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# A NOTE ON $H^1$ MULTIPLIERS FOR LOCALLY COMPACT VILENKIN GROUPS

JAMES E. DALY AND KEITH L. PHILLIPS

ABSTRACT. Kitada and then Onneweer and Quek have investigated multiplier operators on Hardy spaces over locally compact Vilenkin groups. In this note, we provide an improvement to their results for the Hardy space $H^1$ and provide examples showing that our result applies to a significantly larger group of multipliers.

In this note, $G$ denotes a locally compact Abelian group containing a decreasing sequence of open compact subgroups $(G_n)_{n=-\infty}^{\infty}$ such that

(i) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$, and

(ii) $\sup_n\{\text{order}(G_n/G_{n+1})\} < \infty$.

In the case where $G$ is compact, we use the convention that $G_n = G$ if $n \leq 0$. Examples of these groups are the dyadic group on $[0,1)$, the $p$-adic numbers, and more generally, the additive and multiplicative groups of a local field.

Let $\Gamma$ denote the dual group of $G$ and $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}$. The Haar measures $\mu$ on $G$ and $\lambda$ on $\Gamma$ are chosen so that $\mu(G_0) = \lambda(\Gamma_0) = 1$ and consequently, $\mu(G_n) = \big(\lambda(\Gamma_n)\big)^{-1} := (m_n)^{-1}$ for each $n \in Z$. There is a norm on $G$ defined by $|x| = (m_n)^{-1}$ if $x \in G_n \setminus G_{n+1}$. The Fourier transform and inverse Fourier transform respectively are denoted by $\wedge$ and $\vee$, and satisfy

$$(\xi_{G_n})^{\wedge} = \big(\lambda(\Gamma_n)\big)^{-1}\xi_{\Gamma_n} \quad \text{and} \quad (\xi_{\Gamma_n})^{\vee} = \big(\lambda(G_n)\big)^{-1}\xi_{G_n},$$

where $\xi_A$ denotes the characteristic function of a set $A$. The structure of (atomic) Hardy spaces on $G$ has been well studied. See Kitada [4] or Chao-Janson [1] for complete details. A function $a\colon G \longrightarrow \mathbb{C}$ is a 1-*atom*, if for some $n \in Z$ and $x \in G$,

(i) support $(a) \subset I_n := x + G_n$,

(ii) $\|a\|_\infty \leq \big(\mu(I_n)\big)^{-1}$, and

(iii) $\int_G a(x)\,dx = 0$.

A function $f \in L^1(G)$ belongs to $H^1(G)$ if $f$ can be represented as $f = \sum_{i=1}^{\infty} \lambda_i a_i$, where each $a_i$ is a 1-atom, and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. The $H^1$ norm is $\|f\|_{H^1} = \inf(\sum_{i=1}^{\infty} |\lambda_i|)$ with the infimum taken over all such atomic decompositions of $f$. For a distribution $f$ we define the maximal function of $f$ by

$$Mf(x) = \sup_n \Big| f * \big(\mu(G_n)\big)^{-1}\xi_{G_n}(x)\Big|.$$

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The maximal function characterizes $H^1(G)$; that is, $f \in H^1(G)$ if and only if $Mf \in L^1(G)$ with $\|f\|_{H^1} \sim \|Mf\|_{L^1}$. A function $\phi \in L^\infty(\Gamma)$ is a (Fourier) multiplier for $H^1$ if there exists a constant $C > 0$ so that for all $f \in H^1 \cap L^2$,

$$\|(\phi f^\wedge)^\vee\|_{H^1} \le C\|f\|_{H^1}.$$

The corresponding multiplier operator is $T$, defined on $L^2$ by $Tf = (\phi f^\wedge)^\vee$, and the set of such multipliers is denoted by $\mathfrak{M}(H^1)$. For $\phi \in L^\infty(\Gamma)$ and $j \in Z$, set $\phi_j = \phi\xi_{\Gamma_j}$ and $\Delta_j(\phi) = \phi_{j+1} - \phi_j$. Note that $\phi = \sum \Delta_j(\phi)$ distributionally.

