THE MINIMAL DEGREE OF A FAITHFUL QUASI-PERMUTATION REPRESENTATION OF AN ABELIAN GROUP

by HOUSHANG BEHRAVESH

(Received 18 July, 1995)

1. Introduction. Let G be a finite linear group of degree n; that is, a finite group of automorphisms of an n-dimensional complex vector space (or, equivalently, a finite group of non-singular matrices of order n with complex coefficients). We shall say that G is a *quasi-permutation group* if the trace of every element of G is a non-negative rational integer. The reason for this terminology is that, if G is a permutation group of degree n, its elements, considered as acting on the elements of a basis of an n-dimensional complex vector space V, induce automorphisms of V forming a group isomorphic to G. The trace of the automorphism corresponding to an element x of G is equal to the number of letters left fixed by x, and so is a non-negative integer. Thus, a permutation group of degree n has a representation as a quasi-permutation group of degree n. See [**5**].

By a *quasi-permutation matrix* we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices); let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [**1**].

Let $G \cong \prod_{i=1}^{r} C_{m_i}$ where m_i is a prime power. As in [**2**], define $T(G) = \sum_{i=1}^{r} m_i$; when $G = 1$ let $T(G) = 0$. In [**1**] it is proved that $c(G) = q(G) = p(G) = T(G)$ if and only if $G \neq 1$ and G has no direct factor of order 6.

The quantity $p(G)$ for any abelian group depends on the decomposition of G into a direct product of its cyclic subgroups [**2**]. In fact, if $G \neq 1$ is a finite abelian group, then $p(G) = T(G)$.

In this paper $G = \prod_{i=1}^{n} G_i$ will denote the direct product of the subgroups G_i of G $(1 \leq i \leq n)$.

For an abelian group G, the invariants $c(G)$ and $p(G)$ coincide because the Schur indices for abelian groups are trivial. We shall calculate these invariants for an arbitrary abelian group G. In view of [**1**], we need only resolve the case of an abelian group having the cyclic group C_6 as direct factor. Nevertheless our proof applies to an arbitrary finite abelian group.

The main result is that $c(G) = q(G) = T(G) - n$ for an abelian group G, where n is the largest integer such that C_6^n is a direct summand of G.

LEMMA 1.1. *Let G be a finite abelian group and let G be the direct product of its subgroups L and H. Then $T(G) = T(L) + T(H)$.*

Proof. See [**2**].

https://doi.org/10.1017/S0017089500031906 Published online by Cambridge University Press

2. The minimal degree of a faithful quasi-permutation representation of an abelian group. Let χ be an irreducible character of G. Let $m_{\mathbb{Q}}(\chi)$ denote the Schur index of χ in G over $\mathbb{Q}$.

LEMMA 2.1. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Then $m_{\mathbb{Q}}(\chi) \mid \chi(1)$. Moreover when χ is linear, we have $m_{\mathbb{Q}}(\chi) = 1$.*

Proof. See [**3**, Corollary 10.2].

COROLLARY 2.2. *Let G be a finite group and let $m_{\mathbb{Q}}(\chi) = 1$, for all $\chi \in \mathrm{Irr}(G)$. Then $c(G) = q(G)$. In particular if G is a finite abelian group, then $c(G) = q(G)$.*

Proof. This follows from the definitions of $c(G)$ and $q(G)$ together with Lemma 2.1.

LEMMA 2.3. *Let χ be a character of G. Then $\ker \chi = \ker \sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$, where $\Gamma(\chi) = \Gamma(\mathbb{Q}(\chi):\mathbb{Q})$. Moreover χ is faithful if and only if $\sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$ is faithful.*

Proof. It is clear that $\ker(\chi) = \ker(\chi^{\alpha})$, for $\alpha \in \Gamma(\chi)$. However

$$\ker \sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha} = \bigcap_{\alpha \in \Gamma(\chi)} \ker \chi^{\alpha} = \ker \chi.$$

Here are some well known facts about irreducible representations of finite abelian groups over $\mathbb{C}$ and $\mathbb{Q}$. See [**4**].

