21

Basics on the Continuum Limit

In the *continuum limit*, one analyzes the EL equations of the causal action principle for systems of Dirac seas in the presence of classical bosonic fields. As worked out in detail in [45, Chapters 3–5], this limiting case yields the interactions of the standard model and gravity on the level of second-quantized fermionic fields interacting with classical bosonic fields. In this chapter, we explain schematically how the analysis of the continuum limit works and give an overview of the obtained results.

21.1 Causal Fermion Systems in the Presence of External Potentials

In Chapters 15–19, it was explained how to construct and analyze the unregularized kernel of the fermionic projector $\tilde{P}(x,y)$ in Minkowski space in the presence of an external potential $\mathcal{B}$. The general question is whether the causal fermion system corresponding to this kernel satisfies the EL equations corresponding to the causal action principle. Thus, we would like to evaluate the EL equations as stated abstractly in Theorem 7.1.1 for $\tilde{P}(x,y)$. The basic procedure is to form the closed chain (see (5.47)) and to compute its eigenvalues $\lambda_1^{xy}, \ldots \lambda_{2n}^{xy} \in \mathbb{C}$. This, in turn, makes it possible to compute the causal action and the constraints (see (5.35)–(5.39)). Considering the first variations of $P(x,y)$, one then obtains the EL equations.

The main obstacle before one can carry out this program is that, in order to obtain mathematically well-defined quantities, one needs to introduce an *ultraviolet regularization.* As explained in detail in Chapter 5, this regularization is not merely a technical procedure, but it corresponds to implementing a specific microscopic structure of spacetime. In the vacuum, the regularization was introduced with the help of a regularization operator $\mathfrak{R}_\varepsilon$ (see (5.26)). Different choices of regularization operators correspond to different microscopic structures of spacetime. Since the structure of our physical spacetime on the Planck scale is largely unknown, the strategy is to allow for a general class of regularization operators, making it possible to analyze later on how the results depend on the regularization (for more details on this so-called *method of variable regularization*, see [45, §1.2.1]).

In more detail, we proceed as follows. In the vacuum, we can follow the procedure explained in Chapter 5, choosing $\mathcal{H}$ as the subspace of all negative-frequency solutions of the Dirac equation. In preparation for extending this construction to the interacting situation, it is useful to note that the causal action principle can be formulated in terms of the kernel of the fermionic projector given abstractly by (5.45). Therefore, our task is to compute this kernel. It can be obtained alternatively by starting from the unregularized kernel of the fermionic projector constructed in Section 15.4 and introducing a regularization. In the simplest case, working with a regularization that preserves the Dirac equation, that is,

$$\mathfrak{R}_\varepsilon \,:\, \mathcal{H}_m \to \mathcal{H}_m \cap C^0(\mathscr{M}, S\mathscr{M})\,, \tag{21.1}$$

the regularization can be introduced similar to (15.61) by

$$P^\varepsilon := -\mathfrak{R}_\varepsilon \, \pi_{\mathcal{H}} \, \mathfrak{R}_\varepsilon^* \, k_m \,:\, C^\infty_0(\mathscr{M}, S\mathscr{M}) \to \mathcal{H}_m\,. \tag{21.2}$$

For more general regularization operators that do *not* preserve the Dirac equation, one can introduce the regularization by modifying the right-hand side of (5.64) to

$$P^\varepsilon(x,y) := -\big(\mathfrak{R}_\varepsilon \Psi\big)(x)\, \big(\mathfrak{R}_\varepsilon \Psi\big)(y)^*\,, \tag{21.3}$$

where $\Psi : \mathcal{H}_m \to L^2_{\text{loc}}(\mathscr{M}, S\mathscr{M})$ is the unregularized wave evaluation operator, and regularization operator $\mathfrak{R}_\varepsilon \,:\, \mathcal{H}_m \to C^0(\mathscr{M}, S\mathscr{M})$ now maps more generally to continuous wave functions (not necessarily Dirac solutions).

The latter construction has the advantage that it also applies in the presence of an external potential. In a perturbative treatment, it gives rise to the causal perturbation expansion developed in Section 18.2. In this way, we obtain the *regularized kernel* $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential. Following the procedure explained in Chapter 5, we obtain a corresponding causal fermion system. After suitable identifications (as worked out in [45, Section 1.2]), this regularized kernel coincides with the kernel of the fermionic projector as defined abstractly in (5.45).