1. **Multiplier theorems.** In 1989 Onneweer and Quek [5] discussed the sharpness of Kitada's 1987 [4] multiplier theorem for $H^1$:

THEOREM 1. *If $\phi \in L^\infty(\Gamma)$ and*

$$\sum_{j=-\infty}^{\infty} \left\| \big(\Delta_j(\phi)\big)^\vee \right\|_{L^1} < \infty,$$

*then $\phi \in \mathfrak{M}(H^1)$.*

We prove an improvement of this result, but we will first need the following lemma.

LEMMA 2. *Let $T$ denote the multiplier operator with multiplier $\phi$ and $a$ be a 1-atom. Then $T(a) * \xi_{G_n} = \mu(G_n)\left|\sum_{j=-\infty}^{n-1} \big(\Delta_j(\phi)\big)^\vee * a\right|$.*

PROOF. The computations to prove this result are straightforward:

$$\begin{aligned} T(a) * \xi_{G_n} &= (\phi^\vee) * a * \xi_{G_n} \\ &= \Big( \sum_{j=-\infty}^{\infty} \big(\Delta_j(\phi)\big)^\vee * \xi_{G_n} \Big) * a. \end{aligned}$$

For the terms of the sum,

$$\begin{aligned} \big(\Delta_j(\phi)\big)^\vee * \xi_{G_n} &= \phi^\vee * \big(\lambda(\Gamma_{j+1})\xi_{G_{j+1}} - \lambda(\Gamma_j)\xi_{G_j}\big) * \xi_{G_n} \\ &= \phi^\vee * \big(\lambda(\Gamma_{j+1})\xi_{G_{j+1}} * \xi_{G_n} - \lambda(\Gamma_j)\xi_{G_j} * \xi_{G_n}\big). \end{aligned}$$

Using the fact $\xi_{G_k} * \xi_{G_m} = \mu(G_m)\xi_{G_k}$ for $k < m$

$$\big(\Delta_j(\phi)\big)^\vee * \xi_{G_n} = \begin{cases} 0 & \text{if } n \le j \\ \mu(G_n)\big(\Delta_j(\phi)\big)^\vee & \text{if } n > j. \end{cases}$$

Substituting this into the sum, we obtain the desired result. ■

THEOREM 3. *If $\phi \in L^\infty(\Gamma)$ and*

$$\sup_N \Big( \sum_{j=N+1}^{\infty} \int_{(G_N)^c} \left| \big(\Delta_j(\phi)\big)^\vee(x) \right| dx \Big) < \infty,$$

*then $\phi \in \mathfrak{M}(H^1)$.*

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PROOF. Let $a$ be a 1-atom and $T$ denote the multiplier operator with multiplier $\phi$. For $T$ to be bounded on $H^1$ it is sufficient to show that there exists a constant $B$ such that

$$\int_G |MT(a)(x)|\,dx \le B < \infty$$

for all 1-atoms $a$.

Due to the translation invariance of the multiplier operator $T$, we may assume that the support of $a$ is $G_N$ for some $N \in Z$. We have

$$\begin{aligned}\int_G |MT(a)(x)|\,dx &= \int_{G_N} |MT(a)(x)|\,dx + \int_{(G_N)^c} |MT(a)(x)|\,dx \\ &= (1) + (2).\end{aligned}$$

For integral (1), we use the usual $L^2$ argument:

$$\begin{aligned}\int_{G_N} |MT(a)(x)|\,dx &= \int_{G_N} |MT(a)(x)|\xi_{G_N}(x)\,dx \\ &\le \|MT(a)\|_{L^2}\|\xi_{G_N}\|_{L^2} \\ &\le C\|T(a)\|_{L^2}\|\xi_{G_N}\|_{L^2} \\ &\le C\|\phi\|_\infty\|a\|_{L^2}\big(\mu(G_N)\big)^{1/2} \\ &\le C\|\phi\|_\infty\big(\mu(G_N)\big)^{-1/2}\big(\mu(G_N)\big)^{1/2} \\ &= C\|\phi\|_\infty.\end{aligned}$$

