# COMPACT HANKEL OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES

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**Abstract.** We prove the compactness of certain Hankel operators on weighted Bergman spaces of harmonic functions on the unit ball in $\mathbf{R}^n$.

**1. Introduction.** We denote the unit ball in $\mathbf{R}^n$ by $B_n$. Let $w$ be a non-negative integrable function on the interval $[0, 1)$, henceforth called a *weight function*, and consider the *weighted Bergman space* $b^2_w(B_n)$ of harmonic functions $u$ on $B_n$ for which

$$\|u\|_w = \left(\int_{B_n} |u(x)|^2 w(|x|)\, dV(x)\right)^{1/2} < \infty,$$

where $V$ denotes the usual Lebesgue volume measure. We shall show that under mild conditions on the weight function $w$ the space $b^2_w(B_n)$ is a closed linear subspace of $L^2_w(B_n)$, the space of all square-integrable functions on $B_n$ with respect to the measure $w(|x|)\, dV(x)$, so that there exists an orthogonal projection $Q_w$ of $L^2_w(B_n)$ onto $b^2_w(B_n)$. For a function $f \in L^\infty(B_n)$ define the *Hankel* operator $H_f : b^2_w(B_n) \to L^2_w(B_n)$ by

$$H_f u = (I - Q_w)(fu), \qquad u \in b^2_w(B_n).$$

The operator $H_f$ is clearly bounded on $b^2_w(B_n)$ with $\|H_f\| \leq \|f\|_\infty$. In this paper we prove that for every $f$ continuous on the closed unit ball $\bar{B}_n$ the operator $H_f$ is compact on $b^2_w(B_n)$, extending a recent result of M. Jovović [4] to the setting of weighted harmonic Bergman spaces.

In Section 2 we give the preliminaries for the paper. In Section 3 we discuss weighted Bergman spaces and the Bergman projection. In Section 4 we discuss Hankel operators and prove the above mentioned result. We furthermore show that these Hankel operators are in general not Hilbert–Schmidt.

**2. Preliminaries.** We recall that a twice-continuously differentiable function $u$ on $B_n$ is *harmonic* on $B_n$ if $\Delta u \equiv 0$, where $\Delta = D_1^2 + \ldots + D_n^2$ and $D_j$ denotes the partial derivative with respect to the $j$-th coordinate. A polynomial on $\mathbf{R}^n$ is *homogeneous of degree m* (or *m-homogeneous*) if it is a finite linear combination of monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are nonnegative integers such that $\alpha_1 + \ldots + \alpha_n = m$. It is easy to show that a polynomial $p$ on $\mathbf{R}^n$ is homogeneous of degree $m$ if and only if $x \cdot \nabla p(x) = mp(x)$ for all $x \in \mathbf{R}^n$, where $\nabla$ denotes the gradient. Every harmonic function $u$ on $B_n$ can be decomposed as $u = \sum_{k=0}^{\infty} u_k$, where each $u_k$ is a harmonic homogeneous polynomial of degree $k$, and the convergence is uniform on compact subsets of $B_n$. Denote the unit sphere in $\mathbf{R}^n$ by $S_n$. The space $\mathcal{H}_k(S_n)$ of restrictions to $S_n$ of harmonic homogeneous

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polynomials of degree $k$, the so-called *spherical harmonics* of degree $k$, is a (finite-dimensional) Hilbert space with respect to the usual inner product on $L^2(S_n, d\sigma)$, where $\sigma$ denotes the normalized surface-area measure on $S_n$. For each $\eta \in S_n$ the linear functional $p \mapsto p(\eta)$ on the space $\mathcal{H}_k(S_n)$ is uniquely represented by a harmonic $k$-homogeneous polynomial $Z_k(\cdot, \eta)$, called the *zonal harmonic of degree $k$ at $\eta$*. Extending $Z_k$ to a function on $\mathbf{R}^n \times \mathbf{R}^n$ by setting $Z_k(x, y) = |y|^k Z_k(x, y/|y|)$, and using the fact that each zonal harmonic $Z_k(\cdot, \eta)$ is real valued (see pages 78–79 in [**1**]) we have

