, the market-clearing condition is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, which gives $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \left(1-\sum_{h\in\mathcal{B}_t^*} n_{t,h}\right)a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} := p_t^{\mathcal{B}_t^*}$.

The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} < 0\ \forall h \in \mathcal{S}_t^*$, that is iff $\left(f_{t,h} + a\sigma^2\overline{Z}\right)\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \left(1-\sum_{h\in\mathcal{B}_t^*} n_{t,h}\right)a\sigma^2\overline{Z}\ \forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, which simplifies to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}\left(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}\right)$, as stated in Proposition 1.

It remains to show $p_t^{\mathcal{B}_t^*} > p_t^* = \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{1+r}$, where p_t^* is the price if short-selling constraints are absent. Note $(1+r)\left(p_t^{\mathcal{B}_t^*} - p_t^*\right) = \left(1 - \frac{1}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}}\right)a\sigma^2\overline{Z} + \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$ and $\sum_{h\in\mathcal{B}_t^*} n_{t,h} = 1 - \sum_{h\in\mathcal{S}_t^*} n_{t,h}$. Since $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \sum_{h\in\mathcal{S}_t^*} n_{t,h}f_{t,h}$, we obtain:

$$(1+r)\left(p_t^{\mathcal{B}_t^*} - p_t^*\right) = \left(\sum_{h\in\mathcal{S}_t^*} n_{t,h}\right)\left[\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_t^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}}\right] > 0$$

where $\sum\limits_{h\in\mathcal{S}_t^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}} f_{t,h} \leq \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, and $\frac{\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}}{\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}} > \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ is implied by the condition $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) > a\sigma^2\overline{Z}$ above.

Proof of Corollary 1.

The first "if" statement follows from Proposition 1 as $\sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1}) \leq a\sigma^2\overline{Z}$ is equivalent to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$. The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, that is $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, \ldots, \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1, 2\}, \mathcal{B}_t = \{3, \ldots, \tilde{H}_t - 1\}$;. . .;$\mathcal{S}_t = \{1, \ldots, \tilde{H}_t - 1\}, \mathcal{B}_t = \{\tilde{H}_t\}$. For arbitrary sets $\mathcal{S}_t = \{1, \ldots, k\}$, $\mathcal{B}_t = \{k+1, \ldots, \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$, by market-clearing $p_t = \frac{\sum_{h>k} n_{t,h}f_{t,h} - \left[\sum_{h=1}^{k} n_{t,h}\right]a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{k} n_{t,h})(1+r)} := p_t^{(k)}$ and by Proposition 1, the guess is verified if and only if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$. Note that $p_t^{(k^*)} > p_t^* = \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h}f_{t,h}}{1+r}$ for any $k^* \in \{1, \ldots, \tilde{H}_t - 1\}$ follows from the proof of Proposition 1.

It remains to show $p_t^{(k)} < p_t^{(k^*)}\ \forall k < k^*$ and $p_t^{(k)} > p_t^{(k-1)}$ for such k, where $p_t^{(0)} := p_t^*$. Note that $p_t^{(1)}$ solves $\sum_{h>1} n_{t,h}\left(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(1)}\right) = a\sigma^2\overline{Z}$ and p_t^* solves $a\sigma^2\overline{Z} = \sum_{h>1} n_{t,h}\left(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^*\right) + n_{t,1}\left(f_{t,1} + a\sigma^2\overline{z} - (1+r)p_t^*\right)$, where the last term is < 0 since p_t^* is not verified. So $p_t^{(1)} > p_t^*$. For arbitrary k, $p_t^{(k)}$ solves $\sum_{h>k} n_{t,h}\left(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(k)}\right) = a\sigma^2\overline{Z}$ and $p_t^{(k-1)}$ solves $a\sigma^2\overline{Z} = \sum_{h>k} n_{t,h}\left(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(k-1)}\right) + n_{t,k}(f_{t,k} + a\sigma^2\overline{z} - (1+r)p_t^{(k-1)})$, where the last term is < 0 since $p_t^{(k-1)}$ is not verified. So $p_t^{(k)} > p_t^{(k-1)}\ \forall k \in [2, k^*)$. Finally, $p_t^{(k^*)} > p_t^{(k)}\ \forall k < k^*$ follows from the above argument with $k = k^*$. □

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