



The Schwarz Lemma at the Boundary of the Egg Domain B_{p_1, p_2} in \mathbb{C}^n

Xiaomin Tang and Taishun Liu

Abstract. Let $B_{p_1, p_2} = \{z \in \mathbb{C}^n : |z_1|^{p_1} + |z_2|^{p_2} + \dots + |z_n|^{p_2} < 1\}$ be an egg domain in \mathbb{C}^n . In this paper, we first characterize the Kobayashi metric on B_{p_1, p_2} ($p_1 \geq 1, p_2 \geq 1$) and then establish a new type of classical boundary Schwarz lemma at $z_0 \in \partial B_{p_1, p_2}$ for holomorphic self-mappings of B_{p_1, p_2} ($p_1 \geq 1, p_2 > 1$), where $z_0 = (e^{i\theta}, 0, \dots, 0)'$ and $\theta \in \mathbb{R}$.

1 Introduction

Let \mathbb{C}^n be the n -dimensional complex Hilbert space with the inner product and the norm given by

$$\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}, \quad \|z\| = (\langle z, z \rangle)^{\frac{1}{2}},$$

where $z, w \in \mathbb{C}^n$. Let $B^n = \{z \in \mathbb{C}^n : \|z\|^2 = |z_1|^2 + \dots + |z_n|^2 < 1\}$ be the open unit ball in \mathbb{C}^n . The unit sphere is defined by $\partial B^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. Throughout this paper, we write a point $z \in \mathbb{C}^n$ as a column vector in the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

and the symbol $'$ stands for the transpose of vectors or matrices. In what follows, a domain is a connected open subset in \mathbb{C}^n .

For $p_1 \geq 1$ and $p_2 \geq 1$, the egg domain B_{p_1, p_2} is defined by

$$B_{p_1, p_2} = \{z \in \mathbb{C}^n : |z_1|^{p_1} + |z_2|^{p_2} + \dots + |z_n|^{p_2} < 1\}.$$

Denote by $\partial B_{p_1, p_2}$ the boundary of B_{p_1, p_2} . Let $H(B_{p_1, p_2})$ be the set of all holomorphic mappings from B_{p_1, p_2} to \mathbb{C}^n . For $f \in H(B_{p_1, p_2})$, we also write it as $f = (f_1, f_2, \dots, f_n)'$, where f_j is a holomorphic function from B_{p_1, p_2} to \mathbb{C} , $j = 1, \dots, n$. The derivative of $f \in H(B_{p_1, p_2})$ at a point $a \in B_{p_1, p_2}$ is the complex Jacobian matrix of f given by

$$J_f(a) = \left(\frac{\partial f_i}{\partial z_j}(a) \right)_{n \times n}.$$

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Then $J_f(a)$ is a linear mapping from \mathbb{C}^n to \mathbb{C}^n . We set

$$J_{f_i}(a) = \left(\frac{\partial f_i}{\partial z_1}(a), \dots, \frac{\partial f_i}{\partial z_n}(a) \right), \quad i = 1, \dots, n.$$

Let D be the unit disk in the complex plane \mathbb{C} . The classical Schwarz lemma states that a holomorphic function f mapping D into itself, with $f(0) = 0$, satisfies the inequality $|f(z)| \leq |z|$ for any $z \in D$. It is well known that the Schwarz lemma is one of the most important results in the classical complex analysis, which has become a crucial theme in many branches of mathematical research for over a hundred years. Establishing various versions of the Schwarz lemma has attracted the attention of many mathematicians. We refer the reader to [1, 9, 11, 15, 16, 19] for more on this matter. It has been a very natural task to obtain the boundary version of the Schwarz lemma. In the case of one complex variable, the following Schwarz lemma at the boundary is classical.

Theorem 1.1 ([5]) *Let $f: D \rightarrow D$ be a holomorphic function. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$. Moreover, the inequality is sharp.*

If we remove the condition $f(0) = 0$ in Theorem 1.1, then by applying Theorem 1.1 to the holomorphic function

$$g(z) = \frac{1 - \overline{f(0)}}{1 - f(0)} \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)},$$

we have the estimate

$$(1.1) \quad f'(1) \geq \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0.$$

D. Chelst [3] and R. Osserman [14] studied the Schwarz lemma at the boundary of the unit disk. S. G. Krantz [10] explored versions of the Schwarz lemma at the boundary point of a domain. Recently, B. N. Örnek [13] gave some new inequalities of Schwarz inequality at the boundary of the unit disk and obtained the sharpness of these inequalities. On the other hand, in the case of several complex variables, H. Wu [18] proved what is now called the Carathéodory–Cartan–Kaup–Wu theorem, which generalizes the classical Schwarz lemma for holomorphic mappings to higher dimension. This result is stated as follows.

