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TRANSCENDENCE MEASURES BY A METHOD OF MAHLER

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Abstract

Suppose that $f(z)$ is a function of one complex variable satisfying

$$f(z) = a(z)f(z^\rho) + b(z),$$

where ρ is an integer larger than 1 and $a(z)$ and $b(z)$ are rational functions. We consider f evaluated at the algebraic point α and develop a transcendence measure for $f(\alpha)$ under suitable conditions on f and α.

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1. Introduction

Let T: $\mathbf{C} \to \mathbf{C}$ be the transformation defined by $Tz = z^\rho$, where ρ is an integer greater than 1. Suppose that $f(z)$ is a non-rational function of one complex variable which is regular at the origin and which satisfies the functional equation

$$(1) \qquad f(z) = a(z)f(Tz) + b(z),$$

where $a(z)$ and $b(z)$ are rational functions. Further suppose that the coefficients of $f(z)$ in its Taylor series expansion at 0 are algebraic numbers. (Examples of such functions include $f(z) = \prod_{k=0}^{\infty}(1 - T^k z)$ and $f(z) = \sum_{k=0}^{\infty} T^k z$. When $Tz = z^2$, the latter of these is the so-called Fredholm series.)

By a result of Mahler (1929), if α is an algebraic number, $0 < |\alpha| < 1$, for which $f(\alpha)$ is defined, and if $T^k\alpha$ is neither a pole of $b(z)$ nor a zero of $a(z)$ for any k ($k = 0, 1, 2, \ldots$), then $f(\alpha)$ is a transcendental number. Thus for any non-zero polynomial $Q(x)$ with integer coefficients, we have that $Q(f(\alpha)) \neq 0$.

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In this paper we quantify the above result. Specifically, we show that

$$|Q(f(\alpha))| > \exp(-Cd^2(d^2 + \log H)),$$

where d = degree of Q, H = height of Q, and C is an effectively computable constant which does not depend on Q.

We remark that Mahler's original result is more general than indicated above, and that a number of further generalizations, some quite recent, have been effected by Mahler, K. K. Kubota, and Loxton and van der Poorten. These are detailed in the survey article of Loxton and van der Poorten (1977). Also, in the early work of Mahler, $f(z)$ is assumed to be transcendental; however Loxton and van der Poorten (1976) have shown that solutions to (1) are either rational or transcendental.

The remainder of the paper is set out as follows. Section 2 fixes the notation we use. In Section 3, we state and prove our main result, except for giving the proof of one crucial lemma (Lemma 2). This exception is the substance of Section 4. Finally, we offer some brief concluding remarks in Section 5.

2. Notation

Hereafter we abide by the following conventions. For Q a polynomial (in any number of variables) with complex coefficients, we define the height of Q, written $H(Q)$, to be the maximum taken over the absolute values of the coefficients of Q. We denote the degree of Q in the variable z by $\deg_z Q$, and similarly for other variables.

If α is an algebraic number, then the house of α, designated by $\overline{|\alpha|}$, is the maximum taken over the absolute values of all conjugates of α. A denominator for α is a positive integer D such that $D\alpha$ is an algebraic integer. The minimal such D is *the* denominator of α, which we abbreviate as den α. The height and degree of α are (respectively) the height and degree of the minimal polynomial of α, and $H(\alpha)$ and $\deg \alpha$ have the obvious meanings. If $\alpha \neq 0$, then $|\alpha|$ is bounded away from 0 by various functions of $\overline{|\alpha|}$, den α, $H(\alpha)$, and $\deg \alpha$. (See Waldschmidt (1974), Chapter 1.) We refer to any such bound as a Liouville estimate.

For a function $g(z)$ of the complex variable z which is analytic at the origin, we write ord g for the order of $g(z)$ at 0, that is the index of the first non-vanishing power of z appearing in the Taylor series expansion for g at 0.

Lastly C, C', C_0, $C_1, \ldots$ represent positive constants which are computable in terms of $\rho, f(z), a(z), b(z)$ and α. In particular, such constants are independent of the parameters n and k, the polynomial Q, and the algebraic number ξ which subsequently appear.

