

ON SOME “STABILITY” PROPERTIES OF THE FULL C^* -ALGEBRA ASSOCIATED TO THE FREE GROUP F_∞

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Let $C^*(F_\infty)$ be the full C^* -algebra associated to the free group of countably many generators and $S_n C^*(F_\infty)$ be the class of all n -dimensional operator subspaces of $C^*(F_\infty)$. In this paper, we study some stability properties of $S_n C^*(F_\infty)$. More precisely, we will prove that for any E_0, E_1 in $S_n C^*(F_\infty)$, the Haagerup tensor product $E_0 \otimes_h E_1$ and the operator space obtained by complex interpolation E_θ are $(1 + \epsilon)$ -contained in $C^*(F_\infty)$ for arbitrary $\epsilon > 0$. On the other hand, we will show an extension property for *WEP* C^* -algebras.

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0. Introduction and background

This paper is devoted to various questions about “operator spaces”. By an operator space, we mean a closed subspace of $B(H)$ where $B(H)$ denotes the set of all bounded operators on the Hilbert space H . The theory of operator spaces (in particular, the duality theory) was developed recently, cf. [4] and [12, 13]. The morphisms suitable for this category are the completely bounded maps for which [19] is a general reference.

Let E and F be two operator spaces. We denote

$$d_{cb}(E, F) = \inf \{ \|u\|_{cb} \|u^{-1}\|_{cb} \}$$

where the infimum runs over all possible isomorphisms $u : E \rightarrow F$, when E and F are isomorphic. Otherwise, we let $d_{cb}(E, F) = \infty$.

As was shown recently in [16], the class of all n -dimensional operator spaces, denoted by OS_n , is not separable for $d_{cb}(\cdot, \cdot)$. However, it turns out that there exists a separable subclass which includes all the classical spaces of OS_n , namely the class $S_n C^*(F_\infty)$ of all n -dimensional subspaces of $C^*(F_\infty)$. So, it is natural to introduce for E in OS_n the following parameter which compares its operator space structure with all those it can inherit from some embedding into $C^*(F_\infty)$,

$$d_f(E) = \inf \{ d_{cb}(E, F) \}$$

where the infimum runs over all F in $S_n C^*(F_\infty)$. When E is not finite dimensional (f.d. in short), we let

$$d_r(E) = \sup\{d_r(F)\}$$

where the supremum runs over all f.d. subspaces $F \subset E$.

If H is a Hilbert space, $K(H)$ denotes the set of all compact operators on H .

Let $E \subset B(H)$, $F \subset B(K)$ be two concrete realizations of E and F on the Hilbert spaces H and K respectively. Then by definition, $E \otimes_{\min} F$ is the completion of $E \otimes F$ via the embedding $E \otimes F \subset B(H \otimes_2 K)$. Moreover, if E and F are C^* -algebras, $E \otimes_{\min} F$ is again a C^* -algebra.

We say that an operator $u : E \rightarrow F$ is *completely bounded* (c.b. in short) if the map

$$\begin{aligned} id_{K(l_2)} \otimes u : K(l_2) \otimes E &\rightarrow K(l_2) \otimes F \\ T \otimes x &\mapsto T \otimes u(x) \end{aligned}$$

where $id_{K(l_2)}$ is the identity map of $K(l_2)$, can be extended to a bounded operator from $K(l_2) \otimes_{\min} E$ into $K(l_2) \otimes_{\min} F$. The c.b. norm of u is defined as

$$\|u\|_{cb} = \|id_{K(l_2)} \otimes u : K(l_2) \otimes_{\min} E \rightarrow K(l_2) \otimes_{\min} F\|.$$

We say that u is *completely contractive* (resp. *completely isometric*) if the operator $id_{K(l_2)} \otimes u$ is contractive (resp. isometric). By [8], we still have the same definitions if we replace $K(l_2)$ by $B(l_2)$.

Recall that in the Banach space category, a contractive map $u : E \rightarrow F$ is said to be a metric surjection, if it maps the open unit ball of E onto the open unit ball of F . It is very elementary to show that a contractive map u is a metric surjection if and only if the closure of the image of the unit ball of E contains the unit ball of F . In the operator space category, we say that u is a *complete metric surjection* if $id_{K(l_2)} \otimes u : K(l_2) \otimes_{\min} E \rightarrow K(l_2) \otimes_{\min} F$, is a metric surjection and we say that it is a $B(l_2)$ -*metric surjection* if $id_{B(l_2)} \otimes u : B(l_2) \otimes_{\min} E \rightarrow B(l_2) \otimes_{\min} F$, is a metric surjection. It is clear that the $B(l_2)$ -metric surjectivity implies the complete metric surjectivity. But the converse turns to be false in general, see Remark 2.6 below.

If E and F are C^* -algebras or merely operator systems, we say that $u : E \rightarrow F$ is *completely positive* if the map $id_{K(l_2)} \otimes u : K(l_2) \otimes_{\min} E \rightarrow K(l_2) \otimes_{\min} F$ is positive. And we say that u is *unital* if it is unit preserving.

In Section 1, we will study the stability of the parameter $d_r(\cdot)$ under the Haagerup tensor product denoted by \otimes_h . More precisely, we will prove: For any operator spaces E and F , we have

$$d_r(E \otimes_h F) \leq d_r(E)d_r(F).$$

Let us recall the definition of the Haagerup tensor product. Given $x = (x_{i,j})$ in $M_{n,r}(E)$, $y = (y_{i,j})$ in $M_{r,m}(F)$, we define $x \odot y$ in $M_{n,m}(E \otimes F)$ as follows

$$x \odot y = \left(\sum_{k=1}^r x_{i,k} \otimes y_{k,j} \right).$$

And set for u in $M_n(E \otimes F)$,

$$\|u\|_h = \inf\{\|x\|\|y\|\}$$

where the infimum runs over all x in $M_{n,r}(E)$, y in $M_{r,n}(F)$, $r \geq 1$ such that $u = x \odot y$. $E \otimes_h F$ denotes the completion of $E \otimes F$ with respect to this norm. It is well known that the Haagerup tensor product is injective, self-dual, associative but not commutative (cf. [4, 13]).

Let A and B be two C^* -algebras, $u = \sum_{k=0}^r x_k \otimes y_k \in A \otimes B$ and denote by

$$\|u\|_{\max} = \sup \left\{ \left\| \sum_{k=0}^r \pi(x_k) \sigma(y_k) \right\| \right\}$$

where the supremum runs over all commuting C^* -representations $\pi : A \rightarrow B(H)$, $\sigma : B \rightarrow B(H)$ and all Hilbert spaces H . This defines a C^* -norm on $A \otimes B$. Actually, this is the greatest C^* -norm on $A \otimes B$. The maximal tensor product of A and B denoted by $A \otimes_{\max} B$, is the completion of $A \otimes B$ with respect to this C^* -norm (cf. [14], [19, Chapter 10]).

The next result is well known, see e.g. [26, p. 11].

Proposition 0.1. *If A_i, B_i are C^* -algebras, $u_i : A_i \rightarrow B_i$ ($i = 0, 1$) are completely positive maps, then $u_0 \otimes u_1$ extends to a completely positive map $u_0 \otimes u_1 : A_0 \otimes_{\max} A_1 \rightarrow B_0 \otimes_{\max} B_1$ with $\|u_0 \otimes_{\max} u_1\|_{cb} = \|u_0\|_{cb} \|u_1\|_{cb}$.*

The max tensor product is associative, commutative but not injective in general. This means that if $A_1 \subset B_1, A_2 \subset B_2$ are isometric C^* -embeddings, the inclusion map $A_1 \otimes_{\max} A_2 \subset B_1 \otimes_{\max} B_2$ which is clearly contractive, is not necessarily isometric.

A C^* -algebra A is said to be *nuclear* if on $A \otimes B$, there is a unique C^* -norm for any C^* -algebra B . In other words, we have $A \otimes_{\min} B = A \otimes_{\max} B$ for any C^* -algebra B .