Theorem 1.2 ([18]) *Let Ω be a bounded domain in \mathbb{C}^n and let f be a holomorphic self-mapping of Ω that fixes a point $p \in \Omega$. Then*

- (i) *the eigenvalues of $J_f(p)$ all have modulus not exceeding 1;*
- (ii) *$|\det J_f(p)| \leq 1$;*
- (iii) *if $|\det J_f(p)| = 1$, then f is a biholomorphism of Ω .*

A natural question arises. What is a higher dimensional version of the Schwarz lemma at the boundary? It is this problem that motivated our study. In [2], Burns and Krantz obtained a Schwarz lemma at the boundary, which gives a new rigidity result for holomorphic mappings. In [6], Huang further strengthened the Burns–Krantz

result for holomorphic mappings with an interior fixed point. See [7, 8] for more on these matters. Here are two typical results in these papers.

Theorem 1.3 ([2]) *Let Ω be a bounded strongly pseudoconvex domain in \mathbb{C}^n . Let $p \in \partial\Omega$ and let $f: \Omega \rightarrow \Omega$ be a holomorphic mapping such that $f(z) = z + O(|z - p|^4)$ as $z \rightarrow p$. Then $f(z) \equiv z$.*

Theorem 1.4 ([6]) *Let $\Omega \subset\subset \mathbb{C}^n$ ($n > 1$) be a simply connected pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial\Omega$ is a strongly pseudoconvex point. If $f: \Omega \rightarrow \Omega$ is a holomorphic mapping such that $f(z_0) = z_0$ for some $z_0 \in \Omega$ and $f(z) = z + o(|z - p|^2)$ as $z \rightarrow p$, then $f(z) \equiv z$.*

More recently, in [12] we established a version of the boundary Schwarz lemma for holomorphic self-mappings of B^n . The following result is one of the main results in [12].

Theorem 1.5 ([12]) *Let $f: B^n \rightarrow B^n$ be a holomorphic mapping. If f is holomorphic at $z_0 \in \partial B^n$ and $f(z_0) = z_0$, then for the eigenvalues $\lambda, \mu_2, \dots, \mu_n$ of $J_f(z_0)$, the following five statements hold.*

- (i) $\lambda \geq \frac{|1 - \overline{a'} z_0|^2}{1 - \|a\|^2} > 0$, where $a = f(0)$.
- (ii) z_0 is an eigenvector of $\overline{J_f(z_0)'}$ with respect to λ . That is, $\overline{J_f(z_0)'} z_0 = \lambda z_0$.
- (iii) $\mu_j \in \mathbb{C}$ and $|\mu_j| \leq \sqrt{\lambda}$ for $j = 2, \dots, n$.
- (iv) For any μ_j , there exists $\alpha_j \in T_{z_0}^{(1,0)}(\partial B^n) \cap \partial B^n$ such that

$$J_f(z_0)\alpha_j = \mu_j \alpha_j, \quad j = 2, \dots, n.$$

- (v) $|\det J_f(z_0)| \leq \lambda^{\frac{n+1}{2}}$, $|\text{tr} J_f(z_0)| \leq \lambda + \sqrt{\lambda}(n - 1)$.

Here, $T_{z_0}^{(1,0)}(\partial B^n) = \{w \in \mathbb{C}^n : \overline{z_0}' w = 0\}$ is the holomorphic tangent space to ∂B^n at z_0 . Moreover, the inequalities in (i), (iii), and (v) are sharp.

The purpose of this work is to prove the boundary Schwarz lemma at $z_0 = (e^{i\theta}, 0, \dots, 0)' \in \partial B_{p_1, p_2}$ for holomorphic self-mappings of B_{p_1, p_2} . At the same time, we will develop some properties of the Kobayashi metric on B_{p_1, p_2} .

2 Auxiliary Results

In this section, for $p_1 \geq 1$ and $p_2 \geq 1$ we characterize the Kobayashi metric on B_{p_1, p_2} , which will not only be used in the subsequent section but also has its own interest. We begin with some notation and definitions.

A domain $\Omega \subset \mathbb{C}^n$ is said to be *circular* if $e^{i\theta} z \in \Omega$ whenever $z \in \Omega$ and $\theta \in \mathbb{R}$, and a domain $\Omega \subset \mathbb{C}^n$ is said to be *convex* if $tz_1 + (1 - t)z_2 \in \Omega$ whenever $z_1, z_2 \in \Omega$ and $0 \leq t \leq 1$. It is easy to check that B_{p_1, p_2} is a bounded convex circular domain in \mathbb{C}^n . The Minkowski functional $\rho(z)$ of B_{p_1, p_2} is defined by

$$\rho(z) = \inf \left\{ t > 0 : \frac{z}{t} \in B_{p_1, p_2} \right\}, \quad z \in \mathbb{C}^n.$$

It is clear that the Minkowski functional $\rho(z)$ of B_{p_1, p_2} is a Banach norm of \mathbb{C}^n , and

$$B_{p_1, p_2} = \{z \in \mathbb{C}^n : \rho(z) < 1\}$$

is the open unit ball of \mathbb{C}^n as the Banach space with the norm $\rho(z)$ (see [17]). The Minkowski functional $\rho(z)$ of B_{p_1, p_2} is C^1 on \mathbb{C}^n except for some submanifolds of lower dimensions.