The subtle question is whether a chosen regularization of the vacuum also determines the regularization of the kernel $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential. The general answer to this question is no, simply because the interaction introduces additional freedoms for regularizing. Moreover, it is not clear a priori whether the regularized objects should still satisfy the Dirac equation. But at least, in [45, Appendix F] and [41, Appendix D], a canonical procedure is given for *regularizing the light-cone expansion* (see [64] for related constructions in curved spacetime). It consists in taking the formulas of the (unregularized) light-cone expansion (like, e.g., (19.28)–(19.33) in Example 19.2.2) and replacing the singular factors $T^{(n)}$ (like, e.g., (19.34)) by corresponding functions where the singularities on the light cone have been regularized on the scale ε. The precise procedure will be explained in the next section.

21.2 The Formalism of the Continuum Limit

We now give a brief summary of the formalism of the continuum limit. More details can be found in [45, Section 2.4]. The reader interested in the derivation of this formalism is referred to [41, Chapter 4].

Having introduced the regularized kernel of the fermionic projector denoted by $\tilde{P}^\varepsilon(x,y)$, we can form the closed chain

$$A^\varepsilon_{xy} := \tilde{P}^\varepsilon(x,y)\,\tilde{P}^\varepsilon(y,x)\,, \tag{21.4}$$

compute its eigenvalues and proceed by analyzing the EL equations. In the continuum limit, one focuses on the limiting case $\varepsilon \searrow 0$ when the ultraviolet regularization is removed. This limiting case is comparatively easy to analyze. This can be understood from the fact that, in the limit $\varepsilon \searrow 0$, the closed chain A^ε_{xy} becomes singular on the light cone. Therefore, asymptotically for small ε, it suffices to take into account the contributions to A^ε_{xy} on the light cone. These contributions, on the other hand, are captured precisely by the light-cone expansion of the unregularized kernel $\tilde{P}(x,y)$ (see Section 19.2 or the explicit formulas in Example 19.2.2). This is the basic reason why, in the continuum limit, the EL equations can be rewritten as field equations involving fermionic wave functions as well as derivatives of the bosonic potentials.

More specifically, the asymptotics $\varepsilon \searrow 0$ is captured by the *formalism of the continuum limit*, which we now outline (for more details, see [45, Section 2.4] or the derivation of the formalism in [41, Chapter 4]). In the first step, one regularizes the light-cone expansion symbolically by leaving all smooth contributions unchanged, whereas to the singular factors $T^{(n)}$ we employ the replacement rule

$$m^p\,T^{(n)} \to m^p\,T^{(n)}_{[p]}\,. \tag{21.5}$$

Thus, for the formulas of Example 19.2.2, the factors $T^{(n)}$ get an additional index [0]. If the light-cone expansion involves powers of the rest mass, these powers are taken into account in the lower index. The resulting factors $T^{(n)}_{[p]}$ are smooth functions, making all the subsequent computations well defined. The detailed form of these functions does not need to be specified because all we need are the following computation rules. In computations, one may treat the factors $T^{(n)}_{[p]}$ as complex functions. However, one must be careful when tensor indices of factors $\not{\xi}$ are contracted with each other. Naively, this gives a factor ξ^2, which vanishes on the light cone and thus changes the singular behavior on the light cone. In order to describe this effect correctly, we first write every summand of the light-cone expansion such that it involves at most one factor $\not{\xi}$ (this can always be arranged using the anti-commutation relations of the Dirac matrices). We now associate every factor $\not{\xi}$ to the corresponding factor $T^{(n)}_{[p]}$. In short calculations, this can be indicated by putting brackets around the two factors, whereas in the general situation, we add corresponding indices to the factor $\not{\xi}$, giving rise to the replacement rule

$$m^p\,\not{\xi}T^{(n)} \to m^p\,\not{\xi}^{(n)}_{[p]}\,T^{(n)}_{[p]}\,. \tag{21.6}$$

For example, we write the regularized fermionic projector of the vacuum as

$$P^\varepsilon = \frac{\mathrm{i}}{2}\sum_{n=0}^\infty \frac{m^{2n}}{n!}\,\not{\xi}^{(-1+n)}_{[2n]}\,T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\,T^{(n)}_{[2n+1]}\,. \tag{21.7}$$