3. The main result

Our main result is

THEOREM 1. *Let $a(z)$ and $b(z)$ be rational functions and let T be the transformation mapping z onto z^ρ, where ρ is an integer greater than* 1. *Suppose that $f(z)$ is a non-rational solution to the functional equation* (1) *which is analytic at the origin and which has only algebraic coefficients in its power series expansion at* 0. *Assume that α is an algebraic number for which $0 < |\alpha| < 1$, $f(\alpha)$ is defined, and $T^k\alpha$ is neither a pole of $b(z)$ nor a zero of $a(z)$ for any k ($k = 0, 1, 2, \ldots$). Finally, let $Q(x)$ be a non-zero polynomial with integer coefficients and of degree d and height H. Then*

$$|Q(f(\alpha))| > \exp(-Cd^2(d^2 + \log H)),$$

where C is an effectively computable constant not depending on Q.

We actually prove Theorem 1 in the following equivalent form. (See Lang (1966), Chapter 6 for the details of deriving Theorem 1 from Theorem 2.)

THEOREM 2. *Let $a(z)$, $b(z)$, T, $f(z)$, and α be as in Theorem* 1. *If ξ is an algebraic number of degree d and height H, then*

$$|\xi - f(\alpha)| > \exp(-C'd^2(d^2 + \log H)),$$

where C' is an effectively computable constant not depending on ξ.

The fact that $f(z)$ satisfies (1) has several elementary consequences which we require for the proof of Theorem 2. A simple linear algebra argument (given in Kubota (1977), p. 32) permits us to assume that $a(z)$ and $b(z)$ have algebraic coefficients (provided that the coefficients of f are algebraic).

We iterate the functional equation $(k - 1)$ times to get

$$f(z) = \sum_{i=0}^{k-1} a^{(i)}(z)b(T^iz) + a^{(k)}(z)f(T^kz), \tag{2}$$

where $a^{(i)}(z) = a(z)a(Tz) \cdots a(T^{i-1}z)$. We write the power series expansion for $f(z)$ at 0 as

$$f(z) = \sum c_\mu z^\mu, \tag{3}$$

the sum ranging from $\mu = 0$ to $\mu = \infty$. Although (3) may not hold throughout the unit circle, we see that (2) provides an analytic continuation of $f(z)$ to $|z| < 1$. In particular, we note that $f(\alpha)$ fails to be defined for at most finitely many values of α lying in any circle of radius less than 1.

Suppose, for the moment, that $a(z)$ is regular at 0. By (1) and the fact that $f(z)$ is regular at 0, $b(z)$ must also be regular at 0. We may therefore expand $a^{(i)}(z)$ and $b(T^i z)$ in (2) as power series at 0. Since $f(T^k z)$ has, except for c_0, only terms with index at least ρ^k, equation (2) determines the c_μ $(0 < \mu < \rho^k)$ as elements of the field generated by c_0 and the coefficients of $a(z)$ and $b(z)$. Moreover, it follows easily that the c_μ satisfy the growth conditions

$$(4) \qquad |\overline{c_\mu}| \leqslant C_0^{\mu+1}, \quad C_0^{\mu+1} c_\mu \text{ is an algebraic integer.}$$

If now $a(z)$ has a pole of order s at 0, define the function $g(z)$ by

$$z^s g(z) + R(z) = f(z), R(z) = \sum_{\mu=0}^{s-1} c_\mu z^\mu.$$

Then $g(z)$ is regular at 0 and satisfies the functional equation.

$$g(z) = z^{(\rho-1)s} a(z) g(Tz) + z^{-s}(a(z)R(Tz) + b(z) - R(z)).$$

Hence the above analysis is valid for $g(z)$, so that (4) and the assertion that the c_μ all lie in a fixed number field still obtain.

Throughout the remainder of the paper, we assume the hypotheses of Theorem 1. We now construct an auxiliary polynomial which depends on the parameter n.

