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INTEGRAL CLOSURE OF SOME COORDINATE RINGS

DAVID J. SMITH

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1. Introduction

In this paper, some methods are developed for obtaining explicitly a basis for the integral closure of a class of coordinate rings of algebraic space curves.

The investigation of this problem was motivated by a need for examples of integrally closed rings with specified subrings with a view toward examining questions of unique factorization in them. The principal result, giving the elements to be adjoined to a ring of the form $k[x_1, \cdots, x_n]$ to obtain its integral closure, is limited to the rather special case of the coordinate ring of a space curve all of whose singularities are normal. But in numerous examples where the curve has non-normal singularities, the same method, which is essentially a modification of the method of locally quadratic transformations, also gives the integral closure.

2. Terminology and notation

An *algebraic curve* defined over a field k is a one-dimensional subvariety of an n-dimensional affine space with coordinates in a universal domain K whose prime ideal $\mathfrak{P}$ in $k[X_1, \cdots, X_n]$ is absolutely prime, that is, the extension of $\mathfrak{P}$ to $F[X_1, \cdots, X_n]$ is prime for every algebraic field extension F of k. The notions of *point*, *coordinate ring*, and *function field* of the curve are those of Zariski and Samuel (1960; page 22). In particular, a point will be a zero-dimensonal subvariety of the curve. A *place* of the curve is a place of the function field as defined by Chevalley (1951; page 2).

If the constant field k is algebraically closed, a place is defined equivalently as a class of k-isomorphisms of the function field into a power series field $k((t))$ in one variable. More precisely, if the function field K is generated over k by the elements $x_1, \cdots, x_n$, then a *representation* of a place of K/k is a k-isomorphism of K into $k((t))$. A representation is called *primitive* if not all the images of $x_1, \cdots, x_n$ are power series in the same power series of order greater than one. This definition is independent of the choice of generating elements of K over k. Two representations ϕ and ψ are *equivalent* if there is a substitution λ of order one in $k((t))$ such that $\phi = \lambda\psi$. A *place* is an equivalence class of primitive representations. If a system of generators $x_1, \cdots, x_n$ of K/k is fixed, then the n-tuple $(p_1(t), \cdots, p_n(t))$, where $p_i(t)$ is the image of x_i under a primitive representation, is a *parametrization* of the place, so that a place may be equivalently regarded as a class of equivalent parametrizations, or a *minimal quasi-branch* as defined by Semple and Kneebone (1959; page 67). The definition of the *order* of a place on a curve is that of Semple and Kneebone and a *linear* place is a place of order one.

The equivalence of these definitions of place is seen as follows. A place in the latter sense corresponds to the set of elements of K mapped by the isomorphism onto power series of positive order, and this set is the maximal ideal of a valuation ring in K, i.e. a place in the sense of Chevalley. Conversely, given a valuation ring D of K/k (k algebraically closed), one obtains an isomorphism into the power series field $k((t))$ as follows. Let $\xi \in D$. k is the residue class field of D, so for some $a \in k$, $v(\xi - a) > 0$, that is $\xi - a = \pi^r \alpha$ with α a unit in D, π a generator of the maximal ideal, and $r > 0$. Similarly, for some $b \in k$, $v(\alpha - b) > 0$, that is $\alpha - b = \pi^s \beta$, β a unit in D, $s > 0$. Thus

$$\xi = a + b\pi^r + \beta\pi^{r+s}.$$

Continuing in this way, we get a power series

$$\xi(t) = a + bt^r + ct^{r+s} + \cdots$$

and $\xi \to \xi(t)$ is the required isomorphism.

The *centre* of a place on a curve is the intersection of the place with the coordinate ring of the curve, provided that the ring of the place contains the coordinate ring. Otherwise, the place is said not to have finite centre on the curve, or to have its centre *at infinity*. The centre of every place of K/k with finite centre on the curve is a proper prime ideal of $k[x_1, \cdots, x_n]$, and conversely, every prime ideal is the centre of at least one place. Since $k[x_1, \cdots, x_n]$ has degree of transcendence one over k, every non-zero prime ideal is maximal. In case k is algebraically closed, an ideal of $k[x_1, \cdots, x_n]$ is a proper prime ideal if and only if it is of the form $(x_1 - a_1, \cdots, x_n - a_n)$ where $(a_1, \cdots, a_n)$ is a point of the curve, so that in this case the centre of a place on a curve can be regarded as a point on the curve.