The kernel $P(y,x)$ is obtained by taking the conjugate (see (5.53)). The conjugates of the factors $T^{(n)}_{[p]}$ and $\xi^{(n)}_{[p]}$ are the complex conjugates,

$$\overline{T^{(n)}_{[p]}} := \big(T^{(n)}_{[p]}\big)^* \qquad \text{and} \qquad \overline{\xi^{(n)}_{[p]}} := \big(\xi^{(n)}_{[p]}\big)^* . \tag{21.8}$$

One must carefully distinguish between the factors with and without complex conjugation. In particular, the factors $\not\xi^{(n)}_{[p]}$ need not be symmetric, that is, in general,

$$\big(\not\xi^{(n)}_{[p]}\big)^* \neq \not\xi^{(n)}_{[p]} . \tag{21.9}$$

When forming composite expressions, the tensor indices of the factors ξ are contracted to other tensor indices. The factors ξ that are contracted to other factors ξ are called *inner factors*. The contractions of the inner factors are handled with the so-called *contraction rules*,

$$(\xi^{(n)}_{[p]})^j \, (\xi^{(n')}_{[p']})_j = \frac{1}{2} \left(z^{(n)}_{[p]} + z^{(n')}_{[p']} \right), \tag{21.10}$$

$$(\xi^{(n)}_{[p]})^j \, \overline{(\xi^{(n')}_{[p']})_j} = \frac{1}{2} \left(z^{(n)}_{[p]} + \overline{z^{(n')}_{[p']}} \right), \tag{21.11}$$

$$z^{(n)}_{[p]} \, T^{(n)}_{[p]} = -4 \left(n \, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}} \right), \tag{21.12}$$

which are to be complemented by the complex conjugates of these equations. Here, the factors $z^{(n)}_{[p]}$ can be regarded simply as a book-keeping device to ensure the correct application of the rule (21.12). The factors $T^{(n)}_{\{p\}}$ have the same scaling behavior as the $T^{(n)}_{[p]}$, but their detailed form is somewhat different; we simply treat them as a new class of symbols. In cases where the lower index does not need to be specified, we write $T^{(n)}_\circ$. After applying the contraction rules, all inner factors ξ have disappeared. The remaining so-called *outer factors* ξ need no special attention and are treated like smooth functions.

Next, to any factor $T^{(n)}_\circ$, we associate the *degree* $\deg T^{(n)}_\circ$ by

$$\deg T^{(n)}_\circ = 1 - n . \tag{21.13}$$

The degree is additive in products, whereas the degree of a quotient is defined as the difference of the degrees of the numerator and denominator. The degree of an expression can be thought of as describing the order of its singularity on the light cone, in the sense that a larger degree corresponds to a stronger singularity (e.g., the contraction rule (21.12) increments n and thus decrements the degree, in agreement with the naive observation that the function $z = \xi^2$ vanishes on the light cone). Using the formal Taylor series, we can expand in the degree. In all our applications, this will give rise to terms of the form

$$\eta(x,y) \, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad \text{with } \eta(x,y) \text{ smooth} . \tag{21.14}$$

The quotient of the two monomials in this equation is referred to as a *simple fraction*.

A simple fraction can be given a quantitative meaning by considering one-dimensional integrals along curves that cross the light cone transversely away from the origin $\xi = 0$. This procedure is called *weak evaluation on the light cone*. For our purpose, it suffices to integrate over the time coordinate $t = \xi^0$ for fixed $\vec{\xi} \neq 0$. Moreover, using the symmetry under reflections $\xi \to -\xi$, it suffices to consider the upper light cone $t \approx |\vec{\xi}|$. The resulting integrals diverge if the regularization is removed. The leading contribution for small ε can be written as

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} \mathrm{d}t \, \eta(t, \vec{\xi}) \, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)} \, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)} \, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \approx \eta(|\vec{\xi}|, \vec{\xi}) \, \frac{c_{\mathrm{reg}}}{(\mathrm{i}|\vec{\xi}|)^L} \, \frac{\log^r(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}} \,, \tag{21.15}$$

where L is the degree of the simple fraction and c_{reg}, the so-called *regularization parameter*, is a real-valued function of the spatial direction $\vec{\xi}/|\vec{\xi}|$, which also depends on the simple fraction and on the regularization details (the error of the approximation will be specified later). The integer r describes a possible logarithmic divergence. Apart from this logarithmic divergence, the scalings in (21.15) in both ξ and ε are described by the degree.