LEMMA 1. *Let n $(> C_1)$ be an integer. Then there is a polynomial in w and z with integer coefficients, call it $P(w, z)$, having the properties:*

$$(5) \qquad \begin{gathered} 1 \leqslant \deg_w P \leqslant n, \quad \deg_z P \leqslant n, \\ H(P) < \exp(C_2 n^2), \\ \operatorname{ord} P(f(z), z) > n^2/C_1. \end{gathered}$$

PROOF. We rely on the following version of the familiar Siegel's lemma of transcendence proofs (Waldschmidt (1974), p. 10).

Let c_{mr} $(1 \leqslant m \leqslant M, 1 \leqslant r \leqslant N)$ be elements of a number field of degree δ, and let $\sigma_1, \sigma_2, \ldots, \sigma_\delta$ be the distinct embeddings of the number field into $\mathbf{C}$. Suppose that A and D are (rational) integers such that D is a common denominator of the c_{mr} and

$$\sum_{r=1}^{N} |\sigma_h(c_{mr})| < A \qquad (1 \leqslant m \leqslant M, 1 \leqslant h \leqslant \delta).$$

If $N > \delta M$, then the system $\sum_{r=1}^{N} c_{mr} x_r = 0$ $(1 \leqslant m \leqslant M)$ has a non-trivial integral solution $(x_1, \ldots, x_N)$ in which $|x_r|$ $(1 \leqslant r \leqslant N)$ is no greater than $\exp((M\delta/(N - M\delta)) \log DA\sqrt{2}\,)$.

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We think on $P(w, z)$ as a sum of terms of the form $a_{ij}w^i z^j$ $(0 \leqslant i, j \leqslant n)$ and treat the a_{ij}'s as unknowns. Using (3), we write $P(f(z), z)$ as the power series $\sum \beta_l z^l$ and compute that

$$\beta_l = \sum_{i=0}^{n} \sum_{j=0}^{\min(l,n)} a_{ij} c_{l-j,i} \tag{6}$$

where $c_{l-j,i}$ is the coefficient of z^{l-j} in the expansion for $f(z)^i$. In terms of the above form of Siegel's lemma, we seek to solve the system $\beta_l = 0$ $(0 \leqslant l \leqslant n^2/C_1)$ of $M = [n^2/C_1] + 1$ equations by an appropriate choice of the $N = (n+1)^2$ unknowns a_{ij}. If C_1 has a suitable value (say $C_1 = 3\delta$), then $N > \delta M$ and $\delta M/(N - \delta M) < 1$.

Calculating values for A and D is a straightforward matter. It involves estimating the houses and denominators of the $c_{l-j,i}$ by writing the $c_{l-j,i}$ as Cauchy products and appealing to (4). We note that $\deg_w P \geqslant 1$ whenever $n \geqslant C_1$, else P would be a non-zero polynomial in z whose order exceeded its degree.

The key ingredient in our argument is an upper bound for ord $P(f(z), z)$ of the same shape as the lower bound of Lemma 1.

Lemma 2. *Let P be an element of $\mathbf{C}[w, z]$ such that $1 \leqslant \deg_w P \leqslant n$ and $\deg_z P \leqslant n$. Then* ord $P(f(z), z)$ *is at most $C_3 n^2$.*

We introduce a second parameter, k, and consider the number $P(f(T^k\alpha), T^k\alpha)$. The bounds on ord $P(f(z), z)$ enable us to prove

Lemma 3. *Let n and k be integers subject to $n \geqslant C_1$ and $\rho^k \geqslant C_4 n^2$. Construct $P(w, z)$ according to Lemma 1. Then*

$$\exp(-C_5 n^2 \rho^k) < |P(f(T^k\alpha), T^k\alpha)| < \exp(-C_6 n^2 \rho^k).$$

Proof. We observe that (3) holds when $z = T^k\alpha$ if k is larger than some number depending on α and the c_μ. For convenience, set $A_k = P(f(T^k\alpha), T^k\alpha)$. We thus have that

$$A_k = \sum \beta_l (T^k\alpha)^l,$$

where β_l is as in the proof of Lemma 1. Let $\lambda = \text{ord } P(f(z), z)$. From (4), (5) and (6), we find that $\overline{|\beta_l|} < C_7^l l^n$. Liouville estimates coupled with Lemma 2 then yield that $|\beta_\lambda| > \exp(-C_8 n^2)$.