A point of a curve is *simple* if its local ring is regular. If k is algebraically closed, this is equivalent to being the centre of exactly one place of the curve, this place being linear. A point which is not simple is *singular*. A singular point is *normal* if it is the centre of several places all of which are linear and all of which have distinct tangents, where the definition of the tangent of a place is that of Semple and Kneebone, (1959; page 182).

The integral closure of a coordinate ring of a curve Γ is the intersection of all places with finite centre on Γ, and the coordinate ring is integrally closed if and only if every point of Γ is simple.

3. Principal results

LEMMA 1. *Let k be an algebraically closed field, $k[x_1, \cdots, x_n]$ the coordinate ring of an algebraic space curve Γ, and suppose that the point P with coordinates $(a_1, \cdots, a_n)$ is on Γ. Then there is an affine coordinate transformation*

$$Y_i = \sum_j a_{ij}X_j + b_i, \qquad i = 1, \cdots, n$$

such that

(1) *P is at the origin,*

(2) *no tangent to Γ at P lies in the hyperplane $Y_1 = 0$, and*

(3) *no intersection of Γ with $Y_1 = 0$ other than P lies in any other hyperplane $Y_i = 0$.*

PROOF. Let V denote affine n-dimensional space over k, and V' the dual space of V as a vector space. If P is at the origin, the tangents to Γ at P are one-dimensional subspaces, and they correspond to $(n-1)$-dimensional subspaces of V'. Since k is infinite and no vector space over an infinite field is a finite union of proper subspaces, a basis of V' exists containing at least one element which is not in any of these $(n-1)$-dimensional subspaces. The equations of the hyperplanes in V corresponding to these basis elements of V' give a coordinate transformation satisfying (1) and (2).

Having transformed coordinates so that (1) and (2) hold, but continuing to denote the coordinate hyperplanes by $X_i = 0$ for simplicity, let M be the union of the $(n-1)$-dimensional subspaces of V' corresponding to the lines in V through the origin and the points of intersection of Γ with the hyperplane $X_1 = 0$. Let f_1 be a generator of the 1-dimensional subspace of V' corresponding to $X_1 = 0$ in V. Let f_2 be an element of V' not in M, f_3 an element of V' not in M or in the space spanned by f_1 and f_2. Continuing in this way, a basis $\{f_1, \cdots, f_n\}$ of V' is obtained. The equations of the hyperplanes in V corresponding to these basis elements of V' give a coordinate transformation satisfying (1), (2) and (3).

LEMMA 2. *Let k be an algebraically closed field, $k[x_1, \cdots, x_n]$ the coordinate ring of an algebraic space curve Γ of order e with a singular point at the origin. Assume that no line tangent to Γ at the origin lies in the hyperplane $X_1 = 0$, and that the intersections*

$$Q_i = (0, b_{i2}, \cdots, b_{in}); \qquad i = 1, \cdots, m$$

of Γ with the hyperplane $X_1 = 0$ not at the origin do not lie in any of the hyperplanes $X_i = 0$, i different from one. Then for $j = 2, \cdots, n$

$$z_j = (x_j/x_1) \prod_{i=1}^{m} (x_j - b_{ij})^e$$

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is integral over $k[x_1, \cdots, x_n]$. *Furthermore,* $k[x_1, \cdots, x_n, z_2, \cdots, z_n]$ *is the coordinate ring of a curve* Γ' *with the following properties:*

(1) *If* $(a_1, \cdots, a_n, b_2, \cdots, b_n)$ *is a point* Γ', *then the projection* $(a_1, \cdots, a_n)$ *is a point of* Γ.

(2) *A point of* Γ *different from the origin is the projection of one and only one point of* Γ'.