Let $H(D, B_{p_1, p_2})$ be the family of all holomorphic mappings from D into B_{p_1, p_2} . For any $z \in B_{p_1, p_2}$, $\xi \in \mathbb{C}^n$,

$$F_K(z, \xi) = \inf \left\{ \frac{\rho(\xi)}{\rho(f'(0))} = \frac{\|\xi\|}{\|f'(0)\|} : f \in H(D, B_{p_1, p_2}), \right. \\ \left. f(0) = z, f'(0) \text{ and } \xi \text{ have the same direction} \right\}$$

is said to be the infinitesimal form of Kobayashi metric of B_{p_1, p_2} , where $f'(0) = (f_1'(0), \dots, f_n'(0))'$.

Lemma 2.1 For any $z \in B_{p_1, p_2}$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)' \in \mathbb{C}^n$,

$$F_K(z, \xi) \geq \frac{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}.$$

Proof Suppose that $h \in H(D, B_{p_1, p_2})$, $h(0) = z$, and $h'(0)$ and ξ have the same direction. Without loss of generality, we assume that $\xi \neq 0$. Then there exists $\lambda \geq 0$ such that $h'(0) = \lambda \xi$. So we have $\rho(h'(0)) = \rho(\lambda \xi) = \lambda \rho(\xi)$, which implies $\lambda = \frac{\rho(h'(0))}{\rho(\xi)}$. This means

$$h'(0) = \frac{\rho(h'(0))}{\rho(\xi)} \xi.$$

Hence, for any $\zeta \in D$, we obtain

$$h(\zeta) = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + h'(0)\zeta + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \zeta^2 + \dots \\ = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \frac{\rho(h'(0))}{\rho(\xi)} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \zeta + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \zeta^2 + \dots \in B_{p_1, p_2},$$

where $b_j = \frac{1}{2} h_j''(0)$, $j = 1, \dots, n$. It follows that

$$\left| z_1 + \frac{\rho(h'(0))}{\rho(\xi)} \xi_1 \zeta + b_1 \zeta^2 + \dots \right|^{p_1} + \left| z_2 + \frac{\rho(h'(0))}{\rho(\xi)} \xi_2 \zeta + b_2 \zeta^2 + \dots \right|^{p_2} + \dots \\ \dots + \left| z_n + \frac{\rho(h'(0))}{\rho(\xi)} \xi_n \zeta + b_n \zeta^2 + \dots \right|^{p_2} < 1.$$

It is known that there exists a bounded linear functional $T = (a_1, \dots, a_n)$ on the Banach space \mathbb{C}^n with the norm $\rho(z)$ such that $\|T\| = 1$ and

$$\begin{aligned} (2.1) \quad T\left(z_1, \frac{\rho(h'(0))}{\rho(\xi)} \xi_2, \dots, \frac{\rho(h'(0))}{\rho(\xi)} \xi_n\right)' & \\ &= (a_1, a_2, \dots, a_n)\left(z_1, \frac{\rho(h'(0))}{\rho(\xi)} \xi_2, \dots, \frac{\rho(h'(0))}{\rho(\xi)} \xi_n\right)' \\ &= \rho\left(z_1, \frac{\rho(h'(0))}{\rho(\xi)} \xi_2, \dots, \frac{\rho(h'(0))}{\rho(\xi)} \xi_n\right). \end{aligned}$$

Set $\zeta = re^{i\theta}$, where $r \in (0, 1)$ and $\theta \in [0, 2\pi]$. Notice that

$$\begin{aligned} \left|z_1 + \frac{\rho(h'(0))}{\rho(\xi)} \xi_1 r e^{i\theta} + b_1 r^2 e^{2i\theta} + \dots\right|^{p_1} + \left|z_2 e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_2 r + b_2 r^2 e^{i\theta} + \dots\right|^{p_2} + \dots \\ \dots + \left|z_n e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_n r + b_n r^2 e^{i\theta} + \dots\right|^{p_2} < 1. \end{aligned}$$