When analyzing a sum of expressions of the form (21.14), one must know if the corresponding regularization parameters are related to each other. In this respect, the *integration-by-parts rules* are important, which are described symbolically as follows. On the factors $T_\circ^{(n)}$, we introduce a derivation ∇ by

$$\nabla T_\circ^{(n)} = T_\circ^{(n-1)} \,. \tag{21.16}$$

Extending this derivation with the product and quotient rules to simple fractions, the integration-by-parts rules state that

$$\nabla \left(\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)} \, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)} \, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \right) = 0 \,. \tag{21.17}$$

Carrying out the derivative with the product rule, one gets relations between simple fractions. Simple fractions that are not related to each other by the integration-by-parts rules are called *basic fractions*. As shown in [41, Appendix E], there are no further relations between the basic fractions. Thus, the corresponding *basic regularization parameters* are independent.

The abovementioned symbolic computation rules give a convenient procedure to evaluate composite expressions in the fermionic projector, referred to as the *analysis in the continuum limit*: After applying the contraction rules and expanding in the degree, the EL equations can be rewritten as equations involving a finite number of terms of the form (21.14). By applying the integration-by-parts rules, we can arrange that all simple fractions are basic fractions. We evaluate weakly on the light cone (21.15) and collect the terms according to their scaling in ξ. Taking for every given scaling in ξ only the leading pole in ε, we obtain equations that involve linear combinations of smooth functions and basic regularization parameters. We consider the basic regularization parameters as empirical parameters describing the unknown microscopic structure of spacetime. We thus end up with equations involving smooth functions and a finite number of free parameters.

We finally specify the error of the abovementioned expansions. By not regularizing the bosonic potentials and fermionic wave functions, we clearly disregard the

$$\text{higher orders in } \varepsilon/\ell_{\text{macro}} \,. \tag{21.18}$$

Furthermore, in (21.15), we must stay away from the origin, meaning that we neglect the

$$\text{higher orders in } \varepsilon/|\vec{\xi}| \,. \tag{21.19}$$

The higher order corrections in $\varepsilon/|\vec{\xi}|$ depend on the fine structure of the regularization and thus seem unknown for principal reasons. Neglecting the terms in (21.18) and (21.19) also justifies the formal Taylor expansion in the degree. Clearly, leaving out the terms (21.19) is justified only if $|\vec{\xi}| \gg \varepsilon$. Therefore, whenever using the above formalism, we must always ensure that $|\vec{\xi}|$ is much larger than ε.

We finally remark that, when working out the Einstein equations, one must go beyond error terms of the form (21.18) and (21.19). The reason is that the gravitational scales like $\kappa \sim \delta^2 \approx \varepsilon^2$. In order not to lose the relevant terms in the error terms, one must take certain higher-order contributions into account. This is done by using the so-called ι-formalism. Here, we do not enter the details but merely refer the interested reader to [45, §4.2.7].

21.3 Overview of Results of the Continuum Limit Analysis

The formalism of the continuum limit makes it possible to evaluate the EL equations of the causal action for the regularized kernel $\tilde{P}^\varepsilon(x,y)$ in the presence of an external potential $\mathcal{B}$. In order to avoid confusion, we point out that, a priori, the external potential can be chosen arbitrarily; in particular, it does need to satisfy any field equations. We find that the EL equations of the causal action are satisfied in the continuum limit if and only if the potential $\mathcal{B}$ has a specific structure and satisfies dynamical equations. Restricting attention to potentials of this form and complementing the Dirac equation (1.39) by the dynamical equations for $\mathcal{B}$, the potentials are no longer given as external potentials, but instead one gets a coupled system of equations describing a mutual interaction of the Dirac wave functions with classical bosonic fields. The dynamical equations for $\mathcal{B}$ are referred to as the *classical field equations*. In this way, the classical field equations are *derived* from the causal action principle.

We now outline the main results of the continuum limit analysis as obtained in [45, Chapters 3–5]. The main input is to specify the regularized kernel $P^\varepsilon(x,y)$) of the *vacuum*. This involves:

- The fermion configuration in the vacuum, including the masses of the leptons and quarks. Moreover, it is built in that the neutrinos break the chiral symmetry.

- The vacuum kernel should satisfy the EL equations. This poses a few constraints on the regularization operator.

The output of the continuum limit is the following results:

- The structure of the interaction on the level of classical gauge theory.
- The gauge groups and their coupling to the fermions.
- The equations of linearized gravity.