(3) *If* $P_1, \cdots, P_s$ *are the points of* Γ' *vhich project to the origin on* Γ, *then the sum of the orders of the places of* Γ' *centred at* $P_1, \cdots, P_s$ *is less than or equal to the sum of the orders of the places of* Γ *centred at the origin.*

(4) *If* $\mathfrak{P}_1$ *and* $\mathfrak{P}_2$ *are places of* $k(x_1, \cdots, x_n)/k$ *centred at the origin on* Γ *and having distinct tangents, then* $\mathfrak{P}_1$ *and* $\mathfrak{P}_2$ *have distinct centres on* Γ'.

PROOF. Let $\mathfrak{P}$ be any place of the function field $k(x_1, \cdots, x_n)$ of Γ with finite centre, and let v be the order function at $\mathfrak{P}$. Suppose the centre of $\mathfrak{P}$ on Γ is the point $P = (a_1, \cdots, a_n)$. Then $x_1 - a_1$ belongs to $\mathfrak{P}$ so that if a_1 is non-zero, x_1 does not belong to $\mathfrak{P}$. Hence, $v(x_1) = 0$, and

$$v(z_j) = v(x_j) - v(x_1) + e \sum_i v(x_j - b_{ij}) \geqq 0, \qquad j = 2, \cdots, n.$$

If $a_1 = 0$ and not all a_i are zero, then P is one of the points Q_i, and

$$v(x_j - b_{ij}) > 0; \qquad j = 2, \cdots, n.$$

Since x_1 has positive order at some places other than $\mathfrak{P}$ (such as those centred at the origin), $v(x_1)$ is strictly less than the sum of the orders of x_1 at all places with finite centre on $k[x_1, \cdots, x_n]$ which in turn is at most equal to the order e of Γ. Hence,

$$v(z_j) \geqq v(x_j) - v(x_1) + ev(x_j - b_{ij}) \geqq -v(x_1) + e \geqq 0; \qquad j = 2, \cdots, n.$$

If $a_i = 0$ for all i, then consider the parametrization $(p_1(t), \cdots, p_n(t))$ of the place $\mathfrak{P}$ where $p_i(t)$ belongs to the power series ring $k[[t]]$. If

$$p_j(t) = \sum_k p_{jk} t^k$$

has order s_j, then the tangent to $\mathfrak{P}$ has parametric equations

$$X_j = p_{js} t, \text{ where } s = \min\{s_1, \cdots, s_n\}.$$

Since no tangent line to Γ at the origin is contained in the hyperplane $X_1 = 0$, p_{1s} is different from zero, so that

$$s_1 \leqq s_j; \qquad j = 2, \cdots, n.$$

Since

$$v(x_j) = \text{ord}\,[p_j(t)],$$

$$v(x_1) \leqq v(x_j) \text{ and } v(z_j) \geqq v(x_j) - v(x_1) \geqq 0.$$

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Thus z_j has non-negative order at every place with finite centre on $k[x_1, \cdots, x_n]$, so that z_j is integral over $k[x_1, \cdots, x_n]$, $j = 2, \cdots, n$.

Let Γ' be a curve in $(2n-1)$-dimensional affine space having coordinate ring $k[x_1, \cdots, x_n, z_2, \cdots, z_n]$.

If $(a_1, \cdots, a_n, b_2, \cdots b_n)$ is a point of Γ', then the ideal

$$\mathfrak{p} = (x_1 - a_1, \cdots, x_n - a_n, z_2 - b_2, \cdots, z_n - b_n)$$

is a proper prime ideal of $k[x_1, \cdots, x_n, z_2, \cdots, z_n]$, and

$$\mathfrak{p} \cap k[x_1, \cdots, x_n] = (x_1 - a_1, \cdots, x_n - a_n),$$

so that $(a_1, \cdots, a_n)$ is apoint of Γ, and (1) is established.