Recently, Junge and Pisier introduced in [16] a new tensor product in both of the categories of C^* -algebras and operator spaces, namely the *M-tensor product* as follows: Let E and F be two operator spaces, say $E \subset B(H)$ and $F \subset B(K)$. $E \otimes_M F$ is the closure of $E \otimes F$ viewed as a subspace of $B(H) \otimes_{\max} B(K)$. This definition is independent of the given concrete realizations of E and F , it depends only on the operator space structures of E and F .

Proposition 0.2. ([16]) *If E_i, F_i are operator spaces, $u_i : E_i \rightarrow F_i$ ($i = 0, 1$) are c.b. maps, then $u_0 \otimes u_1$ extends to a c.b. map $u_0 \otimes_M u_1 : E_0 \otimes_M E_1 \rightarrow F_0 \otimes_M F_1$ with $\|u_0 \otimes_M u_1\|_{cb} \leq \|u_0\|_{cb} \|u_1\|_{cb}$. Moreover, if u_0 and u_1 are complete isometries, then so is $u_0 \otimes_M u_1$.*

This new tensor product is injective, commutative and the M -norm of a finite rank tensor can be described as the norm of factorization of its associated operator through a subspace of $C^*(F_\infty)$. More precisely, let $u \in E \otimes F$ and $\tilde{u} : E^* \rightarrow F$ its associated operator. Then we have

$$\|u\|_M = \inf\{\|a\|_{cb}\|b\|_{cb}\} \tag{1}$$

where the infimum runs over all operators a, b such that $a : E^* \rightarrow S$ is a weak- $*$ continuous operator, $b : S \rightarrow F$ is an arbitrary operator and where S is a f.d. subspace of $C^*(F_\infty)$. Using (1), it is clear that we have for any f.d. operator space E

$$d_f(E) = \|i_E\|_{E^* \otimes_M E}$$

where i_E is the tensor associated to the identity map of E . Hence by commutativity of the M -tensor product, we get

$$d_f(E^*) = d_f(E). \tag{2}$$

And this implies by duality that for any subspace F of E , we have

$$d_f(E/F) \leq d_f(E). \tag{2'}$$

In particular, this applies when $E = S_1^n$ (the dual of M_n). Note however that (2)' fails in general for infinite dimensional operator spaces even when E/F is f.d.. For instance, if $E = S_1$ (the dual of $K(l_2)$), we have $d_f(S_1) = 1$ (Prop. 2.9) and since any separable operator space is completely isometric to a quotient of S_1 (cf. [3, Corollary 3.2]), (2)' would imply that all operator spaces have $d_f(\cdot)$ equal to 1. Of course, this is false since by the results of [16], there exists a f.d. operator space with $d_f(\cdot) > 1$ and more generally, with $d_f(\cdot)$ arbitrary big. Hence similarly, for any fixed $\lambda > 0$, the inequality: $d_f(E/F) \leq \lambda d_f(E)$ cannot be true for arbitrary $F \subset E$ even when E/F is f.d..

Recall the following important results.

Theorem 0.3.

$$C^*(F_\infty) \otimes_{\min} B(H) = C^*(F_\infty) \otimes_{\max} B(H). \tag{3}$$

This is due to Kirchberg [17], see the recent paper [20] for a much simpler proof. Moreover, it is proved in [16] that in some sense, $C^*(F_\infty)$ is the largest C^* -algebra for which the equality (3) holds. More precisely, we have

Proposition 0.4. *For a C^* -algebra A , the following are equivalent.*

- (i) $A \otimes_{\min} B(H) = A \otimes_M B(H)$ (isometrically).
- (ii) $d_f(A) = 1$.
- (iii) Any f.d. $E \subset A$ is completely isometric to a subspace of $C^*(F_\infty)$.

Remark. It can happen, for a C^* -algebra A , that $d_f(A) = 1$ but A does not embed as a C^* -subalgebra into $C^*(F_\infty)$. For instance, the Cuntz algebra O_2 (cf. [7]). Indeed, O_2 is nuclear hence $d_f(O_2) = 1$. But since $C^*(F_\infty)$ admits a faithful unital representation into an infinite direct sum of matrix algebras (This is due to Choi, see [26, p. 39]), it does not contain any element x such that $x^*x = 1, xx^* \neq 1$ hence cannot contain O_2 as a C^* -subalgebra.

Proposition 0.5. ([16]) *Let $c \geq 1$ be a constant, E an operator space. The following are equivalent.*

- (i) $\forall u \in E \otimes B(l_2), \|u\|_M \leq c\|u\|_{\min}$.
- (ii) $\forall u \in E \otimes F, \|u\|_M \leq c\|u\|_{\min}$ for any operator space F .
- (iii) $d_f(E) \leq c$.

Note that the conditions in Proposition 0.5 imply automatically that, for any operator space F , the identity map $id : E \otimes_{\min} F \rightarrow E \otimes_M F$ has *c.b.* norm less than or equal to c .

In Section 2, we will prove that the complex interpolation method in the f.d. case, preserves the parameter $d_f(\cdot)$. More precisely, we show the following result: $\forall E_0, E_1 \in OS_n$ (assumed compatible) and $\forall 0 \leq \theta \leq 1$, we have

$$d_f(E_\theta) \leq d_f(E_0)^{1-\theta} d_f(E_1)^\theta.$$

Let us recall the definition of E_θ when E_0, E_1 are two compatible operator spaces, i.e., two operator spaces continuously injected into some larger one. Let

$$\Delta = \{x + iy \in \mathbb{C} / 0 \leq x \leq 1, y \in \mathbb{R}\}$$

$$\Delta_j = \{x + iy \in \mathbb{C} / x = j, y \in \mathbb{R}\}$$

for $j = 0, 1$ and consider the following sets of functions

$$G(E_0, E_1) = \left\{ f : \Delta \rightarrow E_0 + E_1 / f = \sum_{k=1}^m f_k x_k, x_k \in E_0 \cap E_1 \text{ where } m \geq 1 \text{ and } \forall 1 \leq k \leq m, \right.$$

$$\left. \begin{aligned} & f_k : \Delta \rightarrow \mathbb{C} \text{ continuous and bounded on } \Delta, \text{ analytic on the interior of} \\ & \Delta \text{ and vanishing at } \infty. \end{aligned} \right\}$$

$$N_\theta(E_0, E_1) = \{f \in G(E_0, E_1) / f(\theta) = 0\}.$$

Equip $G(E_0, E_1)$ with the norm $\|f\| = \max_{j=0,1} \{ \sup_{t \in \mathbb{R}} \|f(j + it)\|_{E_j} \}$. Denote by $F(E_0, E_1)$ the completion of $G(E_0, E_1)$ with respect to this norm and by $S_\theta(E_0, E_1)$ the closure of $N_\theta(E_0, E_1)$ in $F(E_0, E_1)$.

As a Banach space, E_θ is the space $F(E_0, E_1)/S_\theta(E_0, E_1)$ equipped with the quotient norm. Using Stafney’s lemma [25, p. 335], this definition is the same as the classical one given in [2]. It has the advantage to throw into relief these simple dense subspaces we can always restrict ourselves to, which seems to be very convenient for the identifications described in Section 2.

As an operator space, the structure on E_θ is given by the following identification

$$K(l_2) \otimes_{\min} E_\theta = (K(l_2) \otimes_{\min} E_0, K(l_2) \otimes_{\min} E_1)_\theta.$$

All the known results for complex interpolation in the category of Banach spaces extend to the category of operator spaces. In particular, we have for operator spaces the analogue of the duality theorem in Banach spaces (cf. [21]).

The following result was proved in [1].