We shall show that

$$\left|\left(A_k - \beta_\lambda (T^k\alpha)^\lambda\right) / \left(\beta_\lambda (T^k\alpha)^\lambda\right)\right| < \tfrac{1}{2}, \tag{7}$$

whence

$$\left(\tfrac{1}{2}\right)|\beta_\lambda(T^k\alpha)^\lambda| < A_k < \left(\tfrac{3}{2}\right)|\beta_\lambda(T^k\alpha)^\lambda|.$$

The estimates of the lemma then follow immediately from upper and lower bounds for $|\beta_\lambda|$ and λ.

We write the left hand side of (7) as

$$\left|\sum_{l=\lambda+1}^{\infty}(\beta_l/\beta_\lambda)(T^k\alpha)^{l-\lambda}\right|,$$

which is less than or equal to

$$|(T^k\alpha)/\beta_\lambda|\left(\sum_{l<\lambda}|\beta_{\lambda+l+1}|\cdot|T^k\alpha|^l + \sum_{l\geqslant\lambda}|\beta_{\lambda+l+1}|\cdot|T^k\alpha|^l\right).$$

The sum over $l < \lambda$ is trivially less than $\lambda C_7^{2\lambda}(2\lambda)^n < \exp(C_9 n^2)$. When $l \geqslant \lambda$, we have that $l \geqslant n^2/C_1 \geqslant n$ and $l + \lambda + 1 < 3l$, so that $|\beta_{\lambda+l+1}| < C_7^{3l}(3l)^n < (3C_7^3 n)^l$. If $\rho^k > C_4 n^2$, then $3C_7^3 n(T^k\alpha) < \frac{1}{2}$; and the sum over $l \geqslant \lambda$ is majorized by $\Sigma(\frac{1}{2})^l < 1$. Thus we see that the left hand side of (7) is less than

$$\exp\left(-\rho^k|\log|\alpha|| + C_8 n^2 + C_9 n^2 + 1\right).$$

It now is clear that (7) is valid if $\rho^k \geqslant C_4 n^2$.

Equation (2) may be rewritten as

$$f(T^k z) = \left(f(z) - \sum_{i=0}^{k-1} a^{(i)}(z)b(T^i z)\right)\Big/a^{(k)}(z). \tag{8}$$

The number ξ_k defined by

$$\xi_k = \left(\xi - \sum_{i=0}^{k-1} a^{(i)}(\alpha)b(T^i\alpha)\right)\Big/a^{(k)}(\alpha) \tag{9}$$

is algebraic; and a comparison between (8) and (9) suggests that ξ_k is a good approximation to $f(T^k\alpha)$ whenever $|\xi - f(\alpha)|$ is small. We expand upon this idea in

LEMMA 4. *Suppose that $n(\geqslant C_1)$ and $k(\geqslant C_{10})$ are non-negative integers. Define ξ_k by (9). Let P be the polynomial of Lemma 1 and C_5 the constant of Lemma 3. If*

$$|\xi - f(\alpha)| < \exp(-C_{11}n^2\rho^k),$$

then

$$|P(\xi_k, T^k\alpha) - P(f(T^k\alpha), T^k\alpha)| < \exp(-C_5 n^2\rho^k).$$

PROOF. We first observe from (8) and (9) that

$$|\xi_k - f(T^k\alpha)| = |\xi - f(\alpha)|/a^{(k)}(\alpha). \tag{10}$$

A straightforward calculation produces the Liouville estimate

$$|a^{(k)}(\alpha)| > \exp(-C_{12}\rho^k).$$

Next we write $P(w, z) = \sum a_{ij}w^iz^j$ $(0 \leq i, j \leq n)$, so that

$$|P(\xi_k, T^k\alpha) - P(f(T^k\alpha), T^k\alpha)| \leq (n+1)^2 \max |a_{ij}(\xi_k^i - f(T^k\alpha)^i)|.$$