Suppose $P = (a_1, \cdots, a_n)$ is a point of Γ. Then $(x_1 - a_1, \cdots, x_n - a_n)$ is a proper prime ideal of $k[x_1, \cdots, x_n]$, and is hence the contraction of a prime ideal $\mathfrak{p}$ of the integral extension $k[x_1, \cdots, x_n, z_2, \cdots, z_n]$. $\mathfrak{p}$ is of the form

$$\mathfrak{p} = (x_1 - a_1, \cdots, x_n - a_n, z_2 - b_2, \cdots, z_n - b_n).$$

and the point $P' = (a_1, \cdots, a_n, b_2, \cdots, b_n)$ is a point of Γ'. Suppose P is not at the origin. If a_1 is different from zero, then

$$z_j - b_j \equiv \left(\frac{a_j}{a_1}\right) \prod_i (a_j - b_{ij})^e - b_j \qquad (\text{Mod } \mathfrak{p}).$$

Hence

$$b_j = (a_j/a_1) \prod_i (a_j - b_{ij})^e.$$

If $a_1 = 0$, then P is one of the points Q_j. It has already been shown that $z_2, \cdots, z_n$ have positive value at all places centred at Q_j, so since the same is true of $z_2 - b_2, \cdots, z_n - b_n$, it follows that $b_2 = \cdots = b_n = 0$. Thus in either case, $b_2, \cdots, b_n$ are uniquely determined by $a_1, \cdots, a_n$, and (2) is established.

Γ and Γ' have the same function field $K = k(x_1, \cdots, x_n)$. A place of K/k has a representation as a k-isomorphism of K into a power series field $k((t))$, and such a k-isomorphism is completely determined by its action on the elements $x_1, \cdots, x_n$ of K. Hence, a place of K/k is centred at the origin on Γ if and only if it is centred at some point of Γ' lying above the origin on Γ. If $(p_1(t), \cdots, p_n(t))$ is a parametrization of a place of K/k centred at the origin on Γ, then its order on Γ is

$$\min \{\text{ord}[p_i(t)] \,|\, i = 1, \cdots, n\}.$$

Its parametrization on Γ' is

$$(p_1(t), \cdots, p_n(t), q_2(t), \cdots, q_n(t)).$$

where

$$q_j(t) = \frac{p_j(t)}{p_1(t)} \prod_i (p_j(t) - b_{ij})^e,$$

and its order on Γ' is

$$\min\{\text{ord}[p_i(t)], \text{ord}[q_j(t) - q_j(0)] \,|\, i = 1, \cdots, n;\ j = 2, \cdots, n\}$$

which is less than or equal to its order on Γ. Hence, (3) is established. Suppose

$$(p_{i1}(t), \cdots, p_{in}(t)), \qquad i = 1, 2$$

are parametrizations of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ respectively, where

$$p_{ij}(t) = \sum_k p_{ijk} t^k.$$

Let

$$r_i = \min\{\text{ord}[p_{ij}(t)] \,|\, j = 1, \cdots, n\}, \quad i = 1, 2.$$

Then the tangents to $\mathfrak{P}_1$ and $\mathfrak{P}_2$ as places of Γ are given parametrically by

$$X_j = p_{ijr_i} t, \qquad j = 1, \cdots, n, \qquad i = 1, 2,$$

and since no tangent to Γ at the origin lies in the hyperplane $X_1 = 0$,

$$p_{i1r_i} \neq 0, \qquad i = 1, 2.$$

The corresponding parametrizations of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ as places of Γ' are

$$(p_{i1}(t), \cdots, p_{in}(t), q_{i2}(t), \cdots, q_{in}(t)) \qquad i = 1, 2,$$

where

$$q_{ij}(t) = \frac{p_{ij}(t)}{p_{i1}(t)} \prod_k (p_{ij}(t) - b_{kj})^e; \qquad j = 2, \cdots, n.$$

$$q_{ij}(0) = \frac{p_{ijr_i}}{p_{i1r_i}} \prod_k (-b_{kj})^e: \qquad i = 1, 2.$$

If the centes of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ on Γ' coincide, then

$$q_{1j}(0) = q_{2j}(0), \qquad j = 2, \cdots, n.$$

Since none of the points $Q_i = (0, b_{i2}, \cdots, b_{in})$ lies on any hyperplane $X_i = 0$,

$$\prod_k (-b_{kj})^e \neq 0.$$

Hence,

$$\lambda p_{1jr_1} = p_{2jr_2}; \qquad j = 1, \cdots, n$$

where

$$\lambda = \frac{p_{21r_2}}{p_{11r_1}},$$

so the tangents to $\mathfrak{P}_1$ and $\mathfrak{P}_2$ as places of Γ coincide. Thus (4) is established and the proof is complete.