Let G be a finite abelian group, let $\chi \in \mathrm{Irr}(G)$ and let $K = \ker \chi$. Then G/K is isomorphic to a finite subgroup of $\mathbb{C}$. Therefore G/K is cyclic.

Let V be an irreducible $\mathbb{Q}G$-module and let $K_1 = C_G(V)$ be the kernel of the representation of G on V. Let ξ be the corresponding character of V. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\xi = \sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$, where $\Gamma(\chi) = \Gamma(\mathbb{Q}(\chi):\mathbb{Q})$. From Lemma 2.3 we know that $K_1 = \ker \chi$, and so G/K_1 is cyclic.

As in [**1**, p. 303], let $A = \langle a \rangle$ be a cyclic group of order m. Then for each $d \mid m$, there is an irreducible $\mathbb{Q}A$-module $V(d)$ of dimension $\phi(d)$, where ϕ is the Euler totient function. We can take $V(d)$ to be $\mathbb{Q}(\xi_d)$, where ξ_d is a primitive d-th root of unity, and a acts on $V(d)$ as multiplication by ξ_d. Since $\sum_{d \mid m} \phi(d) = m$, the modules $V(d)$ are, up to isomorphism, all the irreducible $\mathbb{Q}A$-modules. Thus, there is exactly one for each divisor d of m.

LEMMA 2.4. *Let $A = \langle a \rangle$ be cyclic of order m and let $d \mid m$. Let χ_d denote the character of $\mathbb{Q}A$-module $V(d)$. Then $\chi_d(a)$ is the sum of the primitive d-th roots of unity, and so is equal to $\mu(d)$, where μ is the Möbius function.*

Proof. Let $S(d) = \chi_d(a)$. We have $S(1) = 1$. Let $f(n) = \sum_{d \mid n} S(d)$. This is the sum of all n-th roots of unity. Therefore

$$\sum_{d \mid n} S(d) = 1 + \varepsilon + \ldots + \varepsilon^{n-1} = \frac{\varepsilon^n - 1}{\varepsilon - 1},$$

where ε is a primitive n-th root of unity. Hence

$$\sum_{d \mid n} S(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Then, by the Möbius inversion formula, we have $S(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = \mu(n)$ for $n \geq 1$.

LEMMA 2.5. *Let A be cyclic of order m and let b be an element of A of order $d \mid m$. Then $\chi_m(b) = \dfrac{\phi(m)}{\phi(d)} \mu(d)$. In particular, χ_m is faithful and is the only faithful character of an irreducible $\mathbb{Q}$-representation of A.*

Proof. See [**1**, Lemma 3.4].

COROLLARY 2.6. *Let $A = \langle a \rangle$ be cyclic of order p^s. Let χ_{p^s} be the character of the $\mathbb{Q}A$-module $V(p^s)$. Then χ_{p^s} is faithful and*

$$\chi_{p^s}(a^i) = \begin{cases} -p^{s-1}, & \text{if } (i, p^s) = p^{s-1}, \\ p^{s-1}(p-1), & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows from Lemma 2.5.

If $n > 1$ is a natural number and $n = p_1^{r_1} \dots p_t^{r_t}$, where the p_i are distinct primes, we define $n^* = \sum_{i=1}^{t} p_i^{r_i}$; define $1^* = 0$. Note that, if $m \mid n$, then $m^* \leq n^*$. Thus, if $G \neq 1$ is a finite cyclic group, then $|G|^* = T(G)$.

LEMMA 2.7. (1) *Let m be a positive integer. Then $2\phi(m) \geq m^*$, unless $m = 6$ when $2\phi(m) = 4$.*

(2) *Let $m = 2^\alpha n$, where $\alpha \geq 0$ and n is odd. Then $\dfrac{p}{p-1} \phi(m) \geq n^*$, for each prime divisor p of n.*

Proof. See [**1**, Lemma 3.5].

Let G_p denote the Sylow p-subgroup of G. We define $\Omega_1(G_p)$ to be $\{z \in G_p : z^p = 1\}$.