Proposition 0.6. *The contractive inclusion $E_\theta \subseteq E^\theta$ is in fact an isometry, where E^θ is the Banach space obtained by the second method of complex interpolation applied to the couple (E_0, E_1) as defined in [2].*

Remark. With the duality theorem, Proposition 0.6 implies the following. Let E_0, E_1 be two compatible Banach spaces such that the intersection $E_0 \cap E_1$ is dense in both E_0 and E_1 . If $E_0^* = E_1^*$ as sets, then we have

$$(E_0, E_1)_\theta^* = (E_0^*, E_1^*)_\theta \quad (\text{isometrically}).$$

In Section 3, we prove an extension property for the C^* -algebras which have the weak expectation property (*WEP* in short). Recall that a C^* -algebra has the *WEP* if there exist completely positive and contractive maps $a : A \rightarrow B(H), b : B(H) \rightarrow A^{**}$ such that ba is the canonical embedding of A into A^{**} .

We will prove the following: Let E_1 be a f.d. operator space with $d_r(E_1) = 1, E_0 \subset E_1$ an arbitrary subspace and A a *WEP* C^* -algebra. Then, any operator $u : E_0 \rightarrow A$, has for arbitrary $\epsilon > 0$, an extension $\tilde{u} : E_1 \rightarrow A$ with $\|\tilde{u}\|_{cb} \leq (1 + \epsilon)\|u\|_{cb}$.

1. “Stability” of $C^*(F_\infty)$ for the Haagerup tensor product

Lemma 1.1. *Let X, Y be two operator spaces. Assume X has a concrete realization in a nuclear C^* -algebra. Then, $d_r(X \otimes_{\min} Y) = d_r(Y)$.*

Proof. If A is a nuclear C^* -algebra, we have by applying Kirchberg’s result (3), the associativity of the minimal and the maximal tensor products the following identities.

$$\begin{aligned} (A \otimes_{\min} C^*(F_\infty)) \otimes_{\min} B(H) &= A \otimes_{\min} (C^*(F_\infty) \otimes_{\min} B(H)) \\ &= A \otimes_{\max} (C^*(F_\infty) \otimes_{\max} B(H)) \\ &= (A \otimes_{\max} C^*(F_\infty)) \otimes_{\max} B(H) \\ &= (A \otimes_{\min} C^*(F_\infty)) \otimes_{\max} B(H). \end{aligned}$$

This means (Proposition 0.4) that $d_f(A \otimes_{\min} C^*(F_\infty))=1$. Then, it is easy to check the inequality $d_f(X \otimes_{\min} Y) \leq d_f(Y)$. And, the equality follows since the converse is trivial. □

More generally, we could not answer the following: Is it true that for any operator spaces X, Y we have $d_f(X \otimes_{\min} Y) \leq d_f(X)d_f(Y)$?

Proposition 1.2. *For any operator spaces X and Y with finite $d_f(\cdot)$ we have*

$$d_f(X \otimes_h Y) \leq d_f(X)d_f(Y).$$

The key of the proof is a result of [6] which gives a nice realization of the Haagerup tensor product using free products of C^* -algebras. First of all, recall the definition of the free product and the unital free product of C^* -algebras.

Free product of C^ -algebras:* Given A_1 and A_2 two C^* -algebras, denote by $\mathbb{C}[A_1, A_2]$ their algebraic free product. This is by definition the non unital algebra generated by the reduced words of letters belonging to A_1 or A_2 . The product of two reduced words in $\mathbb{C}[A_1, A_2]$, say $a_1 a_2 \dots a_n$ and $b_1 b_2 \dots b_m$ is by definition the reduced word obtained from $a_1 \dots a_n b_1 b_2 \dots b_m$ (after juxtaposition) and the adjoint of $a_1 a_2 \dots a_n$ is by definition $a_n^* \dots a_2^* a_1^*$.

If $\pi_i : A_i \rightarrow B(H)$ are C^* -representations for $i = 1, 2$ then, $\pi_1 \star \pi_2$ denotes the $*$ -representation

$$\begin{aligned} \pi_1 \star \pi_2 : \mathbb{C}[A_1, A_2] &\rightarrow B(H) \\ a_1 \dots a_k \dots a_n &\mapsto \pi_{j_1}(a_1) \dots \pi_{j_k}(a_k) \dots \pi_{j_n}(a_n) \end{aligned}$$

where $a_i \in A_{j_i}$ for $i = 1, 2, \dots, n$ and $j_i = 1$ or 2 . By definition, $A_1 \star A_2$ is the completion of $\mathbb{C}[A_1, A_2]$ with respect to the norm

$$\|u\|_\star = \sup \{ \|\pi_1 \star \pi_2(u)\| \}$$

where the supremum runs over all C^* -representations $\pi_i : A_i \rightarrow B(H)$ for $i = 1, 2$ and all Hilbert spaces H .

Unital free product of unital C^ -algebras:* Let $\mathbb{C}_1[A_1, A_2]$ be the unital algebraic free product of the unital C^* -algebras A_1, A_2 . This is by definition the quotient algebra of $\mathbb{C}[A_1, A_2]$ by the relation $\{e_1 = e_2\}$ where e_i denotes the unit of the algebra A_i for $i = 1, 2$.

If $u \in \mathbb{C}_1[A_1, A_2]$, we let

$$\|u\|_* = \sup \{ \|\pi_1 \star \pi_2(u)\| \}$$

where the supremum runs over all C^* -representations $\pi_i : A_i \rightarrow B(H)$ such that $\pi_i(e_i) = id_H$ for $i = 1, 2$ and all Hilbert spaces H . The unital free product of A_1 and A_2 denoted by $A_1 \star A_2$, is the unital C^* -algebra obtained after completion with respect to $\| \cdot \|_*$. Note that in this case, $A_1 \star A_2$ is a quotient of $A_1 \dot{\star} A_2$. See [9] for more on this topic.

In [6], the authors proved that the natural inclusion map

$$\begin{aligned} A_1 \otimes_h A_2 &\hookrightarrow A_1 \dot{\star} A_2 \\ a \otimes b &\mapsto ab \end{aligned}$$

is a complete isometry. Hence, the natural inclusion map $A_1 \otimes_h A_2 \hookrightarrow A_1 \star A_2$ is a complete contraction. It can be derived from the proof of this result that

Lemma 1.3. *In the particular case where A_1 and A_2 are the same unital C^* -algebra, the natural inclusion map $A_1 \otimes_h A_2 \hookrightarrow A_1 \star A_2$ is a complete isometry.*

Proof of Lemma 1.3. To check this, let $\phi : A_1 \otimes_h A_2 \rightarrow B(H)$ be a concrete realization of the operator space $A_1 \otimes_h A_2$ in some $B(H)$. ϕ defines a bilinear map

$$\begin{aligned} A_1 \times A_2 &\rightarrow B(H) \\ (a, b) &\mapsto \phi(a \otimes b) \end{aligned}$$

which is complete contractive in the sense of Christensen–Sinclair. Hence using the representation result of [6], we can find Hilbert spaces L_i , unital C^* -representations $\pi_i : A_i \rightarrow B(L_i)$ for $i = 1, 2$ and “bridging” contractions R, S and T

$$H \xrightarrow{T} L_2 \xrightarrow{S} L_1 \xrightarrow{R} H$$

such that

$$\forall a \in A_1, \forall b \in A_2 \phi(a \otimes b) = R\pi_1(a)S\pi_2(b)T.$$

On $A = A_1 = A_2$, we consider now the unital C^* -representation $\pi = \pi_1 \oplus \pi_2$

$$\begin{aligned} \pi : A &\rightarrow B(L_1 \oplus L_2) \\ a &\mapsto \pi(a) = \begin{pmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{pmatrix}. \end{aligned}$$

Then, we have

$$\phi(a \otimes b) = (R, 0)\pi(a)U\pi(b)\begin{pmatrix} 0 \\ T \end{pmatrix}$$

where U is the unitary operator

$$U = \begin{pmatrix} \sqrt{id_{L_1} - SS^*} & S \\ -S^* & \sqrt{id_{L_2} - S^*S} \end{pmatrix}.$$

Hence,

$$\phi(a \otimes b) = R_1\sigma(a)\pi(b)T_1$$

where $R_1 = (R, 0)U$, $T_1 = \begin{pmatrix} 0 \\ T \end{pmatrix}$ and $\sigma(a) = U^*\pi(a)U$.