$$\int_{S_n} p(\zeta) Z_k(\zeta, y)\, d\sigma(\zeta) = p(y), \tag{2.1}$$

for every harmonic $k$-homogeneous polynomial $p$. Denoting the dimension of $\mathcal{H}_k(S_n)$ by $h_k$, it is easily seen that $Z_k(\eta, \eta) = h_k$, for all $\eta \in S_n$, and thus $Z_k(y, y) = |y|^{2k} h_k$, for all $y \in \mathbf{R}^n$.

Spherical harmonics of distinct degrees are orthogonal; that is,

$$\int_{S_n} p\bar{q}\, d\sigma = 0$$

if $p$ and $q$ are harmonic homogeneous polynomials of distinct degree.

In the sequel the following theorem will play an important role.

THEOREM 2.2. (*Spherical Decomposition Theorem.*) *If $p$ is a homogeneous polynomial of degree $m$, then for each $k = 0, 1, \ldots, [m/2]$ there exist a harmonic homogeneous polynomial $p_{m-2k}$ of degree $m - 2k$, such that*

$$p(x) = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}(x).$$

A constructive proof of the above theorem has recently been given in [**3**]. We observe that another constructive proof may be given as follows. It is elementary to show that for a harmonic $j$-homogeneous polynomial $q$ we have

$$\Delta[|x|^{2i} q] = 2i(n + 2j + 2i - 2)\, |x|^{2i-2} q. \tag{2.3}$$

Assuming that $\sum_{k=0}^{[m/2]-1} |x|^{2k} q_{m-2k-2}$ is the spherical decomposition of $\Delta p$, it follows with the help of (2.3) that

$$\Delta\left[\sum_{k=1}^{[m/2]} |x|^{2k} \frac{q_{m-2k}}{2k(n+2m-2k-2)}\right] = \sum_{k=1}^{[m/2]} |x|^{2k-2} q_{m-2k} = \sum_{k=0}^{[m/2]-1} |x|^{2k} q_{m-2k-2} = \Delta p,$$

so that

$$p_m = p - \sum_{k=1}^{[m/2]} |x|^{2k} \frac{q_{m-2k}}{2k(n+2m-2k-2)}$$

is a harmonic $m$-homogeneous polynomial, and thus $p = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}$ is the spherical

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decomposition of $p$, where $p_{m-2k} = q_{m-2k}/(2k(n+2m-2k-2))$ for $k \geq 1$. We shall use this idea in Section 4 to find explicit formulae for the norms of the Hankel operators associated with the coordinate functions.

**3. Weighted harmonic Bergman spaces.** For a weight function $w$ we introduce the moments

$$\hat{w}(k) = \int_{B_n} |x|^k w(|x|)\, dV(x), \qquad (k = 0, 1, \ldots).$$

We shall assume that $\hat{w}(k) > 0$, for all $k = 0, 1, \ldots$. If $p$ and $q$ are homogeneous harmonic polynomials of degrees $k$ and $l$ respectively then, integrating in polar coordinates, it is easily seen that

$$\langle p, q\rangle_w = \begin{cases} \hat{w}(2k)\displaystyle\int_{S_n} p\bar{q}\, d\sigma, & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

If $u \in b^2_w(B_n)$ has decomposition $u = \sum_{k=0}^{\infty} u_k$, where each $u_k$ is an harmonic $k$-homogeneous polynomial, then it follows from (2.1) and (3.1) that

$$u_k(y) = \frac{1}{\hat{w}(2k)} \langle u_k, Z_k(\cdot, y)\rangle_w.$$

In particular,

$$\|Z_k(\cdot, y)\|^2_w = \langle Z_k(\cdot, y), Z_k(\cdot, y)\rangle_w = \hat{w}(2k) Z_k(y, y) = \hat{w}(2k) h_k |y|^{2k}.$$