LEMMA 2.8. *Let G be a finite abelian group, p a prime and $K_1, \dots, K_s$ subgroups of index p in G. Let*

$$J = \left\{ j : 1 \leq j \leq s, \bigcap_{i=1}^{j-1} K_i \cap \Omega_1(G_p) \not\leq K_j \right\}.$$

Then for each $j \in J$ there is a subgroup W_j, cyclic of order p, such that G is the direct product of the subgroups W_j and the subgroup $\bigcap_{j \in J} K_j$.

Proof. We may assume that $J \neq \emptyset$. Our proof is by induction on s. If $s = 1$, $J = \{1\}$ and $\bigcap_{i=1}^{0} K_i = G$, so that the hypothesis implies that there is an element w of order p in K_1. As $|G : K_1| = p$, G is a direct product of $W = \langle w \rangle$ and K_1.

For $s > 1$, let $J' = J \cap \{1, \dots, s-1\}$. By the induction hypothesis, for each $j \in J'$ there is a subgroup W_j of order p such that g is the direct product of the subgroups W_j and the subgroup $H = \bigcap_{j \in J'} K_j$. If $s \notin J$, we are done. So we assume that $s \in J$. It then suffices to show that H is the direct product of a subgroup of order p and the subgroup

https://doi.org/10.1017/S0017089500031906 Published online by Cambridge University Press

$\bigcap_{j\in J} K_j = K_i \cap H$. As $s \in J$, $\bigcap_{i=1}^{s-1} K_j \cap \Omega_1(G_p) \not\leq K_s$ so that $H \cap \Omega_1(G_p) \not\leq K_s$, and there is an element w of order p in H but not in K_s. As $|G:K_s| = p$ we have $HK_s = G$. It follows that $|H:K_s \cap H| = |HK_s:K_s| = |G:K_s| = p$, so that H is the direct product of $\langle w\rangle$ and $K_s \cap H$, as required.

COROLLARY 2.9. *Let G be a finite abelian group and $K_1, \dots, K_s$ subgroups of index 6 in G. Let $J = \{j : 1 \leq j \leq s, \bigcap_{i=1}^{j-1} K_i \cap \Omega_1(G_p) \not\leq K_j$ for $p = 2, 3\}$. Let n be maximal such that G has a direct summand isomorphic to C_6^n. Then $n \geq |J|$.*

Proof. Let n_p be maximal such that G has a direct summand isomorphic to $C_p^{n_p}$. By the Fundamental Theorem of Finitely Generated Abelian Groups, it suffices to show that $n_2 \geq |J|$ and $n_3 \geq |J|$ as $n = \min\{n_2, n_3\}$.

Let $p = 2$. For each j, with $1 \leq j \leq s$, there is an element x_j of order 3 not in K_j; put $K_j' = K_j\langle x_j\rangle$, a subgroup of index 2 in G. It is clear that $K_j' \cap \Omega_1(G_2) = K_j \cap \Omega_1(G_2)$. Thus we have

$$\bigcap_{i=1}^{j-1} K_1' \cap \Omega_1(G_2) = \bigcap_{i=1}^{j-1} (K_i' \cap \Omega_1(G_2)) = \bigcap_{i=1}^{j-1} (K_i \cap \Omega_1(G_2)) = \bigcap_{i=1}^{j-1} K_i \cap \Omega_1(G_2).$$

If $j \in J$, then $\bigcap_{i=1}^{j-1} K_i' \cap \Omega_1(G_2) \not\leq K_j$ and so

$$\bigcap_{i=1}^{j-1} K_i' \cap \Omega_1(G_2) \not\leq K_j \cap \Omega_1(G_2) = K_j' \cap \Omega_1(G_2),$$

whence

$$\bigcap_{i=1}^{j-1} K_i' \cap \Omega_1(G_2) \not\leq K_j'.$$

The previous lemma implies that G has a direct summand isomorphic to $C_2^{|J|}$, so that $|J| \leq n_2$.

It follows similarly that $|J| \leq n_3$.