Now we have for any $a_i \in A_1$ and $b_i \in A_2$, $1 \leq i \leq n$

$$\begin{aligned} \phi\left(\sum_{i=1}^n a_i \otimes b_i\right) &= R_1\left(\sum_{i=1}^n \sigma(a_i)\pi(b_i)\right)T_1 \\ &= R_1\sigma \star \pi\left(\sum_{i=1}^n a_i b_i\right)T_1. \end{aligned}$$

Recall that the natural inclusion map $A_1 \otimes_h A_2 \hookrightarrow A_1 \star A_2$ is a complete contraction. To prove that it is in fact an isometry, it suffices to see that

$$\begin{aligned} \left\|\sum_{i=1}^n a_i \otimes b_i\right\|_h &= \left\|\phi\left(\sum_{i=1}^n a_i \otimes b_i\right)\right\|_{B(H)} \\ &\leq \|R_1\| \|T_1\| \left\|\sigma \star \pi\left(\sum_{i=1}^n a_i b_i\right)\right\|_{B(L_1 \oplus L_2)} \\ &\leq \left\|\sum_{i=1}^n a_i b_i\right\|_*. \end{aligned}$$

And with the same calculation, we prove that the inclusion map $A_1 \otimes_h A_2 \hookrightarrow A_1 \star A_2$ is a complete isometry. □

Remarks. As observed in [20], the assumption that $A_1 = A_2$ in Lemma 1.3 is not important since we can replace A_1 and A_2 by (say) $A = A_1 \otimes_{\min} A_2$ and embed unitaly A_1 and A_2 in A .

In the particular case of a group C^* -algebra, say $A = C^*(G)$ for some group G , the natural inclusion map

$$C^*(G) \otimes_h C^*(G) \hookrightarrow C^*(G) \star C^*(G)$$

is a complete isometry. On the other hand, it is easy to see that for any groups G_1 and G_2 , we have

$$C^*(G_1) \star C^*(G_2) = C^*(G_1 \star G_2)$$

where $G_1 \star G_2$ is the group free product of G_1 and G_2 . Hence, the inclusion map

$$C^*(G) \otimes_h C^*(G) \hookrightarrow C^*(G \star G)$$

is now a complete isometry.

Proof of Proposition 1.2. By density of $X \otimes Y$ in $X \otimes_h Y$, injectivity of the Haagerup tensor product and Remark 2.8, it suffices to prove that for any f.d. subspaces $E \subseteq X, F \subseteq Y$

$$d_f(E \otimes_h F) \leq d_f(E)d_f(F).$$

Given E, F as above and $\epsilon > 0$, there exist $S_1, S_2 \subseteq C^*(F_\infty)$ f.d. subspaces such that

$$d_{cb}(E, S_1) \leq d_f(E) + \epsilon$$

$$d_{cb}(F, S_2) \leq d_f(F) + \epsilon.$$

$$\begin{aligned} d_f(E \otimes_h F) &\leq d_{cb}(E \otimes_h F, S_1 \otimes_h S_2)d_f(S_1 \otimes_h S_2) \\ &\leq d_{cb}(E, S_1)d_{cb}(F, S_2)d_f(S_1 \otimes_h S_2). \end{aligned}$$

This implies

$$\begin{aligned} d_f(E \otimes_h F) &\leq d_f(E)d_f(F)d_f(C^*(F_\infty) \otimes_h C^*(F_\infty)) \\ &\leq d_f(E)d_f(F)d_f(C^*(F_\infty \star F_\infty)). \end{aligned}$$

And of course $d_f(C^*(F_\infty \star F_\infty)) = 1$ since $F_\infty \star F_\infty$ is isomorphic to F_∞ . □

2. “Stability” of $C^*(F_\infty)$ under complex interpolation

Let (E_0, E_1) be a compatible couple of operator spaces and let $F(E_0, E_1)$ be equipped with the operator space structure given by the embedding below

$$F(E_0, E_1) \rightarrow \bigoplus_{j=0}^1 C_0(\Delta_j, E_j)$$

$$f \mapsto f_0 + f_1$$

where $C_0(\Delta_j, E_j)$ denotes the space of all E_j -valued continuous functions on Δ_j which tend to 0 at ∞ and f_j denotes the restriction of f to Δ_j for $j = 0, 1$ and where $\bigoplus_{j=0}^1 C_0(\Delta_j, E_j)$ denotes the direct sum of $C_0(\Delta_0, E_0)$ and $C_0(\Delta_1, E_1)$ in the sense of l_∞ . Recall that the natural operator space structure on $C_0(\Delta_j, E_j)$ is given by the usual identification

$$C_0(\Delta_j, E_j) = C_0(\Delta_j) \otimes_{\min} E_j.$$

For Section 2, we let \mathcal{K} stand for $K(l_2)$ and \mathcal{B} for $B(l_2)$.

Proposition 2.1. *Let (E_0, E_1) be a compatible couple of f.d. operator spaces with the same dimension, then we have*

$$d_f(E_0) \leq d_f(E_0)^{1-\theta} d_f(E_1)^\theta.$$

The proof of this proposition is based on the following lemmas:

Lemma 2.2. *For any compatible couple of operator spaces (E_0, E_1) , the operator spaces structures on E_0 and $F(E_0, E_1)/S_0(E_0, E_1)$ coincide. In other words, the map*

$$q : F(E_0, E_1) \rightarrow E_0$$

$$f \mapsto f(\theta)$$

is a complete metric surjection.

Lemma 2.3. *Let (E_0, E_1) be a compatible couple of f.d. operator spaces with the same dimension. Then, the map*

$$q : F(E_0, E_1) \rightarrow E_0$$

$$f \mapsto f(\theta)$$

is a \mathcal{B} -metric surjection.

Lemma 2.4. *Let X be an operator space such that $d_f(X) < \infty$ and S any subspace. If the canonical surjection $Q : X \rightarrow X/S$ is a \mathcal{B} -metric surjection, then we have*

$$d_f(X/S) \leq d_f(X).$$

Lemma 2.5. *For any compatible couple of operator spaces (E_0, E_1) , we have*

$$d_f(F(E_0, E_1)) \leq \max\{d_f(E_0), d_f(E_1)\}.$$

Proof of Proposition 2.1. By the first lemma, we see that E_0 and $F(E_0, E_1)/S_0(E_0, E_1)$ are completely isometric. By the second lemma, the map q is a \mathcal{B} -metric surjection. Then, we have by the third one

$$d_f(E_0) \leq d_f(F(E_0, E_1)).$$

Using Lemma 2.5, we get

$$d_f(E_0) \leq \max\{d_f(E_0), d_f(E_1)\}.$$

In the particular case when $d_f(E_0) = d_f(E_1) = 1$, we have $d_f(E_0) = 1$. Whence, the inequality of Proposition 2.1 is satisfied.