Applying the Cauchy-Schwarz inequality we obtain

$$|u_k(y)| \leq (1/\hat{w}(2k))\, \|u_k\|_w\, \|Z_k(\cdot, y)\|_w,$$

and it follows that

$$\begin{aligned} |u(y)| &\leq \sum_{k=0}^{\infty} \frac{1}{\hat{w}(2k)} \|u_k\|_w\, \|Z_k(\cdot, y)\|_w \\ &\leq \left(\sum_{k=0}^{\infty} \|u_k\|^2_w\right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k}\right)^{1/2}. \end{aligned}$$

We conclude that

$$|u(y)| \leq \|u\|_w \left(\sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k}\right)^{1/2}. \tag{3.2}$$

The numbers $h_k$ can be expressed in terms of binomial coefficients (see page 82 or 92 in [**1**]), and it is easily shown that $h_k \approx k^{n-2}$ as $k \to \infty$. The series $\sum_{k=0}^{\infty} (h_k/\hat{w}(2k))\, |y|^{2k}$ has radius of convergence equal to 1, and thus converges uniformly for $|y| \leq r < 1$, for each $0 < r < 1$, if

$$\limsup_{k \to \infty} 1/\sqrt[2k]{\hat{w}(2k)} = 1. \tag{3.3}$$

It follows from (3.2) that $b^2_w(B_n)$ is a closed subspace of $L^2_w(B_n)$ if the weight function

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satisfies (3.3). Using Exercise 3.4 of [5] it is easily shown that condition (3.3) is equivalent to the requirement that, for all $0<\delta<1$, the set $\{r\in(\delta,1):w(r)>0\}$ has positive measure. In the sequel we assume that this condition is satisfied, so that $b^2_w(B_n)$ is a closed linear subspace of $L^2_w(B_n)$.

Furthermore, by uniform convergence and orthogonality of homogeneous harmonic polynomials of distinct degree, for each $0<r<1$ we have

$$\int_{S_n}|u(r\zeta)|^2\,d\sigma(\zeta)=\sum_{k=0}^{\infty}\int_{S_n}|u_k(r\zeta)|^2\,d\sigma(\zeta),$$

and integrating in polar coordinates we obtain

$$\|u\|_w^2=\sum_{k=0}^{\infty}\|u_k\|_w^2. \tag{3.4}$$

Applying formula (3.4) to the function $u-\sum_{k=0}^{m}u_k=\sum_{k=m+1}^{\infty}u_k$ we obtain

$$\left\|u-\sum_{k=0}^{m}u_k\right\|_w^2=\sum_{k=m+1}^{\infty}\|u_k\|_w^2.$$

Thus $\sum_{k=0}^{m}u_k\to u$ in $b^2_w(B_n)$ as $m\to\infty$. Hence the harmonic polynomials are dense in $b^2_w(B_n)$.

Also, if $p$ and $q$ are harmonic homogeneous polynomials of degrees $k$ and $l$, respectively, then

$$\begin{aligned}\langle|x|^{2j}p,q\rangle_w&=nV(B)\int_0^1 r^{n+2k+2j-1}w(r)\,dr\int_{S_n}p\bar{q}\,d\sigma\\&=\hat{w}(2k+2j)\int_{S_n}p\bar{q}\,d\sigma,\end{aligned}$$

and thus

$$\langle|x|^{2j}p,q\rangle_w=\frac{\hat{w}(2k+2j)}{\hat{w}(2k)}\langle p,q\rangle_w. \tag{3.5}$$

It follows from (3.5) and the fact that the harmonic polynomials are dense in $b^2_w(B_n)$ that

$$Q_w[|x|^{2j}p]=\frac{\hat{w}(2k+2j)}{\hat{w}(2k)}p, \tag{3.6}$$

for every harmonic homogeneous polynomial $p$ of degree $k$.

The following result shows that the Bergman projection of a polynomial is a harmonic polynomial of degree less than or equal to that of the original polynomial.