The a_{ij}'s are bounded in Lemma 1 by $\exp(C_2n^2)$, and

$$|\xi_k^i - f(T^k\alpha)^i| \leq |\xi_k - f(T^k\alpha)| n \max(1, |\xi_k|, |f(T^k\alpha)|)^n.$$

Finally, since the coefficients of $f(z)$ satisfy $|c_\mu| < C_0^{\mu+1}$, we have $|f(T^k\alpha)| < 2C_0$ for $k \geq C_{10}$. A similar bound applies to ξ_k because of (10). Combining all these estimates, we conclude that

$$|P(\xi_k, T^k\alpha) - P(f(T^k\alpha), T^k\alpha)| < |\xi - f(\alpha)| \exp(C_{13}(n^2 + \rho^k)).$$

This establishes the lemma.

We require a Liouville estimate for $P(\xi_k, T^k\alpha)$. This is accomplished in

LEMMA 5. *Suppose that n ($\geq C_1$) and k ($\geq C_{14} \log n$) are positive integers. Let ξ_k and $P(w, z)$ be as in the previous lemma. If $P(\xi_k, T^k\alpha) \neq 0$ and if $\rho^k \geq \log H(\xi)$, then*

$$|P(\xi_k, T^k\alpha)| > \exp(-C_{15}dn\rho^k).$$

(Recall that $d = \deg \xi$.)

PROOF. We appeal to the following standard result (Lang (1966), p. 58).

Let $\zeta_1, \ldots, \zeta_m$ be algebraic numbers of degrees $\delta_1, \ldots, \delta_m$ and logarithmic heights $h_1, \ldots, h_m$ respectively. Let δ be the degree of the field $\mathbf{Q}(\zeta_1, \ldots, \zeta_m)$. Let P be a polynomial in $X_1, \ldots, X_m$ with integer coefficients. Denote by N_i the degree of P in X_i. If $P(\zeta_1, \ldots, \zeta_m) \neq 0$, then

$$|P(\zeta_1, \ldots, \zeta_m)| > \exp\left(-\delta\left(\log H(P) + \sum_{i=1}^{m} h_iN_i/\delta_i + 2\sum_{i=1}^{m} N_i\right)\right).$$

Now ξ_k and $T^k\alpha$ are both contained in a field of degree dC_{16} over $\mathbf{Q}$. We regard $P(\xi_k, T^k\alpha)$ as a polynomial of degree at most n in ξ_k and at most $n\rho^k$ in α. The height of P is no greater than $\exp(C_2n^2)$. We need only estimate $H(\xi_k)$ in order to invoke the above result. A tedious, but completely routine calculation (which is somewhat facilitated by bounding the house and denominator of ξ_k separately) reveals that

$$\log H(\xi_k) < C_{17}(\deg \xi_k)(\rho^k + \log H(\xi)).$$

Lemma 5 then follows with a modest amount of arithmetic.

We can now finish the proof of Theorem 2 in short order. We suppose that

$$|\xi - f(\alpha)| \leq \exp(-C_{11}n^2\rho^k), \tag{11}$$

where n and k are sufficiently large to satisfy the hypotheses of all the lemmas (that is $n \geq C_1$, $\rho^k \geq \max(C_{18}n^2, \log H(\xi))$); and we derive a contradiction for $n > C_{19}d$.

Using Lemma 1, we construct $P(w, z)$. From Lemma 4, from the upper bound for $|P(f(T^k\alpha), T^k\alpha)|$ of Lemma 3, and from the triangle inequality, we infer that $|P(\xi_k, T^k\alpha)| < \exp(-C_{20}n^2\rho^k)$. By Lemma 5 this implies that $P(\xi_k, T^k\alpha) = 0$ whenever $n \geq dC_{15}/C_{20}$. However, if $P(\xi_k, T^k\alpha) = 0$, then Lemma 4 gives an upper bound for $|P(f(T^k\alpha), T^k\alpha)|$ which is inconsistent with the lower bound of Lemma 3. Thus (11) is false whenever $n \geq \max(C_1, dC_{15}/C_{20})$, $\rho^k \geq \max(C_{18}n^2, \log H(\xi))$. The assertion of Theorem 2 is immediate from this.