LEMMA 2.10. *Let $G \neq 1$ be a finite abelian group and let n be maximal such that G has a direct summand isomorphic to C_6^n. Also let V be a $\mathbb{Q}G$-module. Suppose that V is faithful for G, but no proper submodule of V is faithful for G. Then G contains an element g such that $\chi_V(g) < 0$ and*

$$\dim V - \chi_V(g) \geq T(G) - n.$$

Proof. Let $V = V_1 \oplus \dots \oplus V_s$, where each V_i is an irreducible $\mathbb{Q}G$-module; let $K_i = C_G(V_i)$ and $K_i^* = \bigcap_{j\neq i} K_j$. Since V is faithful, $\bigcap_{i=1}^{s} K_i = 1$; also, as V has no proper faithful submodule, $K_i^* \neq 1$ if $1 \leq i \leq s$. Let $K_{i,p} = K_i \cap G_p$. Choose a subset $I \subseteq \{1, \dots, s\}$ minimal such that $\bigcap_{i\in I} K_{i,2} = 1$. Renumbering if necessary, we may assume that $I = \{1, \dots, t\}$ for some t. We interpret the case $t = 0$ as corresponding to $G_2 = 1$.

Let $|G/K_i| = n_i$. Then $\dim V_i = \phi(n_i)$ since V_i is the unique faithful module over $\mathbb{Q}$ for the cyclic group G/K_i; namely, V_i is isomorphic to $\mathbb{Q}(\omega)$, where ω is a primitive n_i-th root of unity and the generator of G/K_i acts as multiplication by ω.

https://doi.org/10.1017/S0017089500031906 Published online by Cambridge University Press

For each j, where $1 \le j \le t$, let x_j be an involution in $\bigcap_{\substack{i=1\\ i\ne j}}^{t} K_{i,2}$, and $x = x_1 \dots x_t$. Then x is an involution and acts as an involution on each of $V_1, \dots, V_t$; therefore, it acts as -1 on each of these modules. [See Note (3) in Chapter 1.] Now renumber the V_i so that $V_1, \dots, V_u$ are precisely those on which x acts as -1. Then x acts trivially on $V_{u+1}, \dots, V_s$. For $j = u+1, \dots, s$, choose x_j of prime order in K_j^*, and let $g = xx_{u+1} \dots x_s = x_1 \dots x_t x_{u+1} \dots x_s$. Thus, g acts as -1 on each of $V_1, \dots, V_u$ and as an element of order p_j on V_j if $u+1 \le j \le s$. By Lemma 2.5 we have $\chi_{V_j}(g) = -\dim V_j$ if $1 \le j \le u$, and $\chi_{V_j}(g) = -\dfrac{1}{p_j - 1}\dim V_j$ if $u+1 \le j \le s$. Hence we have $\chi_V(g) < 0$ and

$$\dim V - \chi_V(g) = 2\sum_{j=1}^{u} \dim V_j + \sum_{j=u+1}^{s}\left(1 + \frac{1}{p_j - 1}\right)\dim V_j. \tag{1}$$

For $0 \le j \le s$, define $I_j = \bigcap_{i=1}^{j} K_i$, so that $I_0 = G$. Let

$J_0 = \{j : u+1 \le j \le s\}$,
$J_1 = \{j : 1 \le j \le u,\ |G : K_j| = 6,\ I_{j-1} \cap \Omega_1(G_p) \not\le K_j \text{ for } p = 2, 3\}$,
$J_2 = \{j : 1 \le j \le u,\ |G : K_j| = 6,\ I_{j-1} \cap \Omega_1(G_2) \le K_j\}$,
$J_3 = \{j : 1 \le j \le U,\ |G : K_j| = 6,\ I_{j-1} \cap \Omega_1(G_3) \le K_j,\ i_{j-1} \cap \Omega_1(G_2) \not\le K_j\}$,
and $J_4 = \{j : 1 \le j \ge u,\ |G : K_j| \ne 6\}$.