THEOREM 3.7. *If an $m$-homogeneous polynomial $p$ has spherical decomposition given by* $p(x)=\sum_{k=0}^{[m/2]}|x|^{2k}p_{m-2k}(x)$, *then*

$$Q_w[p]=\sum_{k=0}^{[m/2]}\frac{\hat{w}(2m-2k)}{\hat{w}(2m-4k)}p_{m-2k}.$$

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*Proof.* If $p = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}$ is the spherical decomposition of $p$, then by linearity and (3.6)

$$Q_w[p] = \sum_{k=0}^{[m/2]} Q_w[|x|^{2k} p_{m-2k}] = \sum_{k=0}^{[m/2]} \frac{\hat{w}(2m-2k)}{\hat{w}(2m-4k)} p_{m-2k},$$

proving the result. □

COROLLARY 3.8. *Let $w(r) = (1-r^2)^\lambda$, where $-1 < \lambda < \infty$. If an $m$-homogeneous polynomial $p$ has spherical decomposition given by $p(x) = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}(x)$, then the projection $Q_\lambda[p]$ of $p$ onto $b^2_w(B_n)$ is given by*

$$Q_\lambda[p] = \sum_{k=0}^{[m/2]} \prod_{j=1}^{k} \frac{n+2(m-2k)+2j-2}{n+2(m-2k)+2j+2\lambda} p_{m-2k}.$$

*Proof.* An elementary calculation shows that

$$\hat{w}(2j) = \frac{n}{2} V(B_n) \frac{\Gamma\left(\frac{n}{2}+j\right)\Gamma(\lambda+1)}{\Gamma\left(j+\frac{n}{2}+\lambda+1\right)},$$

and thus

$$\hat{w}(2j) = \frac{n+2j-2}{n+2j+2\lambda} \hat{w}(2j-2), \tag{3.9}$$

for $j \geq 1$. This implies that

$$\frac{\hat{w}(2m-2k)}{\hat{w}(2m-4k)} = \prod_{j=1}^{k} \frac{\hat{w}(2m-4k+2j)}{\hat{w}(2m-4k+2j-2)} = \prod_{j=1}^{k} \frac{n+2(m-2k)+2j-2}{n+2(m-2k)+2j+2\lambda},$$

and the stated result follows from the above theorem. □

REMARKS. 1. Note that as $\lambda \to -1^+$, $Q_\lambda[p]$ converges to the Poisson integral of $p$: $\sum_{k=0}^{[m/2]} p_{m-2k}$.

2. If $\lambda = 0$, then

$$Q_0[p] = \sum_{k=0}^{[m/2]} \frac{n+2m-4k}{n+2m-2k} p_{m-2k},$$

as in [**3**].

**4. Hankel operators.** Let $w$ be a weight function satisfying condition (3.3). We shall consider the Hankel operator $H_{x_1}$ on $b^2_w(B_n)$. Let $p$ be a harmonic $m$-homogeneous polynomial on $\mathbf{R}^n$, where $m \geq 1$. Then $\Delta(x_1 p) = 2D_1 p(x)$. Since $x_1 p$ is homogeneous of degree $m+1$, it follows that $x_1 p$ has spherical decomposition given by

$$x_1 p = p_{m+1} + |x|^2 p_{m-1},$$

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with

$$p_{m-1}(x)=\frac{1}{n+2m-2}D_1p(x),\quad \text{and}\quad p_{m+1}(x)=x_1p(x)-|x|^2p_{m-1}(x).$$

Consequently

$$\begin{aligned}Q_w[x_1p]&=p_{m+1}+\frac{\hat{w}(2m)}{\hat{w}(2m-2)}p_{m-1}\\&=x_1p-|x|^2\frac{1}{n+2m-2}D_1p+\frac{\hat{w}(2m)}{(n+2m-2)\hat{w}(2m-2)}D_1p.\end{aligned}$$

Hence

$$H_{x_1}p=\frac{1}{n+2m-2}\left\{|x|^2D_1p-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}D_1p\right\}. \tag{4.1}$$