In the general case, for any fixed $\epsilon > 0$, there exist $F_j \subseteq C^*(F_\infty)$, $T_j : F_j \rightarrow E_j$ isomorphisms such that $\|T_j\|_{cb} < d_f(E_j) + \epsilon$ and $\|T_j^{-1}\|_{cb} = 1$ for $j = 0, 1$. Consider now the interpolation couple (F_0, F_1) compatible via the injections T_0, T_1 . This means that we will make the couple (F_0, F_1) into a compatible couple in the sense of interpolation by declaring that $x = y$ if and only if $T_0(x) = T_1(y)$, for any $x \in F_0$ and $y \in F_1$. In this viewpoint, T_0 and T_1 (resp. T_0^{-1}, T_1^{-1}) become equal. By complex interpolation, we get an operator $T_\theta : F_\theta \rightarrow E_\theta$ with $\|T_\theta\|_{cb} \leq \|T_0\|_{cb}^{1-\theta} \|T_1\|_{cb}^\theta$ and $\|T_\theta^{-1}\|_{cb} \leq 1$ and we have

$$\begin{aligned} d_f(E_\theta) &\leq d_{cb}(E_\theta, F_\theta) d_f(F_\theta) \\ &\leq \|T_\theta\|_{cb}^{1-\theta} \|T_1\|_{cb}^\theta \\ &\leq (d_f(E_0) + \epsilon)^{1-\theta} (d_f(E_1) + \epsilon)^\theta \end{aligned}$$

since by the particular case $d_f(F_\theta) = 1$. Now, letting ϵ tend to 0, this yields the required inequality. □

Proof of Lemma 2.2. Consider the following injection which is a complete isometry

$$\begin{aligned} \mathcal{K} \otimes_{\min} \left(\bigoplus_{j=0}^1 C_0(\Delta_j, E_j) \right) &\rightarrow \bigoplus_{j=0}^1 C_0(\Delta_j, \mathcal{K} \otimes_{\min} E_j) \\ T \otimes (f.x \oplus g.y) &\mapsto f.(T \otimes x) \oplus g.(T \otimes y) \end{aligned}$$

where $T \in \mathcal{K}$, $f \in C_0(\Delta_0)$, $g \in C_0(\Delta_1)$, $x \in E_0$ and $y \in E_1$. Of course, we use here the usual identification

$$C_0(\Delta, E) = C_0(\Delta) \otimes_{\min} E$$

for any locally compact topological space Δ and any operator space E . The restriction of the previous complete isometry to $\mathcal{K} \otimes_{\min} F(E_0, E_1)$ and the embedding of $F(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1)$ into $\bigoplus_{j=0}^1 C_0(\Delta_j, \mathcal{K} \otimes_{\min} E_j)$ show that $\mathcal{K} \otimes_{\min} F(E_0, E_1)$ and $F(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1)$ can be identified via the isometric isomorphism

$$\begin{aligned} \mathcal{K} \otimes_{\min} F(E_0, E_1) &\rightarrow F(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1) \\ T \otimes (f \cdot x) &\mapsto f \cdot (T \otimes x) \end{aligned} \tag{4}$$

where $T \in \mathcal{K}, f : \Delta \rightarrow \mathbb{C}$ is as in the definition of $G(E_0, E_1)$ and $x \in E_0 \cap E_1$. Moreover, it is easy to check that (extending the notation N_θ to pairs of normed spaces),

$$N_\theta((\mathcal{K} \otimes E_0, \|\cdot\|_{\min}), (\mathcal{K} \otimes E_1, \|\cdot\|_{\min})) = (\mathcal{K} \otimes N_\theta(E_0, E_1), \|\cdot\|_{\min})$$

via (4). And a density argument yields

$$S_\theta(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1) = \mathcal{K} \otimes_{\min} S_\theta(E_0, E_1).$$

Remark that the previous identifications remain valid with \mathcal{B} instead of \mathcal{K} . Hence, we have

$$\begin{aligned} \mathcal{K} \otimes_{\min} E_\theta &\equiv (\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1)_\theta \\ &\equiv F(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1) / S_\theta(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1) \\ &\equiv (\mathcal{K} \otimes_{\min} F(E_0, E_1)) / (\mathcal{K} \otimes_{\min} S_\theta(E_0, E_1)) \\ &\equiv \mathcal{K} \otimes_{\min} (F(E_0, E_1) / S_\theta(E_0, E_1)). \end{aligned} \quad \square$$

Proof of Lemma 2.3. First recall that for an operator space E , we have (cf. [11, Proposition 4.1])

$$\begin{aligned} (\mathcal{K} \otimes_{\min} E)^* &\equiv S_1 \widehat{\otimes} E^* \\ (\mathcal{K} \otimes_{\min} E)^{**} &\equiv CB(S_1, E^{**}) \end{aligned}$$

where $S_1 = \mathcal{K}^*$, with the dual operator space structure and $\widehat{\otimes}$ denotes the projective tensor product of operator spaces in the sense of [4] and [12]. When E is f.d., we get

$$(\mathcal{K} \otimes_{\min} E)^{**} \equiv \mathcal{B} \otimes_{\min} E.$$

Now, apply successively the remark after Proposition 0.6 to the couples $(\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1)$ and $(S_1 \widehat{\otimes} E_0^*, S_1 \widehat{\otimes} E_1^*)$ to obtain

$$\begin{aligned}
 \mathcal{B} \otimes_{\min} E_0 &\equiv (\mathcal{K} \otimes_{\min} E_0)^{**} \\
 &\equiv (\mathcal{K} \otimes_{\min} E_0, \mathcal{K} \otimes_{\min} E_1)_0^{**} \\
 &\equiv (S_1 \widehat{\otimes} E_0^*, S_1 \widehat{\otimes} E_1^*)_0^* \\
 &\equiv (\mathcal{B} \otimes_{\min} E_0^*, \mathcal{B} \otimes_{\min} E_1^*)_0 \\
 &\equiv F(\mathcal{B} \otimes_{\min} E_0, \mathcal{B} \otimes_{\min} E_1) / S_\theta(\mathcal{B} \otimes_{\min} E_0, \mathcal{B} \otimes_{\min} E_1) \\
 &\equiv (\mathcal{B} \otimes_{\min} F(E_0, E_1)) / (\mathcal{B} \otimes_{\min} S_\theta(E_0, E_1)).
 \end{aligned}$$

This implies that the canonical surjection q is a \mathcal{B} -metric surjection. □

Proof of Lemma 2.4. By Proposition 0.5, it suffices to prove that the identity map $\mathcal{B} \otimes_{\min} X/S \rightarrow \mathcal{B} \otimes_M X/S$ is bounded with norm less or equal to $d_f(X)$. Let $u \in \mathcal{B} \otimes_{\min} X/S$. Without loss of generality, we can assume $u \in \mathcal{B} \otimes X/S$. By assumption, Q is a \mathcal{B} -metric surjection. Hence, for arbitrary $\epsilon > 0$, we can find a lifting \tilde{u} of u in $\mathcal{B} \otimes_{\min} X$ such that $\|\tilde{u}\|_{\min} < (1 + \epsilon)\|u\|_{\min}$. The identity map $\mathcal{B} \otimes_{\min} X \rightarrow \mathcal{B} \otimes_M X$ is bounded with norm less or equal to $d_f(X)$. Hence, $\|\tilde{u}\|_M \leq d_f(X)\|\tilde{u}\|_{\min}$. By Proposition 0.2, we have $\|u\|_M \leq \|Q\|_{cb}\|\tilde{u}\|_M$. Finally, we get $\|u\|_M \leq (1 + \epsilon)d_f(X)\|u\|_{\min}$. Then, letting ϵ tend to 0, we obtain the required inequality. □

Remark 2.6. By Lemma 2.4 and the comments after (2)', there exists a complete metric surjection which is not a \mathcal{B} -metric surjection. Equivalently, let $\lambda > 0$ be fixed. We say that $u : E \rightarrow F$ is a (λ, \mathcal{B}) -metric surjection if $id_{\mathcal{B}} \otimes u$ maps the open unit ball of $\mathcal{B} \otimes_{\min} E$ onto a set containing the open ball of radius λ^{-1} of $\mathcal{B} \otimes_{\min} F$. The same proof as for Lemma 2.4 shows that for any (λ, \mathcal{B}) -metric surjection $Q : X \rightarrow X/S$ where $S \subset X$ with $d_f(X) < \infty$, we have $d_f(X/S) \leq \lambda d_f(X)$. Hence, again by the comments after (2)', there exists a complete metric surjection which is not a (λ, \mathcal{B}) -metric surjection.