This shows that

$$\begin{aligned} \eta = \left(z_1 + \frac{\rho(h'(0))}{\rho(\xi)} \xi_1 r e^{i\theta} + b_1 r^2 e^{2i\theta} + \dots, z_2 e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_2 r \right. \\ \left. + b_2 r^2 e^{i\theta} + \dots, \dots, z_n e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_n r + b_n r^2 e^{i\theta} + \dots\right)' \in B_{p_1, p_2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \Re\left\{a_1\left(z_1 + \frac{\rho(h'(0))}{\rho(\xi)} \xi_1 r e^{i\theta} + b_1 r^2 e^{2i\theta} + \dots\right) \right. \\ \left. + a_2\left(z_2 e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_2 r + b_2 r^2 e^{i\theta} + \dots\right) + \dots \right. \\ \left. \dots + a_n\left(z_n e^{-i\theta} + \frac{\rho(h'(0))}{\rho(\xi)} \xi_n r + b_n r^2 e^{i\theta} + \dots\right)\right\} \\ \leq |T(\eta)| \leq \|T\| \rho(\eta) < 1. \end{aligned}$$

Taking the integral with θ on $[0, 2\pi]$ for the inequality above, we obtain

$$(2.2) \quad \Re\left\{a_1 z_1 + a_2 \frac{\rho(h'(0))}{\rho(\xi)} \xi_2 r + \dots + a_n \frac{\rho(h'(0))}{\rho(\xi)} \xi_n r\right\} < 1.$$

Let $r \rightarrow 1^-$. Then (2.1) and (2.2) imply

$$\rho\left(z_1, \frac{\rho(h'(0))}{\rho(\xi)} \xi_2, \dots, \frac{\rho(h'(0))}{\rho(\xi)} \xi_n\right) \leq 1.$$

It follows that

$$\begin{aligned} |z_1|^{p_1} + \left|\frac{\rho(h'(0))}{\rho(\xi)} \xi_2\right|^{p_2} + \dots + \left|\frac{\rho(h'(0))}{\rho(\xi)} \xi_n\right|^{p_2} = \\ |z_1|^{p_1} + \left|\frac{\rho(h'(0))}{\rho(\xi)}\right|^{p_2} (|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2}) \leq 1. \end{aligned}$$

This gives

$$\frac{\rho(h'(0))}{\rho(\xi)} \leq \frac{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}.$$

By the definition of the Kobayashi metric, we get

$$F_K(z, \xi) \geq \frac{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}.$$

The proof is complete. ■

Lemma 2.2 For any $z = (z_1, 0, \dots, 0)' \in B_{p_1, p_2}$, $\xi = (0, \xi_2, \dots, \xi_n)' \in \mathbb{C}^n$,

$$F_K(z, \xi) = \frac{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}.$$

Proof Without loss of generality, we assume that $\xi \neq 0$. Take

$$h(\zeta) = z + \zeta(1 - |z_1|^{p_1})^{\frac{1}{p_2}} \frac{\xi}{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}, \quad \zeta \in D.$$

Then $h: D \rightarrow \mathbb{C}^n$ is a holomorphic mapping, and

$$|h_1(\zeta)|^{p_1} + |h_2(\zeta)|^{p_2} + \dots + |h_n(\zeta)|^{p_2} = |z_1|^{p_1} + |\zeta|^{p_2}(1 - |z_1|^{p_1}) < 1.$$

This means that $h \in H(D, B_{p_1, p_2})$ and $h(0) = z$. Moreover,

$$h'(0) = \frac{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}} \xi \quad \text{and} \quad \xi$$

have the same direction. Hence,

$$F_K(z, \xi) \leq \frac{\rho(\xi)}{\rho(h'(0))} = \frac{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}.$$

On the other hand, by Lemma 2.1, we obtain

$$F_K(z, \xi) \geq \frac{(|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2})^{\frac{1}{p_2}}}{(1 - |z_1|^{p_1})^{\frac{1}{p_2}}}.$$

This gives the desired result. ■

The following lemma characterizes the contraction property of the Kobayashi metric, which is also a version of the Schwarz lemma.

Lemma 2.3 ([4]) Let $\phi: B_{p_1, p_2} \rightarrow B_{p_1, p_2}$ be a holomorphic mapping. Then for any $z \in B_{p_1, p_2}$, $\xi \in \mathbb{C}^n$,

$$F_K(\phi(z), J_\phi(z)\xi) \leq F_K(z, \xi).$$

3 Main Results

In this section, we present the main results of this article. First, we set up the following notations and definitions. We then provide a generalization of Theorem 1.1.

For $p_1 > 1$ and $p_2 > 1$, take

$$g(z) = |z_1|^{p_1} + |z_2|^{p_2} + \dots + |z_n|^{p_2}, \quad z \in \mathbb{C}^n.$$

Then the gradient of $g(z)$ is

$$\begin{aligned} \nabla g(z) &= 2 \left(\frac{\partial g}{\partial z_1}(z), \frac{\partial g}{\partial z_2}(z), \dots, \frac{\partial g}{\partial z_n}(z) \right)' \\ &= \left(p_1 |z_1|^{p_1-2} z_1, p_2 |z_2|^{p_2-2} z_2, \dots, p_2 |z_n|^{p_2-2} z_n \right)', \end{aligned}$$

and $\nabla g(z)$ is continuous on \mathbb{C}^n . This shows that B_{p_1, p_2} is a domain with C^1 boundary. So we have the following proposition.