If $q$ is a harmonic homogeneous polynomial of degree $k$, then

$$\begin{aligned}\langle H_{x_1}p,H_{x_1}q\rangle_w&=\langle H_{x_1}p,x_1q\rangle_w\\&=\frac{1}{n+2m-2}\left\{\langle|x|^2D_1p,x_1q\rangle_w-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\langle D_1p,x_1q\rangle_w\right\}\\&=\frac{1}{n+2m-2}\left\{\langle x_1D_1p,|x|^2q\rangle_w-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\langle x_1D_1p,q\rangle_w\right\}.\end{aligned}$$

Similar formulae hold for $\langle H_{x_j}p,H_{x_j}q\rangle_w$, $(j=2,\ldots,n)$. Adding these formulae, and making use of $\sum_{j=1}^{n}x_jD_jp=mp$, we obtain

$$\sum_{j=1}^{n}\langle H_{x_j}p,H_{x_j}q\rangle_w=\frac{m}{n+2m-2}\left\{\langle p,|x|^2q\rangle_w-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\langle p,q\rangle_w\right\}.$$

It follows that

$$\begin{aligned}\sum_{j=1}^{n}\langle H_{x_j}p,H_{x_j}q\rangle_w&=\frac{m}{n+2m-2}\left\{\langle|x|^2p,q\rangle_w-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\langle p,q\rangle_w\right\}\\&=\frac{m}{n+2m-2}\left\{\frac{\hat{w}(2m+2)}{\hat{w}(2m)}\langle p,q\rangle_w-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\langle p,q\rangle_w\right\}\\&=\frac{m}{n+2m-2}\left\{\frac{\hat{w}(2m+2)}{\hat{w}(2m)}-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\right\}\langle p,q\rangle_w.\end{aligned}$$

It is easy to prove that the operators $H_{x_1},\ldots,H_{x_n}$ are unitarily equivalent on $b^2_w(B_n)$. In fact, if $1<j\le n$ and $U_j$ is the mapping defined on $L^2_w(B_n)$ by $(U_jg)(x)=g(\tilde{x})$, where $\tilde{x}$ is the vector obtained from $x$ by interchanging its first and $j$th coordinate, then $U_j$ is a unitary operator on $L^2_w(B_n)$ mapping $b^2_w(B_n)$ into itself, and $H_{x_j}U_jg=U_jH_{x_1}g$, for all $g\in b^2_w(B_n)$ (which is easily verified by using (4.1) and the analogous formula for $H_{x_j}$). In particular, we have

$$\|H_{x_1}p\|^2_w=\frac{m}{n(n+2m-2)}\left(\frac{\hat{w}(2m+2)}{\hat{w}(2m)}-\frac{\hat{w}(2m)}{\hat{w}(2m-2)}\right)\|p\|^2_w, \tag{4.2}$$

for every harmonic $m$-homogeneous polynomial $p$ with $m \geq 1$.

Note that (4.2) implies that $\hat{w}(2m+2)/\hat{w}(2m) \geq \hat{w}(2m)/\hat{w}(2m-2)$, which can also be verified directly using the Cauchy-Schwarz inequality: also $\hat{w}(2m)^2 \leq \hat{w}(2m-2)\hat{w}(2m+2)$. It follows from (3.3) that $\lim_{m\to\infty} \hat{w}(2m+2)/\hat{w}(2m) = 1$. That $H_{x_1}$ is compact on $b^2_w(B)$ is proved as follows. Write $\mathscr{V}_k$ for the space of all harmonic polynomials of degree at most $k$. Let $S_k$ denote the operators defined on $b^2_w(B_n)$ such that $S_k p = H_{x_1} p$ if $p \in \mathscr{V}_k$ and $S_k p = 0$ if $p \in b^2_w(B) \ominus \mathscr{V}_m$. We shall estimate $\|H_{x_1} - S_k\|$. Write $u = \sum_{m=0}^{\infty} u_m$, where each $u_m$ is a harmonic $m$-homogeneous polynomial. Then, using (4.2), Cauchy-Schwarz and (3.4), we have

$$\begin{aligned}
\|(H_{x_1} - S_k)u\|_w &\leq \sum_{m=k+1}^{\infty} \|H_{x_1} u_m\|_w \\
&\leq \sum_{m=k+1}^{\infty} \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \|u_m\|_w \\
&\leq \frac{1}{2} \left\{ \sum_{m=k+1}^{\infty} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \left\{ \sum_{m=k+1}^{\infty} \|u_m\|_w^2 \right\}^{1/2} \\
&\leq \frac{1}{2} \left\{ \sum_{m=k+1}^{\infty} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \|u\|_w,
\end{aligned}$$