Define subgroups M_j of G as follows:

$$M_j = \begin{cases} K_j, & \text{if } j \in J_1 \cup J_4,\\ G_2 K_j, & \text{if } j \in J_0 \cup J_2,\\ G_3 K_j, & \text{if } j \in J_3. \end{cases}$$

Let $m_j = |G : M_j|$ so that:

(a) for $j \in J_0$, m_j is the maximal odd divisor of n_j and so, by Lemma 2.7(2), $\dfrac{p_j}{p_j - 1}\dim V_j \ge m_j^*$;

(b) for $j \in J_1$, $m_j = n_j = 6$ so that $\dim V_j = \phi(6) = 2$ while $m_j^* = 5$;

(c) for $j \in J_2$, $m_j = 3$, $n_j = 6$ so that $\dim V_j = 2$ while $m_j^* = 3$;

(d) for $j \in J_3$, $m_j = 2$, $n_j = 6$ so that $\dim V_j = 2$ while $m_j^* = 2$;

(e) for $j \in J_4$, $m_j = n_j \ne 6$ so that $\dim V_j = \phi(n_j)$ and so, by Lemma 2.7(1), $2\dim V_j \ge m_j^*$.

It follows that

$$2\sum_{j=1}^{u} \dim V_j + \sum_{j=u+1}^{s}\left(1 + \frac{1}{p_j - 1}\right)\dim V_j \ge \sum_{j \in J_1} m_j^* - |J_1| + \sum_{j \notin J_1} m_j^*.$$

It follows from Corollary 2.9 that $n \ge |J_1|$, so that we have

$$\dim V - \chi_V(g) \ge \sum_{j=1}^{s} m_j^* - n. \tag{2}$$

We next show that $\bigcap_{j=1}^{s} M_j = 1$. Suppose that this is not the case and that m is an

https://doi.org/10.1017/S0017089500031906 Published online by Cambridge University Press

element of prime order p in this intersection. As $\bigcap_{j=1}^{s} K_j = 1$, there is a minimal index j for which $m \notin K_j$.

Suppose that $p = 2$. Then $j \notin J_1 \cup J_4$ as here $M_j = K_j$. Also $j \notin J_3$ as here $M_j = G_3K_j$ and so $M_j/K_j \cong G_3/G_3K_j$, a 3-group. If $j \in J_0$, then $m \in K_i$, for $1 \le i \le t$, so that $m \in I_t \cap G_2$ which is trivial; this is a contradiction. If $j \in J_2$, then by the minimality of j, $m \in I_{j-1} \cap \Omega_1(G_2)$ so that $m \in K_j$, by the definition of J_2, again a contradiction.

Suppose that $p = 3$. As before $j \notin J_0 \cup J_1 \cup J_2 \cup J_4$. If $j \in J_3$, then $m \in I_{j-1} \cap \Omega_1(G_3) \le K_j$, a contradiction.

The case $o(m) \ge 5$ also leads to a contradiction as the Sylow p-subgroup of M_j is contained in K_j for each $p \ne 2, 3$ and for all j, $1 \le j \le s$.

As $\bigcap_{j=1}^{s} M_j = 1$, G can be embedded as a subgroup of the direct product $\mathrm{Dr}_{j=1}^{s}\, G/M_j$. However from [**2**],

$$T(G) \le T(\mathrm{Dr}_{j=1}^{s}\, G/M_j) = \sum_{j=1}^{s} T(G/M_j) = \sum_{j=1}^{s} m_j^*.$$

From (2), we deduce the inequality

$$\dim V - \chi_V(g) \ge T(G) - n,$$

as required.

THEOREM 2.11. *Let $G \ne 1$ be a finite abelian group and let n be maximal such that G has a direct summand isomorphic to C_6^n. Then*

$$c(G) = q(G) = T(G) - n.$$

Proof. By Corollary 2.2 we have $c(G) = q(G)$.