Proposition 3.1 *Let $p_1 > 1$, $p_2 > 1$ and $z \in \partial B_{p_1, p_2}$. Then the tangent space $T_z(\partial B_{p_1, p_2})$ to $\partial B_{p_1, p_2}$ at z is*

$$T_z(\partial B_{p_1, p_2}) = \left\{ \alpha \in \mathbb{C}^n : \Re \left[p_1 |z_1|^{p_1-2} \overline{z_1} \alpha_1 + p_2 \sum_{j=2}^n |z_j|^{p_2-2} \overline{z_j} \alpha_j \right] = 0 \right\},$$

and the holomorphic tangent space $T_z^{(1,0)}(\partial B_{p_1, p_2})$ to $\partial B_{p_1, p_2}$ at z is

$$T_z^{(1,0)}(\partial B_{p_1, p_2}) = \left\{ \alpha \in \mathbb{C}^n : p_1 |z_1|^{p_1-2} \overline{z_1} \alpha_1 + p_2 \sum_{j=2}^n |z_j|^{p_2-2} \overline{z_j} \alpha_j = 0 \right\}.$$

In particular, when $p_1 \geq 1$, $p_2 > 1$ and $z_0 = (e^{i\theta}, 0, \dots, 0)' \in \partial B_{p_1, p_2}$, where $\theta \in \mathbb{R}$, we have

$$T_{z_0}(\partial B_{p_1, p_2}) = \{ \alpha \in \mathbb{C}^n : \Re \overline{z_0}' \alpha = 0 \}, \quad T_{z_0}^{(1,0)}(\partial B_{p_1, p_2}) = \{ \alpha \in \mathbb{C}^n : \alpha_1 = 0 \}.$$

Theorem 3.2 *Let $f: B_{p_1, p_2} \rightarrow B_{p_1, p_2}$ be a holomorphic mapping and let $z_0 = (e^{i\theta}, 0, \dots, 0)' \in \partial B_{p_1, p_2}$, where $p_1 \geq 1$, $p_2 > 1$ and $\theta \in \mathbb{R}$. If f is holomorphic at z_0 and $f(z_0) = z_0$, then for the eigenvalues $\lambda, \mu_2, \dots, \mu_n$ of $J_f(z_0)$, the following five statements hold.*

- (i) $\lambda \geq \frac{|1 - \overline{f_1(0)} e^{i\theta}|^2}{1 - |f_1(0)|^2} > 0$.
- (ii) z_0 is an eigenvector of $\overline{J_f(z_0)'}'$ with respect to λ . That is, $\overline{J_f(z_0)'}' z_0 = \lambda z_0$.
- (iii) $\mu_j \in \mathbb{C}$ and $|\mu_j| \leq \lambda^{\frac{1}{p_2}}$ for $j = 2, \dots, n$.
- (iv) For any μ_j , there exists $\alpha_j \in T_{z_0}^{(1,0)}(\partial B_{p_1, p_2}) \cap \partial B_{p_1, p_2}$ such that

$$J_f(z_0) \alpha_j = \mu_j \alpha_j, \quad j = 2, \dots, n.$$

- (v) $|\det J_f(z_0)| \leq \lambda \lambda^{\frac{n-1}{p_2}}$, $|\text{tr} J_f(z_0)| \leq \lambda + \lambda^{\frac{1}{p_2}} (n-1)$.

Moreover, the inequalities in (i), (iii), and (v) are sharp.

Proof The proof is divided into five steps.

Step 1. Suppose that f is holomorphic in a neighborhood V of z_0 . Then $f(\partial B_{p_1, p_2} \cap V)$ and $\partial B_{p_1, p_2}$ are tangent at z_0 . This means that the tangent space and holomorphic tangent space to $f(\partial B_{p_1, p_2} \cap V)$ at $f(z_0) = z_0$ are contained in $T_{z_0}(\partial B_{p_1, p_2})$ and $T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$, respectively. Notice that for any $\alpha \in T_{z_0}(\partial B_{p_1, p_2})$, $J_f(z_0)\alpha$ is a tangent vector of $f(\partial B_{p_1, p_2} \cap V)$ at $f(z_0) = z_0$. Then $J_f(z_0)\alpha \in T_{z_0}(\partial B_{p_1, p_2})$. This gives $\Re \overline{z_0}' J_f(z_0)\alpha = 0$ for any $\alpha \in T_{z_0}(\partial B_{p_1, p_2})$. So there exists $\lambda \in \mathbb{R}$ such that $\overline{z_0}' J_f(z_0) = \lambda \overline{z_0}'$. That is

$$(3.1) \quad \overline{J_f(z_0)'} z_0 = \lambda z_0.$$

It follows that λ is an eigenvalue of $\overline{J_f(z_0)'}$, and z_0 is an eigenvector of $\overline{J_f(z_0)'}$ with respect to λ . Since λ is a real number, we know that λ is also an eigenvalue of $J_f(z_0)$. The proof of (ii) is complete.