In [45], the continuum limit is worked out in three steps for systems of increasing complexity. In Chapter 3, a system formed of a sum of three Dirac seas is considered. This configuration, referred to as a *sector*, can be thought of as a simplified model describing the three generations of charged leptons (e, μ, τ). In the continuum limit, we obtain the following results for the interaction as described by the causal action principle:

- The fermions interact via an *axial gauge field*.
- This axial gauge field is massive, with the mass determined by the masses of the fermions and the regularization.
- We find that the field equations for the axial gauge field arise in the continuum limit only if the number of generations equals three. For one or two generations, the resulting equations are overdetermined, whereas for more than three generations, the equations are under-determined (which means in particular that there is no well-posed Cauchy problem).
- We obtain nonlocal corrections to the classical field equations described by integral kernels that decay on the Compton scale. It seems that these nonlocal corrections capture certain features of the underlying quantum field theory. But the detailed connection has not been worked out.
- There is no gravitational field and no Higgs field.

In [45, Chapter 4], a system formed as a direct sum of two sectors is considered. This system is referred to as a *block*. The first sector looks as in Chapter 3. In the second sector, however, the chiral symmetry is broken. This system can be regarded as a model for the leptons, including the three generations of neutrinos. In the continuum limit, we obtain the following results for the interaction as described by the causal action principle:

- The fermions interact via an SU(2) gauge field, which couples only to one chirality (say, the left-handed fermions).
- The corresponding gauge field is again massive.
- Moreover, the fermions interact linearly via the linearized Einstein equations, where the coupling constant is related to the regularization length.

Finally, in [45, Chapter 5], a realistic system involving leptons and quarks is considered. To this end, one considers a direct sum of eight sectors, one of which with broken chiral symmetry (the neutrino sector). These eight sectors form pairs, referred to as blocks. The block containing the neutrino sector describes the leptons, whereas the other three blocks describe the quarks. Moreover, we obtain the following results:

- The fermions interact via the gauge group $\mathrm{U}(1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)$. The corresponding gauge fields couple to the fermions as in the standard model. The SU(2)-field couples only to the left-handed component and is massive. The other gauge fields are massless.
- Moreover, the fermions interact linearly via the linearized Einstein equations.
- The EL equations corresponding to the causal action principle coincide with those of the standard model after spontaneous symmetry breaking, plus linearized gravity.
- There are scalar degrees of freedom that can be identified with the Higgs potential. However, the corresponding dynamical equations have not yet been worked out.
- Again, the fermions interact linearly via the linearized Einstein equations, where the coupling constant is related to the regularization length. Taking into account that the causal action principle is diffeomorphism invariant, we obtain the Einstein equations, up to possible higher-order corrections in curvature (which scale in powers of (δ^2 Riem), where δ is the Planck length and Riem is the curvature tensor). Thus, including error terms, the derived Einstein equations take the form (see [45, Theorems 4.9.3 and 5.4.4])

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} + \Lambda\, g_{jk} = G\, T_{jk} + \mathcal{O}\big(\delta^4\, \mathrm{Riem}^2\big), \tag{21.20}$$

  where T_{jk} is the energy-momentum tensor and G is the gravitational coupling constant.

We conclude this section by discussing a few aspects of the derivation of these results. We begin with the system of one sector as considered in [45, Chapter 3]. In this case, the kernel of the fermionic projector is the sum of $g \in \mathbb{N}$ Dirac seas of masses $m_1, \ldots, m_g$, that is,

$$P(x,y) = \sum_{\beta=1}^{g} P_{m_\alpha}(x,y)\,, \tag{21.21}$$

where again

$$P_m(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta\big(k^2 - m^2\big)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)}\,. \tag{21.22}$$

In order to perturb the system by gauge potentials, we first introduce the kernel of the *auxiliary fermionic projector* $P^{\mathrm{aux}}(x,y)$, which is obtained from $P(x,y)$ by replacing the sums with direct sums,

$$P^{\mathrm{aux}}(x,y) = \bigoplus_{\beta=1}^{g} P_{m_\alpha}(x,y), \tag{21.23}$$

this means that $P^{\text{aux}}(x,y)$ is represented by a $(4g \times 4g)$-matrix. The auxiliary kernel satisfies the Dirac equation

$$\Bigg(\mathrm{i}\partial\!\!\!/_x - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_g \end{pmatrix}\Bigg) P^{\text{aux}}(x,y) = 0\,. \tag{21.24}$$