4. The proof of Lemma 2.

For each non-negative integer m, we define a subset $\mathbb{S}_m$ of $\mathbf{C}[w, z]$ as follows: The polynomial $S(w, z)$ is in $\mathbb{S}_m$ if and only if

$$1 \leq \deg_w S \leq n - m, \qquad \deg_z S \leq n\rho^m, \quad \text{and} \tag{12}$$

$$\operatorname{ord} S(f(T^m z), z) > C_3 n\rho^m(\deg_w S), \tag{13}$$

where the value of C_3 will be specified presently. We note that $\mathbb{S}_n = \varnothing$ because of (12). We shall prove that $\mathbb{S}_0 = \varnothing$ by showing that if $\mathbb{S}_m \neq \varnothing$ $(0 \leq m < n)$, then $\mathbb{S}_{m+1} \neq \varnothing$. This will establish Lemma 2, since $P(w, z)$ would be in $\mathbb{S}_0$ if Lemma 2 failed to hold.

Assume that $\mathbb{S}_m \neq \varnothing$ for some m, $(0 \leq m < n)$, and let S be an element of $\mathbb{S}_m$. Set $r = \deg_w S$ and write $c(z)$ for the denominator of $a(z)b(z)$. We define the polynomials S_1 and S_2 by

$$S_1(w, z) = c(T^m z)^r S(wa(T^m z) + b(T^m z), z),$$
$$S_2(w, z) = S(w, Tz).$$

We first demonstrate that S_1 and S_2 have a common factor (say Q) as polynomials in w, and next that either Q or S_2/Q is in $\mathbb{S}_{m+1}$.

It is easy to verify that S_1 and S_2 are both of degree r in w, that

$$\deg_z S_1 \leq C_{21}\rho^m(r + n) \leq 2C_{21}\rho^m n,$$

and that

$$\deg_z S_2 \leq n\rho^{m+1}.$$

Because of (1) (with z replaced by $T^m z$), it is also apparent that

$$\text{(14)} \qquad \operatorname{ord} S_1(f(T^{m+1}z), z) = \rho^m r \operatorname{ord} c(z) + \operatorname{ord} S(f(T^m z), z) > C_3 n \rho^m r,$$

$$\text{(15)} \qquad \operatorname{ord} S_2(f(T^{m+1}z), z) = \rho \operatorname{ord} S(f(T^m z), z) > C_3 n \rho^{m+1} r,$$

We claim that ord $S(f(T^m z), z)$ is finite. To see this, suppose to the contrary. Then $f(T^m z)$ is algebraic and hence (since it satisfies a functional equation like (1)) rational. Write $f(T^m z) = p(z)/q(z)$, where p and q are relatively prime polynomials chosen so that either $q(0) = 1$ or else $p(0) = 1$. Let η be a primitive ρ^m-th root of unity. We have that $p(z)/q(z) = p(\eta z)/q(\eta z)$, whence (by unique factorization) $p(z) = p(\eta z)$ and $q(z) = q(\eta z)$. Thus $f(z) = p'(z)/q'(z)$, where $p'(T^m z) = p(z)$ and $q'(T^m z) = q(z)$, clearly a contradiction. This permits us to subtract (14) from (15) to obtain

$$\text{(16)} \qquad \begin{aligned} &\operatorname{ord} S_2(f(T^{m+1}z), z) - \operatorname{ord} S_1(f(T^{m+1}z), z) \\ &\qquad > (\rho - 1) C_3 n \rho^m r - p^m r \operatorname{ord} c(z) > n\rho^{m+1}, \end{aligned}$$

the latter inequality upon assuming that $C_3 > \rho + \operatorname{ord} c(z)$.