It is clear from the above proof that the choice of elements to be adjoined to form the desired integral extension may be modified by replacing e by any sufficiently large integer, and hence that the actual form of the elements to be adjoined may be simpler than those appearing in the statement of the Lemma.

If the origin is a normal s-fold singular point of an algebraic curve Γ defined over an algebraically closed field k, and the axes are suitably chosen, then the curve Γ' defined in the previous lemma has s distinct simple points projecting to the origin on Γ, and the other points of Γ and Γ' correspond in such a way that the set of places with centre at some point not at the origin on Γ is precisely the same as the set of places centred at the corresponding point on Γ'.

If all the singularities of Γ are normal, then all singularities of Γ' are normal, and the procedure described in the preceding results may be successively applied to obtain a finite sequence of curves, the last of which has no singular points. Its coordinate ring is thus integrally closed and integral over that of Γ. Hence it is the integral closure of the coordinate ring of Γ.

A simplification of the procedure outlined above is obtained by partially combining the successive steps as follows:–

THEOREM 1. *Let k be an algebraically closed field, $k[x_1, \cdots, x_n]$ the coordinate ring of an algebraic curve Γ all of whose singular points $P_1, \cdots, P_s$ are normal. For $i = 1, \cdots, s$, let*

$$Y_j = \sum_{k=1}^{n} t_{ijk} X_k + c_{ij}, \qquad j = 1, \cdots, n$$

be a coordinate transformation chosen so that P_i is at the origin, no tangent to Γ at P_i lies in the hyperplane $Y_1 = 0$, and no intersection

$$Q_{ik} = (0, b_{ik2}, \cdots, b_{ikn})$$

of Γ with $Y_1 = 0$ other than the origin lies on any other hyperplane $Y_i = 0$. For $i = 1, \cdots, s$ and $j = 1, \cdots, n$, define

$$x_{ij} = \sum_{k=1}^{n} t_{ijk} x_k + c_{ij}.$$

Then if

$$z_{ij} = \frac{x_{ij}}{x_{i1}} \prod_{k} (x_{ij} - b_{ikj})^e,$$

where e is the order of Γ, then $R = k[x_1, \cdots, x_n, z_{12}, \cdots, z_{sn}]$ is the integral closure of $k[x_1, \cdots, x_n]$.

PROOF. By Lemma 2, $z_{i2}, \cdots, z_{in}$ are integral over $k[x_{i1}, \cdots, x_{in}]$ which is equal to $k[x_1, \cdots, x_n]$. Hence R is integral over $k[x_1, \cdots, x_n]$. R is the coordinate ring of a curve all of whose singularities, if any, are normal, so to show that R is integrally closed, it suffices to show that no two distinct places of $k(x_1, \cdots, x_n)/k$ have the the same centre on R. If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are two places having the same centre on R, then they have the same centre on $k[x_1, \cdots, x_n]$, say $(x_1 - a_1, \cdots, x_n - a_n)$, where $(a_1, \cdots, a_n)$ is a point of Γ. If this point is simple, then it is the centre of only one place, so that $\mathfrak{P}_1 = \mathfrak{P}_2$. Otherwise, $(a_1, \cdots, a_n)$ is one of the points P_i. Since

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$$k[x_1, \cdots, x_n] = k[x_{i1}, \cdots, x_{in}],$$

and since $\mathfrak{P}_1$ and $\mathfrak{P}_2$ have the same centre on R, they have the same centre on $k[x_{i1}, \cdots, x_{in}, z_{i2}, \cdots, z_{in}]$ and hence, by Lemma 2, their tangents, as places of the transform of Γ by the equations

$$Y_j = \sum_k t_{ijk} X_k + c_{ij},$$

coincide. Since all singular points of Γ are normal, $\mathfrak{P}_1 = \mathfrak{P}_2$, and the proof is complete.

The following examples are applications of some of the preceding methods.