Therefore, it can be perturbed as usual by inserting a potential $\mathcal{B}$ into the Dirac equation

$$\Bigg(i\partial\!\!\!/_x + \mathcal{B}(x) - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_g \end{pmatrix}\Bigg) \tilde{P}^{\text{aux}}(x,y) = 0, \tag{21.25}$$

where $\mathcal{B}(x)$ is a matrix potential acting on $\mathbb{C}^{4g}$. The perturbed kernel $\tilde{P}^{\text{aux}}$ can be computed with the methods explained in Chapters 18 and 19. Finally, we obtain the perturbed kernel of the fermionic projector by summing over the generation indices in an operation referred to as the *sectorial projection*,

$$\tilde{P}(x,y) := \sum_{\alpha,\beta=1}^{g} \tilde{P}^\alpha_\beta(x,y)\,. \tag{21.26}$$

After introducing an ultraviolet regularization, this kernel can be analyzed in the EL equations of the causal action principle, exactly as outlined in Section 21.2.

In order to gain the largest possible freedom in perturbing the system, the operator $\mathcal{B}$ should be chosen as general as possible. For this reason, in [45, Chapter 4], a general class of potential was considered, including nonlocal potentials (i.e., integral operators). A general conclusion of the analysis is that, in order to satisfy the EL equations, the potential $\mathcal{B}$ must be local, that is, a differential operator or a multiplication operator by a potential that may involve left- and right-handed potentials but also bilinear, scalar or pseudo-scalar potentials,

$$\mathcal{B}(x) = \chi_L A\!\!\!/_R(x) + \chi_R A\!\!\!/_L(x) + \sigma^{ij}\,\Lambda_{ij}(x) + \Phi(x) + \mathrm{i}\Gamma\,\Xi(x), \tag{21.27}$$

where each of the potentials is a $g \times g$-matrix acting on the generations, and Γ is the pseudo-scalar matrix, which in physics textbooks is often denoted by γ^5. Analyzing the continuum limit for such multiplication operators, one gets the abovementioned results.

One feature that at first sight might be surprising is that, despite local gauge symmetry, we get *massive gauge fields*. In order to understand how this comes about, we need to consider local gauge symmetries in connection with the chiral gauge potentials in (21.27). On the fundamental level of the causal fermion system, *local gauge transformations* arise from the freedom in choosing bases of the spin spaces (see (5.90) and (5.91) in Section 5.9). In the present setting with four-component Dirac spinors, the local gauge transformations take the form

$$\psi(x) \to U(x)\,\psi(x) \qquad \text{with} \qquad U(x) \in \mathrm{U}(2,2)\,, \tag{21.28}$$

where U(2, 2) is the group of unitary transformations of the spinors at the spacetime point x. The causal action principle is gauge invariant in the sense that the causal action is invariant under such gauge transformations. The group U(2, 2) can be used to describe gravity as a gauge theory (for details, see Section 4.2 or [38]). Restricting attention to flat spacetime, the main interest is that U(2, 2) contains the gauge group U(1) of electrodynamics as a subgroup. In other words, the causal action principle is gauge invariant under local phase transformations

$$\psi(x) \to \mathrm{e}^{-\mathrm{i}\Lambda(x)}\, \psi(x), \tag{21.29}$$

with a real-valued function Λ.

The chiral potentials in (21.27) also give rise to generalized phase transformations. This can be seen, for example, by working out the leading term to the light-cone expansion (similar to (19.28) for the electromagnetic potential). One finds that the chiral gauge potentials lead to phase transformations of the left- and right-handed components of the wave functions, that is,

$$\psi(x) \to U(x)\, \psi(x) \qquad \text{with} \qquad U(x) := \chi_L\, \mathrm{e}^{-\mathrm{i}\Lambda_L(x)} + \chi_R\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)}, \tag{21.30}$$

again with real-valued functions Λ_L and Λ_R. The point is that this transformation is *not unitary* with respect to the spin inner product because the chirality flips when taking the adjoint

$$U^* = \chi_R\, \mathrm{e}^{\mathrm{i}\Lambda_L(x)} + \chi_L\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)} \quad \text{but} \quad U^{-1} = \chi_L\, \mathrm{e}^{\mathrm{i}\Lambda_L(x)} + \chi_R\, \mathrm{e}^{-\mathrm{i}\Lambda_R(x)}, \tag{21.31}$$

note that $\chi_L^* = (1-\Gamma)^*/2 = (1+\Gamma)/2 = \chi_R$ because $\Gamma^* = -\Gamma$. Therefore, as soon as $A_L \neq A_R$, the generalized phase transformation $U(x)$ in (21.30) is *not* a local transformation of the form (21.28). Consequently, the local transformation in (21.30) does *not* correspond to a symmetry of the causal action principle. Therefore, it is not a contradiction if these gauge potentials arise in the effective field equations as mass terms.