Step 2. Take $g(\zeta) = \overline{z_0}' f(\zeta z_0) = e^{-i\theta} f_1(\zeta z_0)$, $\zeta \in D$. Then $g: D \rightarrow D$ is a holomorphic function, and g is holomorphic at 1 with $g(1) = 1$. Moreover, (3.1) yields

$$g'(1) = e^{-i\theta} J_{f_1}(z_0) z_0 = \overline{z_0}' J_f(z_0) z_0 = \lambda.$$

Thus, by (1.1) we obtain

$$\lambda = g'(1) \geq \frac{|1 - \overline{g(0)}|^2}{1 - |g(0)|^2} = \frac{|1 - \overline{f_1(0)} e^{i\theta}|^2}{1 - |f_1(0)|^2} > 0.$$

The proof of (i) is complete.

Step 3. Since for any $\alpha \in T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$, we have $J_f(z_0)\alpha \in T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$. Hence, $J_f(z_0)$ is a linear transformation on the $(n - 1)$ -dimensional complex vector space $T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$. This shows that there are $\mu_2, \dots, \mu_n \in \mathbb{C}$ such that μ_2, \dots, μ_n are the all eigenvalues of the linear transformation $J_f(z_0)$ on $T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$. So there exist eigenvectors $\alpha_j \in T_{z_0}^{(1,0)}(\partial B_{p_1, p_2}) \cap \partial B_{p_1, p_2}$ such that

$$J_f(z_0)\alpha_j = \mu_j \alpha_j, \quad j = 2, \dots, n.$$

The proof of (iv) is complete.

Take the eigenvector $\xi \in T_{z_0}^{(1,0)}(\partial B_{p_1, p_2}) \cap \partial B_{p_1, p_2}$ of $J_f(z_0)$ with respect to μ_j , $j = 2, \dots, n$. Then

$$\xi = (0, \xi_2, \dots, \xi_n)', \quad |\xi_2|^{p_2} + \dots + |\xi_n|^{p_2} = 1 \text{ and } J_f(z_0)\xi = \mu_j \xi, \quad j = 2, \dots, n.$$

It follows that

$$(3.2) \quad |\mu_j|^{p_2} = |J_{f_2}(z_0)\xi|^{p_2} + \dots + |J_{f_n}(z_0)\xi|^{p_2}.$$

For any $t \in (0, 1)$, by Lemmas 2.1–2.3, we have

$$\begin{aligned} \frac{|J_{f_2}(tz_0)\xi|^{p_2} + \dots + |J_{f_n}(tz_0)\xi|^{p_2}}{1 - |f_1(tz_0)|^{p_1}} &\leq [F_K(f(tz_0), J_f(tz_0)\xi)]^{p_2} \\ &\leq [F_K(tz_0, \xi)]^{p_2} \\ &= \frac{|\xi_2|^{p_2} + \dots + |\xi_n|^{p_2}}{1 - t^{p_1}} = \frac{1}{1 - t^{p_1}}. \end{aligned}$$

This implies

$$(3.3) \quad |J_{f_2}(tz_0)\xi|^{p_2} + \dots + |J_{f_n}(tz_0)\xi|^{p_2} \leq \frac{1 - |f_1(tz_0)|^{p_1}}{1 - t^{p_1}}.$$

Notice that $f_1(tz_0) = e^{i\theta} \overline{z_0}' f(tz_0)$ and

$$f(tz_0) = z_0 - J_f(z_0)z_0(1-t) + O(|t-1|^2) \quad (t \rightarrow 1^-).$$

Then

$$\begin{aligned} f_1(tz_0) &= e^{i\theta} [1 - \overline{z_0}' J_f(z_0)z_0(1-t) + O(|t-1|^2)] \\ &= e^{i\theta} [1 - \lambda(1-t) + O(|t-1|^2)] \quad (t \rightarrow 1^-). \end{aligned}$$

This gives

$$|f_1(tz_0)|^{p_1} = |1 - \lambda(1-t) + O(|t-1|^2)|^{p_1} = 1 - p_1\lambda(1-t) + O(|t-1|^2) \quad (t \rightarrow 1^-).$$

Thus,

$$\lim_{t \rightarrow 1^-} \frac{1 - |f_1(tz_0)|^{p_1}}{1 - t^{p_1}} = \lim_{t \rightarrow 1^-} \frac{p_1\lambda(1-t)}{1 - t^{p_1}} = \lambda.$$

This, together with (3.2) and (3.3), yields

$$|\mu_j| \leq \lambda^{\frac{1}{p_2}}, \quad j = 2, \dots, n.$$

The proof of (iii) is complete. Moreover, (v) can be easily obtained from (iii).