Proof of Lemma 2.5. $C_0(\Delta_j)$ is a commutative C^* -algebra hence a nuclear one. Proposition 1.1 implies that $d_f(C_0(\Delta_j) \otimes_{\min} E_j) = d_f(E_j)$ so $d_f(C_0(\Delta_j, E_j)) = d_f(E_j)$. Then, to prove Lemma 2.5, it suffices to check that we have for arbitrary operator spaces X_0 and X_1

$$d_f(X_0 \oplus X_1) = \max\{d_f(X_0), d_f(X_1)\}. \tag{5}$$

Indeed,

$$\begin{aligned}
 d_f(F(E_0, E_1)) &\leq d_f(C_0(\Delta_0, E_0) \oplus C_0(\Delta_1, E_1)) \\
 &= \max\{d_f(C_0(\Delta_0, E_0)), d_f(C_0(\Delta_1, E_1))\} \\
 &= \max\{d_f(E_0), d_f(E_1)\}.
 \end{aligned}$$

Let us sketch (5). Assume $X_i \subset \mathcal{B}_i = B(H_i)$ for $i = 0, 1$. Since $\mathcal{B}_0 \oplus \mathcal{B}_1$ is injective, we

have an isometric inclusion

$$\mathcal{B} \otimes_{\max} (\mathcal{B}_0 \oplus \mathcal{B}_1) \subset \mathcal{B} \otimes_{\max} \mathcal{B}(H_0 \oplus H_1).$$

This implies,

$$\mathcal{B} \otimes_{\max} (\mathcal{B}_0 \oplus \mathcal{B}_1) = \mathcal{B} \otimes_M (\mathcal{B}_0 \oplus \mathcal{B}_1).$$

On the other hand, it is very easy to show that

$$(\mathcal{B} \otimes_{\max} \mathcal{B}_0) \oplus (\mathcal{B} \otimes_{\max} \mathcal{B}_1) = \mathcal{B} \otimes_{\max} (\mathcal{B}_0 \oplus \mathcal{B}_1).$$

This is in fact true for arbitrary C^* -algebras. Hence

$$\begin{aligned} \mathcal{B} \otimes_M (X_0 \oplus X_1) &\equiv (\mathcal{B} \otimes_M X_0) \oplus (\mathcal{B} \otimes_M X_1) \\ &\cong (\mathcal{B} \otimes_{\min} X_0) \oplus (\mathcal{B} \otimes_{\min} X_1) \\ &\equiv \mathcal{B} \otimes_{\min} (X_0 \oplus X_1) \end{aligned}$$

and we have

$$\begin{aligned} d_f(X_0 \oplus X_1) &= \|id : \mathcal{B} \otimes_{\min} (X_0 \oplus X_1) \rightarrow \mathcal{B} \otimes_M (X_0 \oplus X_1)\|_{cb} \\ &= \left\| id : \bigoplus_{j=0}^1 \mathcal{B} \otimes_{\min} X_j \rightarrow \bigoplus_{j=0}^1 \mathcal{B} \otimes_M X_j \right\|_{cb} \\ &= \max_{j=0,1} \{ \|id : \mathcal{B} \otimes_{\min} X_j \rightarrow \mathcal{B} \otimes_M X_j\|_{cb} \} \\ &= \max\{d_f(X_0), d_f(X_1)\}, \end{aligned}$$

Whence (5). □

Proposition 2.7. *Let X be an operator space, $S \subset X$ and $Q : X \rightarrow X/S$ the canonical surjection. Suppose that the kernel of $id_{\mathcal{B}} \otimes Q : \mathcal{B} \otimes_{\min} X \rightarrow \mathcal{B} \otimes_{\min} X/S$ is exactly $\mathcal{B} \otimes_{\min} S$. Then the following are equivalent.*

- (i) Q is a \mathcal{B} -metric surjection.
- (ii) For any $F \subset X/S$ f.d. and any $\epsilon > 0$, there exists $E \subset X$ f.d. such that $Q|_E : E \rightarrow F$ is a $(1 + \epsilon)$ - \mathcal{B} -metric surjection.

Proof. (ii) \Rightarrow (i) is trivial. We sketch only (i) \Rightarrow (ii). Let F be as in (ii) and assume $F^* \subset \mathcal{B}$ completely isometrically. Let $i_F \in F^* \otimes_{\min} F$ be the tensor associated to id_F . Say $i_F = \sum_1^n e_i^* \otimes e_i$ where $\{e_i, 1 \leq i \leq n\}$ is a basis of F and $\{e_i^*, 1 \leq i \leq n\}$ is the dual basis. Assume $e_i = Q(x_i)$ and let $u = \sum_1^n e_i^* \otimes x_i$. Then u is a lifting of i_F for the operator

$i_B \otimes Q$. By (i), there exists $u_0 \in \ker(id_B \otimes Q)$ such that $\|u - u_0\|_{\min} < 1 + \epsilon$. Since $\ker(id_B \otimes Q) = \mathcal{B} \otimes_{\min} S$, we can assume $u_0 \in \mathcal{B} \otimes S$, say $u_0 = \sum_1^{n'} T_i \otimes s_i$ where $T_i \in \mathcal{B}$, $s_i \in S$. Let E be the subspace of X spanned by the x_i 's and the s_i 's. Then E is as in (ii). Indeed, $Q|_E : E \rightarrow F$ and if $v \in \mathcal{B} \otimes_{\min} F, \tilde{v} : F^* \rightarrow \mathcal{B}$ its associated operator, we can write

$$\begin{aligned} v &= (\tilde{v} \otimes id_F)(i_F) \\ &= (\tilde{v} \otimes id_F)(id_B \otimes Q)(u - u_0) \\ &= (id_B \otimes Q|_E)(V \otimes id_E)(u - u_0), \end{aligned}$$

where V is an extension of \tilde{v} to \mathcal{B} with $\|V\|_{cb} = \|\tilde{v}\|_{cb} = \|v\|_{\min}$. Hence, $(V \otimes id_E)(u - u_0)$ is a preimage of v with

$$\begin{aligned} \|(V \otimes id_E)(u - u_0)\|_{\min} &\leq \|V\|_{cb} \|u - u_0\|_{\min} \\ &\leq \|v\|_{\min} \|u - u_0\|_{\min} \\ &\leq (1 + \epsilon) \|v\|_{\min}. \end{aligned}$$

□

Note that if we go back to the proof of Lemma 2.3, we see that

$$\begin{aligned} q : F(E_0, E_1) &\rightarrow E_\theta \\ f &\mapsto f(\theta) \end{aligned}$$

is a \mathcal{B} -metric surjection with $\ker(id_B \otimes q) = \mathcal{B} \otimes_{\min} S_\theta(E_0, E_1)$. Hence, for any $\epsilon > 0$, there exists a f.d. subspace E of $F(E_0, E_1)$ such that $d_{cb}(E_\theta, E/E \cap S_\theta(E_0, E_1)) < 1 + \epsilon$. This implies that, in the f.d. case, all the “natural” operator spaces are $(1 + \epsilon)$ -completely isomorphic to a quotient of f.d. subspaces of $C_0(\Delta_0, E_0) \oplus_\infty C_0(\Delta_1, E_1)$.

Now, we will consider $d_f(E)$ for several examples of “natural” infinite dimensional operator spaces. Let us first specify what we mean by “natural” operator spaces.

The space l_2 can be provided with several natural operator space structures as $R = B(l_2^*, \mathbb{C})$ and $C = B(\mathbb{C}, l_2)$ which naturally form a couple of interpolation, we let $x = y$ for $x \in R, y \in C$ if they correspond to the same vector in l_2 . OH denotes l_2 equipped with the unique operator space structure for which $OH^* = \overline{OH}$, where \overline{OH} is the operator space complex conjugate of OH . Equivalently, we have $OH = (R, C)_{\frac{1}{2}}$ (cf. [21] for more details).