More specifically, the relative phases between left- and right-handed potentials do come up in the closed chain $A_{xy} = P(x,y)P(y,x)$, as one sees immediately from the fact that, if $P(x,y)$ is vectorial, then the chirality flips at the corresponding factor, that is,

$$\begin{aligned} &\chi_L A_{xy} = \chi_L P(x,y)\, \chi_R P(y,x) \\ &\to\; \exp\Big(-\mathrm{i}\big(\Lambda_L(x) - \Lambda_R(x)\big)\Big) \exp\Big(\mathrm{i}\big(\Lambda_L(y) - \Lambda_R(y)\big)\Big)\, A_{xy}\,. \end{aligned} \tag{21.32}$$

Working out the corresponding contribution to the EL equations in the continuum limit, one finds that the axial current and a corresponding axial mass term come up in the effective field equations. The coupling constant and the bosonic mass depend on the detailed form of the regularization. But they can be computed for specific choices of the regularization, as is exemplified in [45, Chapter 3] for a hard cutoff in momentum space and the $i\varepsilon$-regularization.

We now move on to the system of two sectors as analyzed in [45, Chapter 4]. The vacuum is described by a kernel of the fermionic projector $P(x,y)$ being a direct sum of two summands, each of which is of the form (21.21), where we choose

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the number of generations as $g = 3$. Hence, $P(x, y)$ is a 8×8-matrix. Replacing the sums by direct sums, one obtains the corresponding auxiliary kernel $P^{\text{aux}}(x, y)$ (being represented by a 24×24-matrix). In order to account for the observational fact that neutrinos are left-handed particles, one must *break the chiral symmetry* of one of the sectors (the *neutrino sector*). To this end, we assume that the regularization of the neutrino sectors is different from that of the other sector (the *charged sector*) by contributions which are not left-right invariant. The relevant length scale is denoted by $\delta \gtrsim \varepsilon$. This procedure is very general and seems the right thing to do because the regularization effects on the scale δ are also needed in order to obtain the correct form of the curvature term in the Einstein equations. In fact, the obtained linearized Einstein equations involve the coupling constant $G \sim \delta^2$. As briefly mentioned at the end of Section 10.2, the derivation of the Einstein equations uses the ι-formalism, which goes beyond the standard formalism of the continuum limit.

The system analyzed in [45, Chapter 5] is obtained similarly by adding direct summands to $P(x, y)$ describing the three generations of quarks. We begin with eight sectors. These eight sectors form pairs, giving rise to four blocks. We conclude by outlining how this mechanism of *spontaneous block formation* comes about. For this purpose, we return to the gauge phases as already mentioned in (21.28) and (21.30). We already saw in (21.32) that, if the kernel of the fermionic projector is vectorial, then the relative phases (i.e., the difference of left- and right-handed gauge phases) show up in the eigenvalues of the closed chain. Such phase factors drop out of the causal Lagrangian because of the absolute values in (5.35). However, the situation becomes more involved if the kernel of the fermionic projector is *not* vectorial. Indeed, expanding the vacuum kernel in powers of the rest mass, the zero-order contribution to $P(x, y)$ is vectorial, whereas the first-order contribution is scalar (more generally, one sees from (19.1) that the even orders in the mass are vectorial, whereas the odd orders are scalar). As a consequence, the absolute values of the eigenvalues $|\lambda_i^{xy}|$ depend in a rather complicated way on the chiral gauge phases. Moreover, considering a direct sum of Dirac seas, one must keep into account that the gauge phases in the formulas for $P(x, y)$ (and similarly in composite expressions) must be replaced by generalized phases, which can be described in terms of ordered exponentials of the gauge potentials. Evaluating the causal Lagrangian (5.35), one gets conditions for the chiral gauge phases. In simple terms, these conditions can be stated by demanding that matrices formed of ordered exponentials of the gauge potentials must have degeneracies. Qualitatively speaking these degeneracies mean that the left-handed gauge potential must be the same in each block, and this condition even makes it possible to explain why such blocks form. A more detailed and more precise explanation can be found in [45, Chapter 5].