For integral (2), using Lemma 2

$$\begin{aligned}\int_{(G_N)^c} |MT(a)(x)|\,dx &= \int_{(G_N)^c} \sup_n\Big|T(a) * \big(\mu(G_n)\big)^{-1}\xi_{G_n}(x)\Big|\,dx \\ &= \int_{(G_N)^c} \sup_n\Big|\sum_{j=-\infty}^{n-1} \big(\Delta_j(\phi)\big)^\vee * a(x)\Big|\,dx \\ &= \int_{(G_N)^c} \sup_n\Big|\sum_{j=N+1}^{n-1} \big(\Delta_j(\phi)\big)^\vee * a(x)\Big|\,dx\end{aligned}$$

with the last equality following from the fact that $\big(\Delta_j(\phi)\big)^\vee * a(x) = 0$ for $j \le N$ as the support of $a$ is contained in $G_N$. Continuing,

$$\begin{aligned}&\le \int_{(G_N)^c} \sum_{j=N+1}^{\infty} \Big|\big(\Delta_j(\phi)\big)^\vee * a(x)\Big|\,dx \\ &= \int_{(G_N)^c} \sum_{j=N+1}^{\infty} \Big|\int_G \big(\Delta_j(\phi)\big)^\vee(t)a(x-t)\,dt\Big|\,dx \\ &\le \sum_{j=N+1}^{\infty} \int_G \Big|\big(\Delta_j(\phi)\big)^\vee(t)\Big| \int_{(G_N)^c} |a(x-t)|\,dx\,dt.\end{aligned}$$

Each of the integrals over $G$ is split into one over $G_N$ and the other over $(G_N)^c$. For the integrals over $G_N$, $t \in G_N$ and $x \in (G_N)^c$ imply $x - t \in (G_N)^c$ and $a(x-t) = 0$. Thus

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$$\begin{aligned}\sum_{j=N+1}^{\infty}\int_G\left|\left(\Delta_j(\phi)\right)^{\vee}(t)\right|\int_{(G_N)^c}|a(x-t)|\,dx\,dt &= \sum_{j=N+1}^{\infty}\int_{(G_N)^c}\left|\left(\Delta_j(\phi)\right)^{\vee}(t)\right|\int_{(G_N)^c}|a(x-t)|\,dx\,dt\\ &\le \sum_{j=N+1}^{\infty}\int_{(G_N)^c}\left|\left(\Delta_j(\phi)\right)^{\vee}(t)\right|dt,\end{aligned}$$

the last inequality follows as $\|a\|_{L^1}\le 1$. Combining these estimates, we obtain

$$\begin{aligned}\int_G|MT(a)(x)|\,dx &= \int_{G_N}|MT(a)(x)|\,dx+\int_{(G_N)^c}|MT(a)(x)|\,dx\\ &\le C\|\phi\|_\infty+\sum_{j=N+1}^{\infty}\int_{(G_N)^c}\left|\left(\Delta_j(\phi)\right)^{\vee}(t)\right|dt.\end{aligned}$$