We view S_1 and S_2 as polynomials in w and consider their resultant, call it $\mathbf{R}(S_1, S_2)$, which is a polynomial in z. On one hand we may expand $\mathbf{R}(S_1, S_2)$ as a determinant. In this form $\mathbf{R}(S_1, S_2)$ consists of a sum of various products, each non-zero product containing r factors chosen from among the coefficients of S_1 and r factors chosen from among the coefficients of S_2. We see, therefore, that

$$\deg \mathbf{R}(S_1, S_2) < r(2C_{21}\rho^m n) + r(n\rho^{m+1}).$$

On the other hand,

$$\text{(17)} \qquad \mathbf{R}(S_1, S_2) = S_1 Q_1 + S_2 Q_2,$$

where Q_1 and Q_2 are polynomials in w and z both of degree less than r in w. (See Lang (1965), p. 136.) When $w = f(T^{m+1}z)$, the right hand side of (17) has (by (14) and (15)) order at least $C_3 n r \rho^m$. We choose C_3 to be larger than $(2C_{21} + \rho)$ and deduce that $\mathbf{R}(S_1, S_2)$ must be identically zero upon comparing its order to its degree. This implies that S_1 and S_2 have a common factor, say $Q(w, z)$, of positive degree in w.

We write

$$\text{(18)} \qquad S_1 = QT_1, \quad S_2 = QT_2,$$

and note that $\deg_z Q \leqslant n\rho^{m+1}$ and $\deg_z T_2 \leqslant n\rho^{m+1}$ (since $\deg_z S_2 \leqslant n\rho^{m+1}$). We assert, furthermore, that both $\deg_w Q$ and $\deg_w T_2$ lie between 1 and $r - 1$ inclusively. Because $\deg_w Q \geqslant 1$ and $r = \deg_w Q + \deg_w T_2$, it suffices to show that $\deg_w T_2 \neq 0$. From (18) we have the equation $T_2 S_1 = T_1 S_2$. If $\deg_w T_2 = 0$, then we let $w = f(T^{m+1}z)$ in this equation and compute orders. We find that

$$\operatorname{ord} T_2 = \operatorname{ord} T_1 + \operatorname{ord} S_2(f(T^{m+1}z), z) - \operatorname{ord} S_1(f(T^{m+1}z), z),$$

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so that ord $T_2 > n\rho^{m+1}$ by (16). The order of T_2 is thus larger than its degree, whence $T_2 \equiv 0$, an obvious absurdity in light of (18).

Finally, we point out that either ord $Q(f(T^{m+1}z), z) > C_3 n\rho^{m+1}(\deg_w Q)$ or else ord $T_2(f(T^{m+1}z), z) > C_3 n\rho^{m+1}(\deg_w T_2)$. For if both inequalities were simultaneously violated, we could add their negations to get ord $S_2(f(T^{m+1}z), z) < C_3 n\rho^{m+1}(\deg_w S_2)$ in contradiction to (15). Thus $\mathcal{S}_{m+1}$ is non-empty since it contains either Q or T_2. This completes the proof of Lemma 2.

5. Concluding remarks

For fixed d, Theorem 1 is best possible in its dependence on H. Although the dependence on d is presumably not so good, it is comparable to that of other transcendence measures. The constant C can be given explicitly in terms of ρ, C_0, $\deg \alpha$, $H(\alpha)$, the degrees of $a(z)$ and $b(z)$, and the heights and degrees of the coefficients of $a(z)$ and $b(z)$. However the attendant technical complications make such an exercise seem pointless.

It is worth noting that Lemma 2 readily generalizes to the case where $P \in \mathbb{C}[w, z_1, z_2, \ldots, z_m]$, f is a function of $z_1, \ldots, z_m$, and T is a transformation of the type considered by Mahler (1929). Unfortunately this does not allow us to generalize Theorem 1 except for very special T. The difficulty lies in determining which term of $P(f(T^k(z_1, \ldots, z_m)), T^k(z_1, \ldots, z_m))$ controls its asymptotic behaviour.

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