21.4 Exercises

Exercise 21.1 This exercise explains in a simple example how the *regularization of the Hadamard expansion* works.

(a) Consider the singular term of the first summand of the Hadamard expansion (19.1) in Minkowski space,

$$\lim_{\nu \searrow 0} \frac{1}{\xi^2 - i\nu\, \xi^0}, \tag{21.33}$$

where again $\xi := y - x$. A simple method to remove the pole is not to take the limit $\nu \searrow 0$, but instead to set $\nu = 2\varepsilon$,

$$\frac{1}{\xi^2 - 2i\varepsilon\, \xi^0}\,. \tag{21.34}$$

Show that this regularization can be realized by the replacement

$$\xi^0 \to \xi^0 - i\varepsilon\,, \tag{21.35}$$

up to a multiplicative error of the order

$$\Big(1 + \mathcal{O}\Big(\frac{\varepsilon^2}{\xi^2}\Big)\Big)\,. \tag{21.36}$$

The basic concept behind the regularized Hadamard expansion is to regularize all singular terms in this way, leaving all smooth functions unchanged. This gives a consistent formalism is one works throughout with error terms of the form (21.36). *Hint:* This is the so-called iε-regularization introduced in [45, Section 2.4]. For details in curved spacetime, see [64].

(b) Show that for kernels written as Fourier transforms

$$K(x,y) = \int_M \frac{d^4p}{(2\pi)^4}\, \hat{K}(p)\, e^{-ip(y-x)} \tag{21.37}$$

(with $\hat{K}$ supported in say the lower half plane $\{p^0 < 0\}$), the replacement rule (21.34) amounts to inserting a convergence-generating factor $e^{\varepsilon p^0}$ into the integrand.

Exercise 21.2 The goal of this exercise is to explore *weak evaluation on the light cone* in a simple example.

(a) Show that, setting $t = \xi^0$ and choosing polar coordinates with $r = |\vec{\xi}|$, regularizing the pole in (21.33) according to (21.34) gives the function

$$\frac{1}{(t - i\varepsilon)^2 - r^2}\,. \tag{21.38}$$

(b) As a simple example of a composite expression, we take the absolute square of the regularized function

$$\frac{1}{\big|(t - i\varepsilon)^2 - r^2\big|^2}\,. \tag{21.39}$$

Show that this expression is ill defined in the limit $\varepsilon \searrow 0$ even as a distribution.

(c) Use the identity

$$\frac{1}{(t-\mathrm{i}\varepsilon)^2-r^2} = \frac{1}{(t-\mathrm{i}\varepsilon-r)(t-i\varepsilon+r)} = \frac{1}{2r}\left(\frac{1}{t-\mathrm{i}\varepsilon-r} - \frac{1}{t-\mathrm{i}\varepsilon+r}\right) \tag{21.40}$$

to rewrite the integrand in (21.39) in the form

$$\sum_{p,q=0}^{1} \frac{\eta_{p,q}(t,r,\varepsilon)}{(t-\mathrm{i}\varepsilon-r)^p\,(t+\mathrm{i}\varepsilon-r)^q}\,, \tag{21.41}$$

with functions $\eta_{p,q}(t,r,\varepsilon)$, which in the limit $\varepsilon \searrow 0$ converge to smooth functions. Compute the functions $\eta_{p,q}$.

(d) We now compute the leading contributions and specify what we mean by "leading." First compute the following integrals with residues:

$$I_0(\varepsilon) := \int_{-\infty}^{\infty} \frac{1}{(t-\mathrm{i}\varepsilon-r)\,(t+\mathrm{i}\varepsilon-r)}\;\mathrm{d}t\,. \tag{21.42}$$

Show that

$$\int_{-\infty}^{\infty} \frac{\eta_{1,1}(t,r)}{(t-\mathrm{i}\varepsilon-r)^2\,(t+\mathrm{i}\varepsilon-r)^2}\;\mathrm{d}t = I_0(\varepsilon)\,\eta_{2,2}(r,r) + \mathcal{O}(\varepsilon)\,. \tag{21.43}$$

Explain in which sense this formula is a special case of the weak evaluation formula (21.15).