EXAMPLE 1. Let k be the field of complex numbers. Let $k[x, y]$ denote the coordinate ring of the curve Γ defined over k by

$$f(x, y) = (x^2 + y^2)^2 + 3x^2y - y^3.$$

Since f has subdegree three and degree four, the origin is a threefold point and all other points are simple. The tangents to Γ at the origin are the lines with equation $Y = 0$ and $Y = \pm\sqrt{3}X$, so that the origin is a normal singular point of Γ. (0, 1) is the only intersection of Γ with the axis $X = 0$ other than the origin. Hence, the integral closure of $k[x, y]$ is $k[x, y, z]$ where $z = y(y-1)^4/x$.

EXAMPLE 2. Let k be the field of complex numbers. Let $k[x, y]$ denote the coordinate ring of the curve Γ defined over k by

$$f(X, Y) = 2(Y^4 - 2Y^3 - 3Y^2) + (X^2 - 4)^2.$$

Γ has normal double points with non-vertical tangents at $(-2, 0)$ and $(2, 0)$ and all other points are simple. The coordinate transformations

$$X' = X + 2, \quad Y' = Y \text{ and } X' = X - 2, \quad Y' = Y,$$

locate these points respectively at the origin of the transformed systems. In both these coordinate systems, the intersections of Γ with $X' = 0$ other than the origin are the points $(0, -1)$ and $(0, 3)$. Hence, the integral closure of $k[x, y]$ is $k[x, y, z_1, z_2]$ where

$$z_1 = y(y-3)^4(y+1)^4/(x+2) \text{ and } z_2 = y(y-3)^4(y+1)^4/(x-2).$$

4. Auxiliary results and examples

In case the field of constants is not algebraically closed, the following results, the proofs of which are direct, are sometimes useful.

THEOREM 2. *Let k be an arbitrary field and $f(X, Y)$ an irreducible polynomial in $k[X, Y]$ of subdegree n and degree m.*

(1) *If $k[x, y] = k[X, Y]/(f)$, and $f(0, Y) = Y^n p(Y)$ where $p(Y)$ belongs to $k[Y]$, then $yp(y)/x$ is integral over $k[x, y]$.*

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(2) *If*

$$f(X, Y) = f_n(X, Y) + \cdots + f_m(X, Y)$$

where f_i *is homogeneous of degree* i, *then* $g(X, Y) = f_n(1, Y) + Xf_{n+1}(1, Y) + \cdots + X^{m-n}f_m(1, Y)$ *is irreducible.*

EXAMPLE 3. Let k be a field with characteristic different from two.

$$f(X, Y) = X^4 + X^2Y^2 - 2X^2Y - XY^2 + Y^2$$

is irreducible over k and defines a curve Γ_1 with a non-normal double point at the origin and no other intersections with the Y-axis. Denote its coordinate ring by $k[x, y]$. Then y/x is integral over $k[x, y]$. Now $k[x, y, y/x] = k[x, y/x]$ which is the coordinate ring of the curve Γ_2 defined by the polynomial

$$g(X, Y) = X^2 + X^2Y^2 - 2XY - XY^2 + Y^2.$$

$g(X, Y)$ is irreducible by Theorem 2. Γ_2 has a double point at the origin and no other intersections with the Y-axis, so that y/x^2 is integral over $k[x, y/x]$. $k[x, y/x, y/x^2] = k[x, y/x^2]$ which is the coordinate ring of the curve Γ_3 defined by

$$h(X, Y) = 1 + X^2Y^2 - 2Y - XY^2 + Y^2.$$

Again, $h(X, Y)$ is irreducible, but Γ_3 is free of singular points. Hence the integral closure of $k[x, y]$ is $k[x, y/x^2]$.

The next example illustrates that, although the first part of Theorem 2 appears in some respects to be a stronger form of Lemma 2, the conditions in Lemma 2 may not in fact be relaxed.

EXAMPLE 4. Let k be the field of complex numbers. The curve Γ defined by

$$f(X, Y) = X^2 - Y^3$$

fails to satisfy the conditions in Lemma 2 on orientation of axes, and the element y/x of its coordinate ring $k[x, y]$ is in fact not integral over $k[x, y]$. On the other hand, by Theorem 2 $z = y^2/x$ is integral over $k[x, y]$. Moreover, $k[x, y, z] = k[z]$ which, being a polynomial ring, is integrally closed and hence the integral closure of $k[x, y]$.

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Department of Mathematics
Auckland University
New Zealand