This is the desired result. ∎

2**. Comparison of multiplier theorems.** The sufficient condition of Theorem 3 appears in Kitada [4] in one of the computations, but not as a sufficient condition for a multiplier $\phi$. Onneweer and Quek in [5] also noted this sufficient condition in one of their proofs, but made use of the stronger condition $\sum_{j=-\infty}^{\infty}\left\|\left(\Delta_j(\phi)\right)^{\vee}\right\|_{L^1}<\infty$. For bounded $\phi$, this condition implies that $\phi$ satisfies the sufficiency condition of Theorem 3 as is easily seen by the following:

$$\begin{aligned}\sum_{j=N+1}^{\infty}\int_{(G_N)^c}\left|\left(\Delta_j(\phi)\right)^{\vee}(x)\right|dx &\le \sum_{j=N+1}^{\infty}\int_G\left|\left(\Delta_j(\phi)\right)^{\vee}(x)\right|dx\\ &\le \sum_{j=-\infty}^{\infty}\int_G\left|\left(\Delta_j(\phi)\right)^{\vee}(x)\right|dx\\ &= \sum_{j=-\infty}^{\infty}\left\|\left(\Delta_j(\phi)\right)^{\vee}\right\|_{L^1}.\end{aligned}$$

To see that Theorem 3 is strictly better, consider the dyadic group $D$ on $[0,1)$. For $n>0$ $G_n$ corresponds to $[0,2^{-n})$, $\Gamma$ is the Walsh functions $\{\omega_k\}_{k=0}^{\infty}$, and $\Gamma_{-j}=\{\omega_k: 0\le k<2^j\}$. Let $\{b_j\}$ be a bounded sequence and define $\phi$ by

$$\phi(n)=b_j \quad \text{for } 2^{j-1}\le n<2^j$$

(that is, $\omega_n\in\Gamma_{-j+1}\setminus\Gamma_{-j}$). The multiplying sequence $\{\phi(n)\}$ is constant on dyadic rings. For $j\ge 1$,

$$\left(\Delta_{-j}(\phi)\right)^{\vee}(x)=\sum_{n=2^{j-1}}^{2^j-1}\phi(n)w_n(x)=b_j\sum_{n=2^{j-1}}^{2^j-1}w_n(x).$$

It is well known that $\sum_{n=2^{j-1}}^{2^j-1}w_n(x)=2^{j-1}\{\xi_{[0,2^{-j})}(x)-\xi_{[2^{-j},2^{-j+1})}(x)\}\equiv h_j$. A calculation gives $\|h_j\|_{L^1}=1$ for all $j$. Thus

$$\sum_{j=0}^{\infty}\left\|\left(\Delta_{-j}(\phi)\right)^{\vee}\right\|_{L^1}=\sum_{j=1}^{\infty}\left\|\left(\Delta_{-j}(\phi)\right)^{\vee}\right\|_{L^1}=\sum_{j=1}^{\infty}|b_j|\,\|h_j\|_{L^1}=\sum_{j=1}^{\infty}|b_j|.$$

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So the multiplying sequence $\{\phi(n)\}$ must be in $l^1$ for the Onneweer-Quek condition to be satisfied. However this multiplying sequence $\{\phi(n)\}$ does satisfy the conditions of Theorem 3:

$$\sum_{j=N+1}^{\infty} \int_{(G_N)^c} \left| \big(\Delta_j(\phi)\big)^{\vee}(t) \right| dt = \sum_{j=N+1}^{\infty} b_j \int_{(G_N)^c} |h_j(t)|\, dt = 0$$

as $h_j$ is supported on $G_N = [0, 2^{-N})$ for $j > N$. Thus $T$ is bounded on $H^1$ and its operator norm depends only on $\|\phi\|_\infty$.

It is well known and is easily seen independently of Theorem 3 that multipliers constant on dyadic blocks are bounded on $H^1$. Using the Littlewood-Paley square function characterization of $H^1$, we verify again the boundedness of $T$ on $H^1$ with norm dependent upon only $\|\phi\|_\infty$ as follows:

$$\begin{aligned}
\|T(f)\|_{H^1} &= \int_0^1 \Big( \sum_{j=1}^{\infty} \Big| \sum_{n=2^{j-1}}^{2^j-1} \phi(n) \langle f, w_n \rangle w_n(x) \Big|^2 \Big)^{1/2} dx \\
&\leq \int_0^1 \Big( \sum_{j=1}^{\infty} |b_j| \Big( \sum_{n=2^{j-1}}^{2^j-1} \langle f, w_n \rangle w_n(x) \Big)^2 \Big)^{1/2} dx \\
&\leq \|b_j\|_\infty \int_0^1 \Big( \sum_{j=1}^{\infty} \Big( \sum_{n=2^{j-1}}^{2^j-1} \langle f, w_n \rangle w_n(x) \Big)^2 \Big)^{1/2} dx \\
&= C\|\phi\|_\infty \|f\|_{H^1}.
\end{aligned}$$

In [2] we use a dyadic version of Theorem 3 to prove a conjecture of Simon [6] concerning the characterization of $H^1$ on the dyadic group $[0, 1)$ by certain square functions. Let $W_j(f)$ and $\sigma_j(f)$ denote the $j$-th partial sum and Cesaro sum of the Walsh series of $f$, respectively. Verification that the square function

$$S(f) = \Big( \sum_{n=1}^{\infty} \left| W_{2^n}(f) - \sigma_{2^n}(f) \right|^2 \Big)^{1/2}$$

gives an equivalent norm on $H^1$ $\big(\|f\|_{H^1} \sim \|S(f)\|_{L^1}\big)$ is equivalent to the verification of the boundedness on $H^1$ of the multiplier operators corresponding to the sequences $\phi$ and $\phi^{-1}$ where

$$\phi(n) = n/2^j \quad \text{for } 2^{j-1} \leq n < 2^j.$$

These two sequences are shown to satisfy the conditions of Theorem 3 while they do not satisfy the conditions of Theorem 1. Our attempt to settle the Simon conjecture and related square function issues led us to improve multiplier theorems of this type.

There is an important subclass of multipliers $\phi$ for which the conditions similar to those discussed here have been explored in depth. These are the homogeneous multipliers that extend the concept of a Calderon-Zygmund singular integral operator from the Euclidean setting to the 0-dimensional one. For a $p$-adic field or $p$-series field, the multiplier $\phi$ is said to be homogeneous of degree 0 if $\phi(px) = \phi(x)$. Due to the homogeneity of $\phi$ we need only verify the condition of Theorem 3 for $N = 0$. Without loss of generality, assume that $\phi$ satisfies $\int_{|t|=1} \phi(t)\,dt = 0$. The smoothness condition of Theorem 3 for $\phi$ is then the finiteness of

$$\sum_{j=1}^{\infty} \int_{|t|>1} \left| \left(\Delta_j(\phi)\right)^{\vee}(t) \right| dt,$$

and this is equivalent to the smoothness condition of Theorem 6.4 for $p = 1$ of our work in [3]. In fact the condition stated above appears in this form in the final lines of the proof of 6.4 of [3]. In the notation of [3] the space $LA_1^+$ is exactly those homogeneous multipliers for which this expression is finite. Thus for the Hardy space $H^1$, Theorem 3 generalizes this multiplier result for homogeneous degree zero multipliers on local fields to arbitrary multipliers for a general locally compact Vilenkin group.

In [7] W. S. Young obtains a Marcinkiewicz Multiplier Theorem for $L^r$, $1 < r < \infty$, for Vilenkin groups with bounded order. Hardy spaces are not considered there. An interesting part of the Young construction is that the differences considered for the Marcinkiewicz type result are over dyadic blocks, even though the Vilenkin group has no algebraic dyadic structure. For example, for the 3-adic field it follows that control over differences $|\phi(k+1) - \phi(k)|$ summed over the dyadic blocks gives a good multiplier theorem even though the underlying algebraic structure suggests using the 3-adic blocks.

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*Department of Mathematics*
*University of Colorado*
*Colorado Springs, CO 80907*
*USA*
*email: jedaly@math.uccs.edu*
*keith@pyramid.uccs.edu*

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