The space l_∞ has, as a C^* -algebra, a natural structure given by any isometric C^* -embedding into some $B(H)$. This structure is unique up to a complete isometry. The space l_1 is equipped with the structure given by the usual embedding $l_1 \hookrightarrow l_\infty^*$, where l_∞^* is the dual operator space. And the spaces l_p are equipped with the complex interpolated operator space structure: $l_p = (l_\infty, l_1)_\theta, \theta = \frac{1}{p}$. More generally, we let for any operator space E

$$\begin{aligned}
 l_\infty^0(E) &= l_\infty \otimes_{\min} E \\
 l_1(E) &= l_1 \widehat{\otimes} E \\
 l_p(E) &= (l_\infty^0(E), l_1(E))_\theta, \quad p = \frac{1}{\theta}.
 \end{aligned}$$

Recall that the operator space S_1 denotes the standard dual of \mathcal{K} . And let the Schatten classes S_p be equipped with the operator space structure given by $S_p = (\mathcal{K}, S_1)_\theta$, $\theta = \frac{1}{p}$. More generally, we define for an operator space E

$$\begin{aligned}
 \mathcal{K}(E) &= \mathcal{K} \otimes_{\min} E \\
 S_1(E) &= S_1 \widehat{\otimes} E \\
 S_p(E) &= (\mathcal{K}(E), S_1(E))_\theta, \quad p = \frac{1}{\theta}.
 \end{aligned}$$

Moreover, if (E_0, E_1) is a pair of compatible operator spaces, we have

$$\begin{aligned}
 l_p(E_\theta) &= (l_\infty^0(E_0), l_1(E_1))_\theta \\
 S_p(E_\theta) &= (\mathcal{K}(E_0), S_1(E_1))_\theta.
 \end{aligned}$$

And we still keep the same definitions for the n -dimensional versions of all these spaces which were introduced and discussed in detail in [22].

Remark 2.8. Let E be an operator space. Let E_n be an increasing family (or more generally a directed net) of subspaces of E such that $\bigcup_n E_n$ is dense in E . Then, we have $d_f(E) = \sup_n \{d_f(E_n)\}$. Indeed, by a simple perturbation argument, for any f.d. subspace $F \subset E$ and any $\epsilon > 0$, there exist an integer n and a subspace $\tilde{F} \subset E_n$ such that $d_{cb}(F, \tilde{F}) < 1 + \epsilon$.

Proposition 2.9. For any $0 < \theta < 1$, $p = \frac{1}{\theta}$

- (i) $d_f((R, C)_\theta) = 1$. In particular, $d_f(OH) = 1$.
- (ii) $d_f(S_p) = 1$, $d_f(l_p) = 1$. More generally, for any compatible couple of f.d. operator spaces E_0, E_1 with the same dimension, we have

$$\begin{aligned}
 d_f(S_p(E_\theta)) &\leq d_f(E_0)^{1-\theta} d_f(E_1)^\theta \\
 d_f(l_p(E_\theta)) &\leq d_f(E_0)^{1-\theta} d_f(E_1)^\theta.
 \end{aligned}$$

Proof. (i) is a direct consequence of Proposition 2.1 in the f.d. case. For the general case, recall that R and C are homogeneous (cf. [21]). Hence, the projections on the n -first coordinates are completely contractive and any n -dimensional subspace of R (resp. C) is canonically completely isometric to R_n (resp. C_n). Then, it is easy to

see that $\bigcup_n (R_n, C_n)_\theta$ is dense in $(R, C)_\theta$. Whence by Remark 2.8, $d_f((R, C)_\theta) = 1$. In the particular case of $\theta = \frac{1}{2}$, this gives a second proof to the fact that $d_f(OH) = 1$, already proved in [16].

(ii) For S_p , this follows as in (i), from the fact that $\bigcup_n S_p^n$ is dense in S_p and that for each n , we have $d_f(M_n) = 1$, $d_f(S_1^n) = d_f(M_n) = 1$ by (2), hence $d_f(S_p^n) = 1$. More generally, $\bigcup_n S_p^n(E_\theta)$ is dense in $S_p(E_\theta)$ (cf. [22]) and for each n , we have completely isometrically $S_p^n(E_\theta) = (M_n(E_\theta), S_1^n(E_1))_\theta$. Hence,

$$\begin{aligned} d_f(S_p^n(E_\theta)) &\leq d_f(M_n(E_\theta))^{1-\theta} d_f(S_1^n(E_1))^\theta \\ &\leq d_f(M_n \otimes_{\min} E_\theta)^{1-\theta} d_f(S_1^n \widehat{\otimes} E_1)^\theta \\ &\leq d_f(E_\theta)^{1-\theta} d_f(E_1)^\theta \end{aligned}$$

since

$$d_f(M_n \otimes_{\min} E_\theta) = d_f(E_\theta)$$

by Lemma 1.1 and

$$\begin{aligned} d_f(S_1^n \widehat{\otimes} E_1) &= d_f(M_n \otimes_{\min} E_1^*) \\ &= d_f(E_1^*) \\ &= d_f(E_1) \end{aligned}$$

by applying successively (2). This gives $d_f(S_p(E_\theta)) \leq d_f(E_\theta)^{1-\theta} d_f(E_1)^\theta$ (again by Remark 2.8). On the other hand, the spaces l_p and $l_p(E_\theta)$ can be viewed as the diagonals of S_p and $S_p(E_\theta)$ respectively (cf. [22]). Whence, $d_f(l_p) = 1$ and $d_f(l_p(E_\theta)) \leq d_f(E_\theta)^{1-\theta} d_f(E_1)^\theta$. \square

Let E, F be two operator spaces. As a Banach space, $E \oplus_\infty F$ denotes as usual the direct sum of E and F in the sense of l_∞ . As an operator space, it is equipped with the operator space structure corresponding to the identification between $\mathcal{K} \otimes_{\min} (E \oplus F)$ and $(\mathcal{K} \otimes_{\min} E) \oplus_\infty (\mathcal{K} \otimes_{\min} F)$. $E \oplus_1 F$ is the operator space, subspace of the standard dual $(E^* \oplus_\infty F^*)^*$. And for $1 < p < \infty$, the direct sum of E and F in the l_p -sense is by definition

$$E \oplus_p F = (E \oplus_\infty F, E \oplus_1 F)_\theta, \theta = \frac{1}{p}.$$

This definition was introduced in [22] more generally, for a family of operator spaces.

Proposition 2.10. *For any operator spaces E, F we have*

$$d_f(E \oplus_p F) = \max\{d_f(E), d_f(F)\}.$$

Proof. Since the direct sum \oplus_p is injective (cf. [22]), we can reduce to f.d. operator spaces. In this case, the equality of Proposition 2.10 follows from Proposition 2.1, the equality (5) and the stability of $d_f(\cdot)$ by duality. \square

Remark. For $1 \leq p < \infty$, Proposition 2.10 can be easily extended to a general family of operator spaces $(E_i)_{i \in I}$. Namely, if $l_p(E_i)_{i \in I}$ denotes the direct sum of the spaces E_i in the l_p -sense as defined in [22], we have similarly

$$d_f(l_p(E_i)_{i \in I}) = \sup_{i \in I} \{d_f(E_i)\}.$$

Note however that this generalization fails for $p = \infty$. For instance, B has a completely isometric embedding into the direct sum $\oplus_n M_n$ (in the sense of l_∞) where the M_n 's are the $n \times n$ matrices and $d_f(M_n) = 1$ for all n , but $d_f(B) = \infty$.

3. Some remarks on the C^* -algebras with WEP

Recall that a C^* -algebra A is called *WEP* if the inclusion $A \rightarrow A^{**}$ factors completely positively and contractively through $B(H)$. Actually, by a recent result of Haagerup [15], it suffices that the inclusion $A \rightarrow A^{**}$ factors completely boundedly through $B(H)$. Equivalently, A is *WEP* if it has the extension property for the max tensor product, meaning that for any C^* -algebras B, C with C containing A as a C^* -subalgebra, the inclusion map $A \otimes_{\max} B \subset C \otimes_{\max} B$ is isometric. It is not hard to see that a C^* -algebra A is *WEP* if and only if the unitization of A is *WEP*.

A C^* -algebra A is said to be *approximately injective* if for any pair $S_0 \subset S_1$ of f.d. operator systems, any completely positive map $u_0 : S_0 \rightarrow A$ can be nearly extended to a completely positive map $u_1 : S_1 \rightarrow A$, meaning $\|u_1|_{S_0} - u_0\| < \epsilon$. In [10] where this property was introduced, it is proved that a unital C^* -algebra A is approximately injective if and only if for any pair $S_0 \subset S_1$ of f.d. operator systems, any unital completely positive map $u_0 : S_0 \rightarrow A$ and any $\epsilon > 0$, there exists a unital completely positive map $u_1 : S_1 \rightarrow A$ with $\|u_1|_{S_0} - u_0\| < \epsilon$. Moreover, a C^* -algebra is approximately injective if and only if its unitization is approximately injective.