Hence

$$\begin{aligned}
\|H_{x_1} - S_k\| &\leq \frac{1}{2} \left\{ \sum_{m=k+1}^{\infty} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \\
&\leq (1-\rho_k)^{1/2},
\end{aligned}$$

where $\rho_k = \hat{w}(2k+2)/\hat{w}(2k)$, and it follows that $S_k \to H_{x_1}$ as $k \to \infty$. Since each of the $S_k$ is of finite rank, the operator $H_{x_1}$ must be compact on $b^2_w(B_n)$. In fact, we have the following result.

THEOREM 4.3. *Let $w$ be a weight function satisfying* (3.3). *Then, for every $f$ in $C(\bar{B}_n)$, the Hankel operator $H_f$ is compact on $b^2_w(B_n)$.*

*Proof.* That $\mathscr{A} = \{f \in C(\bar{B}_n) : H_f \text{ is compact on } b^2_w(B_n)\}$ is a closed algebra can be proved by the same argument as given in [**2**]. We have just shown that $H_{x_1}$ is compact on $b^2_w(B_n)$ and, since each of the operators $H_{x_j}$ is unitarily equivalent to $H_{x_1}$, we conclude that $x_j \in \mathscr{A}$, for each $j$. This implies that $\mathscr{A}$ contains all polynomials and by the Stone-Weierstrass Theorem $\mathscr{A} = C(\bar{B}_n)$. □

It is interesting to note that the Hankel operator $H_{x_1}$ is in general not Hilbert-Schmidt. In fact, we have the following result, similar to the situation on the weighted Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. (See [**6**].) It shows that for $n > 2$ the Hankel operator $H_{x_1}$ is not Hilbert-Schmidt on $b^2_w(B_n)$ for the indicated weight functions $w$.

THEOREM 4.4. *Let $w(r) = (1-r^2)^\lambda$, where $-1 < \lambda < \infty$. Then $H_{x_1}$ does not belong to the Schatten $\gamma$-class of $b^2_w(B_n)$ if $\gamma \leq n-1$.*

*Proof.* For $2 \le \gamma < \infty$ we have the inequality

$$\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \ge \langle H_{x_1}^* H_{x_1} p, p \rangle_w^{\gamma/2},$$

for every $p \in b_w^2(B_n)$ of unit norm (by Proposition 6.3.3 in [7]), and it follows from (4.2) that

$$\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \ge \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{\gamma/2},$$

for every $p \in b_w^2(B_n)$ of unit norm. Summing over an orthonormal set $h_m$ of $m$-homogeneous harmonic polynomials, and subsequently summing over all $m \ge 1$ we obtain

$$\begin{aligned} \|H_{x_1}\|_\gamma^\gamma &= \operatorname{trace}((H_{x_1}^* H_{x_1})^{\gamma/2}) \\ &\ge \sum_{m=1}^{\infty} \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{\gamma/2} h_m. \end{aligned}$$

Using (3.9) we have

$$\|H_{x_1}\|_\gamma^\gamma \ge \sum_{m=1}^{\infty} \left\{ \frac{4(\lambda+1)m}{n(n+2m-2)(n+2m+2\lambda+2)(n+2m+2\lambda)} \right\}^{\gamma/2} h_m.$$

Since $h_m \approx m^{n-2}$, the assumption that $H_{x_1}$ belongs to the Schatten $\gamma$-class, implies that $\sum_{m=1}^{\infty} m^{n-2-\gamma} < \infty$, and thus $\gamma > n-1$. □

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