Now let V be a faithful quasi-permutation representation of G over $\mathbb{Q}$ of minimal degree. Then $q(G) = \dim V$. Write $V = V_1 \oplus W$, where V_1 is a faithful $\mathbb{Q}G$-module with no proper faithful submodules for G. By Lemma 2.10, there is $g \in G$ such that

$$\dim V_1 - \chi_{V_1}(g) \ge T(G) - n$$

and $\chi_{V_1}(g) < 0$. Since $\chi_V(g) \ge 0$ we have

$$\begin{aligned} 0 \le \chi_V(g) = \chi_{V_1}(g) + \chi_W(g) &\le \dim V_1 - (T(G) - n) + \chi_W(g) \\ &\le \dim V_1 - (T(G) - n) + \dim W. \end{aligned}$$

Hence

$$\dim V = q(G) \ge T(G) - n. \tag{4}$$

Now we show that there exists a quasi-permutation module U over $\mathbb{Q}$ for G such that $\dim U = T(G) - n$ and, since $T(G) - n$ is the minimal value, we have $q(G) = T(G) - n$.

Let $G = \prod_{i=1}^{s} G_i$. Here $G_i \cong C_6$, where p_i is a prime, m_i is a positive integer for $i = 1, \ldots, n$, and $G_i \cong C_{p_i^{m_i}}$ for $i = n+1, \ldots, s$. Let $K_i = \prod_{j \ne i} G_j$ for $i = 1, \ldots, s$. Then

https://doi.org/10.1017/S0017089500031906 Published online by Cambridge University Press

$\bigcap_{i=1}^{s} K_i = 1$ and G/K_i is cyclic. Let χ_i be the faithful irreducible $\mathbb{Q}$-character of $G/K_i \cong G_i$; (see Lemma 2.5). Let V_i be its module and let $U_1 = \bigoplus_{i=1}^{s} V_i$. Then, by Lemma 2.5 and Corollary 2.6, we have:

$$\dim V_i = 2 \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad \min\{\chi_i(g) : g \in G\} = -2 \quad \text{for} \quad i = 1, \ldots, n;$$
$$\dim V_i = p_i^{m_i - 1}(p_i - 1) \quad \text{for} \quad i = n+1, \ldots, s, \quad \text{and} \quad \min\{\chi_i(g) : g \in G\} = -p_i^{m_i - 1}$$

for $i = n+1, \ldots, s$. Hence $\dim U_1 = \sum_{i=1}^{s} \dim V_i$ and

$$A = \min\{\chi_{U_1}(g) : g \in G\} \geq \sum_{i=1}^{s} \min\{\chi_i(g) : g \in G\}.$$

Let $l = -A$ and let $l\mathbb{Q}$ denote the direct sum of l copies of the trivial module. Let $U = U_1 + l\mathbb{Q}$, so that U is a faithful quasi-permutation module over $\mathbb{Q}$. Hence by (4) we have $T(G) - n \leq q(G)$ and, by the definition of $q(G)$, we have $q(G) \leq \dim U$. It follows that

$$T(G) - n \leq \dim U = \dim(U_1 \oplus l\mathbb{Q}) \leq \dim U_1 - \sum_{i=1}^{s} \min\{\chi_i(g) : g \in G\} = T(G) - n.$$

Hence $\dim U = T(G) - n$, as required.

ACKNOWLEDGEMENT. This paper is a part of a Ph.D thesis submitted to the University of Manchester. The work was done under the supervision of Professor Brian Hartley (1992–94). Also I would like to express my sincere gratitude to Dr. Robert Sandling for help in preparing this paper.

REFERENCES

1. J. M. Burns, B. Goldsmith, B. Hartley and R. Sandling, On quasi-permutation representations of finite groups, *Glasgow Math. J.* **36** (1994), 301–308.

2. M. Hoffman, An invariant of finite abelian groups. *Amer. Math. Monthly* **94** (1987), 664–666.

3. I. M. Isaacs, *Character theory of finite groups* (Academic Press, New York, 1976).

4. S. K. Sehgal, *Topics in group rings* (Marcel Dekker, New York, 1978).

5. W. J. Wong, Linear groups analogous to permutation groups, *J. Austral. Math. Soc. Ser. A* **3** (1963), 180–184.

6. W. J. Wong, On linear p-groups, *J. Austral. Math. Soc. Ser. A* **4** (1964), 174–178.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANCHESTER
MANCHESTER M13 9PL
ENGLAND

Present address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF URMIA
URMIA
IRAN