Step 4. We claim that $\lambda, \mu_2, \dots, \mu_n$ are the all eigenvalues of the linear transformation $J_f(z_0)$ on the n -dimensional complex vector space \mathbb{C}^n .

Assume that $\alpha_2, \dots, \alpha_n$ is a standard orthogonal basis of $T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$. Then $z_0, \alpha_2, \dots, \alpha_n$ becomes a standard orthogonal basis of \mathbb{C}^n . Write

$$U = (z_0, \alpha_2, \dots, \alpha_n).$$

Then U is a unitary square matrix of order n . Since $\overline{J_f(z_0)'} z_0 = \lambda z_0$, we have

$$\overline{J_f(z_0)'}(z_0, \alpha_2, \dots, \alpha_n) = (z_0, \alpha_2, \dots, \alpha_n) \begin{pmatrix} \lambda & \overline{B}' \\ 0 & \overline{V}' \end{pmatrix},$$

where V is a complex square matrix of order $(n-1)$ and B is an $(n-1) \times 1$ complex matrix. It follows that

$$\overline{U}' J_f(z_0) = \begin{pmatrix} \lambda & 0 \\ B & V \end{pmatrix} \overline{U}' \quad \text{or} \quad J_f(z_0)U = U \begin{pmatrix} \lambda & 0 \\ B & V \end{pmatrix}.$$

That is,

$$(3.4) \quad J_f(z_0)(z_0, \alpha_2, \dots, \alpha_n) = (z_0, \alpha_2, \dots, \alpha_n) \begin{pmatrix} \lambda & 0 \\ B & V \end{pmatrix}.$$

Hence,

$$(3.5) \quad J_f(z_0)(\alpha_2, \dots, \alpha_n) = (\alpha_2, \dots, \alpha_n)V.$$

Notice that $J_f(z_0)$ is a linear transformation on the $(n-1)$ -dimensional complex vector space $T_{z_0}^{(1,0)}(\partial B_{p_1, p_2})$. This, together with (3.5), shows that the all roots of the characteristic polynomial $\det(xI_{n-1} - V)$ of $J_f(z_0)$ are just μ_2, \dots, μ_n . If $\lambda \notin \{\mu_2, \dots, \mu_n\}$, then $\lambda, \mu_2, \dots, \mu_n$ are the all eigenvalues of the linear transformation $J_f(z_0)$ on \mathbb{C}^n .

This means that the claim holds. Suppose that $\lambda = \mu_{i_0}$, where μ_{i_0} is a root of order k of $\det(xI_{n-1} - V)$, and notice that $J_f(z_0)$ is also a linear transformation on \mathbb{C}^n . Then (3.4) and (3.5) imply that the characteristic polynomial of $J_f(z_0)$ is just

$$\det \left[xI_n - \begin{pmatrix} \lambda & 0 \\ B & V \end{pmatrix} \right] = (x - \lambda) \det(xI_{n-1} - V).$$

Thus, $\lambda = \mu_{i_0}$ is a root of order $(k + 1)$ of the characteristic polynomial of $J_f(z_0)$. Therefore, $\lambda, \mu_2, \dots, \mu_n$ are the all eigenvalues of the linear transformation $J_f(z_0)$ on \mathbb{C}^n .

Step 5. We claim that the inequalities in (i), (iii), and (v) are sharp, and we break the proof into two cases.

Case 1. $f_1(0) = 0$. Without loss of generality, we assume that the positive integer $m \geq 2$. Set the positive integer k such that $\frac{p_2}{p_1}k \geq m - 1$. Take

$$f(z) = (e^{-i(m-1)\theta} z_1^m, e^{-ik\theta} m^{\frac{1}{p_2}} z_1^k z_2, \dots, e^{-ik\theta} m^{\frac{1}{p_2}} z_1^k z_n)'.$$

Then for any $z \in B_{p_1, p_2}$, we have

$$\begin{aligned} |f_1(z)|^{p_1} + |f_2(z)|^{p_2} + \dots + |f_n(z)|^{p_2} &= |z_1|^{p_1 m} + m|z_1|^{p_2 k} (|z_2|^{p_2} + \dots + |z_n|^{p_2}) \\ &< |z_1|^{p_1 m} + m|z_1|^{p_2 k} (1 - |z_1|^{p_1}) \\ &= |z_1|^{p_1 m} + m|z_1|^{p_1 \frac{p_2}{p_1} k} (1 - |z_1|^{p_1}) \\ &= x^m + mx^{\frac{p_2}{p_1} k} (1 - x), \end{aligned}$$

where $x = |z_1|^{p_1} \in [0, 1)$. Notice that

$$\begin{aligned} x^m + mx^{\frac{p_2}{p_1} k} (1 - x) &\leq x^m + mx^{m-1} (1 - x) < x^m + (1 + x + \dots + x^{m-1})(1 - x) \\ &= x^m + (1 - x^m) = 1. \end{aligned}$$

Hence, $f: B_{p_1, p_2} \rightarrow B_{p_1, p_2}$ is a holomorphic mapping, and f is holomorphic at z_0 with $f(z_0) = z_0$. Moreover, we obtain

$$J_f(z_0) = \begin{pmatrix} m & 0 & \dots & 0 \\ 0 & m^{\frac{1}{p_2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m^{\frac{1}{p_2}} \end{pmatrix}.$$

This means that the inequalities are sharp in (iii) and (v).

Case 2. $f_1(0) \neq 0$. Take

$$f(z) = (e^{i\theta}(1 - r) + rz_1, r^{\frac{1}{p_2}} z_2, \dots, r^{\frac{1}{p_2}} z_n)',$$

where $r \in (0, 1)$. Then for each $z \in B_{p_1, p_2}$, we get

$$\begin{aligned} |f_1(z)|^{p_1} + |f_2(z)|^{p_2} + \dots + |f_n(z)|^{p_2} &= |e^{i\theta}(1 - r) + rz_1|^{p_1} + r(|z_2|^{p_2} + \dots + |z_n|^{p_2}) \\ &< ((1 - r) + r|z_1|)^{p_1} + r(1 - |z_1|^{p_1}) \\ &= (1 - r + rx)^{p_1} + r(1 - x^{p_1}), \end{aligned}$$

where $x = |z_1| \in [0, 1)$. Set

$$g(x) = (1 - r + rx)^{p_1} + r(1 - x^{p_1}), \quad x \in [0, 1].$$

Then $g'(x) = p_1 r(1 - r + rx)^{p_1 - 1} - p_1 r x^{p_1 - 1}$. Since $1 - r + rx \geq x$ and $p_1 - 1 \geq 0$, we know that $g'(x) \geq 0$ for all $x \in [0, 1]$. This shows that $g(x)$ is an increasing function on $[0, 1]$. Thus, $g(x) \leq g(1) = 1$ for all $x \in [0, 1)$. Hence, $f: B_{p_1, p_2} \rightarrow B_{p_1, p_2}$ is a holomorphic mapping. Moreover, f is holomorphic at z_0 with $f(z_0) = z_0$, $f_1'(0) \neq 0$ and

$$J_f(z_0) = \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r^{\frac{1}{p_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r^{\frac{1}{p_2}} \end{pmatrix}.$$

It follows that the inequalities are sharp in (iii) and (v).

Finally, we claim that the inequality is sharp in (i). Take

$$f(z) = \left(\frac{z_1 - r e^{i\theta}}{1 - r e^{-i\theta} z_1}, 0, \dots, 0 \right)',$$

where $r \in (0, 1)$. Then $f: B_{p_1, p_2} \rightarrow B_{p_1, p_2}$ is a holomorphic mapping, and f is holomorphic at z_0 with $f(z_0) = z_0$. Furthermore,

$$J_f(z_0) = \begin{pmatrix} \frac{1+r}{1-r} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This gives $\lambda = \frac{1+r}{1-r}$. By a straightforward calculation, we obtain

$$\frac{|1 - \overline{f_1(0)} e^{i\theta}|^2}{1 - |f_1(0)|^2} = \frac{(1+r)^2}{1-r^2} = \frac{1+r}{1-r} = \lambda,$$

which implies that the inequality is sharp in (i). The proof is complete. ■

Remark 3.3 From the proof of Theorem 3.2 it is clear that we need only to assume that the mapping f is C^1 up to the boundary of B_{p_1, p_2} near z_0 .

Remark 3.4 When $n = 1$ and $z_0 = 1$, Theorem 3.2 reduces to Theorem 1.1, which extends the boundary Schwarz lemma for holomorphic self-mappings of the unit disk to the egg domain B_{p_1, p_2} ($p_1 \geq 1, p_2 > 1$).

Remark 3.5 If $p_1 = p_2 = 2$, then Theorem 3.2 is just Theorem 1.5 at the special point $z_0 = (e^{i\theta}, 0, \dots, 0)' \in \partial B^n$.

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Department of Mathematics, Huzhou University, Huzhou, Zhejiang 313000, P.R. China
 e-mail: txm@hutc.zj.cn lts@ustc.edu.cn