As shown in [10], any approximately injective C^* -algebra has the *WEP*. But the converse is definitely not true. Indeed, by the equivalence of the conjectures (A2) \Leftrightarrow (A4) of [17] and the results of [16], there exists a *WEP* C^* -algebra which is not approximately injective. Actually, there exists a unital one, since the *WEP* and the approximate injectivity are shared simultaneously between any C^* -algebra and its unitization.

Various extension properties of *WEP* C^* -algebras have been studied in [5] and [18]. In particular, in [5], an extension property for a *WEP* C^* -algebra A is proved for completely positive maps of the form $u : S \rightarrow A$ defined on a subspace S of M_n . The goal of this section is to show that an extension property is valid more generally in the completely bounded setting for maps defined on a subspace $S \subset E$ of a f.d. operator space E with $d_f(E) = 1$. We should point out however that this extension property

cannot be true for arbitrary pairs $S \subset E$ of f.d. operator spaces. Indeed, this would imply by Lemma 3.1 that any unital WEP C^* -algebra is approximately injective which is not true.

Lemma 3.1. *If a unital C^* -algebra A satisfies the following property: for any f.d. operator spaces $E_0 \subset E_1$, any map $u_0 : E_0 \rightarrow A$ and any $\epsilon > 0$, there exists a map $u_1 : E_1 \rightarrow A$ extending u_0 with $\|u_1\|_{cb} \leq (1 + \epsilon)\|u_0\|_{cb}$, then A is approximately injective.*

Proof. Let $E_0 \subset E_1$ be f.d. operator systems and $u_0 : E_0 \rightarrow A$ a completely positive map with $u_0(1) = 1$. Then $\|u_0\|_{cb} = 1$. Given $\epsilon > 0$, there exists $u_1 : E_1 \rightarrow A$ an extension of u_0 such that $\|u_0\|_{cb} \leq 1 + \epsilon$ by our assumption on A . Such a map can be assumed self-adjoint. If not, it can be replaced by $\frac{1}{2}(u_1 + u_1^*)$ where $u_1^*(x) = (u_1(x^*))^* \forall x \in E_1$. Theorem 2.5 of [10] implies that u_1 is nearly completely positive, meaning that there exists a completely positive map $u : E_1 \rightarrow A$ such that $u(1) = 1$ and $\|u - u_1\| < f(\epsilon)$ where $f(\epsilon)$ tends to 0 when ϵ does. Finally, u is the approximate extension needed for u_0 . □

Proposition 3.2. *Let A be a WEP C^* -algebra, $E_0 \subset E_1$ a pair of f.d. operator spaces and $\epsilon > 0$. Then, any operator $u : E_0 \rightarrow A$ has an extension $\tilde{u} : E_1 \rightarrow A$ with*

$$\|\tilde{u}\|_{cb} \leq (1 + \epsilon)d_f(E_1)\|u\|_{cb}.$$

Proof. A has the WEP. Hence, we can find contractive completely positive maps, $a : A \rightarrow B(H)$, $b : B(H) \rightarrow A^{**}$ such that $j_A = ba$, where j_A is the canonical injection of A into A^{**} . Assume E_1^* is a subspace of $C^*(F_\infty)$. Let $j : E_1^* \rightarrow C^*(F_\infty)$ be the canonical inclusion map. We want to prove that the map

$$\begin{aligned} CB(E_1, A) &\rightarrow CB(E_0, A) \\ T &\mapsto T|_{E_0} \end{aligned}$$

is a metric surjection. Equivalently, the closure of the image of the open unit ball of $CB(E_1, A)$ contains the open unit ball of $CB(E_0, A)$. We suppose given $u : E_0 \rightarrow A$ with $\|u\|_{cb} < 1$. By the operator space version of the Hahn–Banach theorem (cf. [19]), we may extend the operator au to an operator $v : E_1 \rightarrow B(H)$ with $\|v\|_{cb} = \|au\|_{cb} < 1$. Then the map bv is an “extension” of u with the same *c.b.* norm, but taking its values in A^{**} .

The maps

$$id_{C^*(F_\infty)} \otimes b : C^*(F_\infty) \otimes_{\max} B(H) \rightarrow C^*(F_\infty) \otimes_{\max} A^{**}$$

and the inclusion

$$C^*(F_\infty) \otimes_{\max} A^{**} \subset (C^*(F_\infty) \otimes_{\max} A)^{**}$$

are clearly completely contractive. Denote by $\hat{v} \in E_1^* \otimes B(H)$, $\widehat{bv} \in E_1^* \otimes A^{**}$ the tensors corresponding to the operators v and bv respectively. Then,

$$\begin{aligned} \|j \otimes id_{B(H)}(\widehat{bv})\|_{(C^*(F_\infty) \otimes_{\max} A)^{**}} &\leq \|j \otimes id_{B(H)}(\hat{v})\|_{C^*(F_\infty) \otimes_{\max} A^{**}} \\ &= \|(j \otimes id_{B(H)})(id_{E_1^*} \otimes b)(\hat{v})\|_{C^*(F_\infty) \otimes_{\max} A^{**}} \\ &= \|(id_{C^*(F_\infty)} \otimes b)(j \otimes id_{B(H)})(\hat{v})\|_{C^*(F_\infty) \otimes_{\max} A^{**}} \\ &\leq \|j \otimes id_{B(H)}(\hat{v})\|_{C^*(F_\infty) \otimes_{\max} B(H)}. \end{aligned}$$

And using (3), we get

$$\begin{aligned} \|j \otimes id_{B(H)}(\widehat{bv})\|_{(C^*(F_\infty) \otimes_{\max} A)^{**}} &= \|j \otimes id_{B(H)}(\hat{v})\|_{C^*(F_\infty) \otimes_{\min} B(H)} \\ &= \|\hat{v}\|_{E_1^* \otimes_{\min} B(H)} \\ &= \|v\|_{cb} \\ &< 1. \end{aligned}$$

Let $E_0^* \otimes A, E_1^* \otimes A$ be equipped, each with the norm induced by its embedding in $C^*(F_\infty) \otimes_{\max} A$ for the sequel. The previous inequalities give

$$\|\widehat{bv}\|_{(E_1^* \otimes A)^{**}} < 1.$$

Hence, there exists a net $(\hat{f}_i)_{i \in I}$ of elements of $E_1^* \otimes A$ such that $\|\hat{f}_i\|_{E_1^* \otimes A} < 1$ and $\hat{f}_i \rightarrow \widehat{bv}$ for the topology $\sigma((E_1^* \otimes A)^{**}, (E_1^* \otimes A)^*)$. Equivalently, for any x in E_1 , $f_i(x) \rightarrow bv(x)$ for $\sigma(A^{**}, A^*)$, where the f_i 's are the operators associated to the \hat{f}_i 's. When x is in E_0 , $bv(x) = u(x)$ hence, $f_i(x) \rightarrow u(x)$ for $\sigma(A, A^*)$. In terms of tensors, this means since E_0 is f.d., that $q \otimes id_A(\hat{f}_i) \rightarrow \hat{u}$ for the topology $\sigma((E_0^* \otimes A), (E_0^* \otimes A)^*)$, where $q : E_1^* \rightarrow E_0^*$ is the restriction map. Then, using Mazur's theorem, we may find in the convex set generated by $(q \otimes id_A(\hat{f}_i))_i$ a net $(q \otimes id_A(\hat{g}_i))_i$ converging to \hat{u} for the norm topology of $E_0^* \otimes A$. Equivalently, $\|g_i - u\| \rightarrow 0$ where as expected, g_i is the operator corresponding to the tensor \hat{g}_i . Note that the \hat{g}_i 's are in the convex set generated by the \hat{f}_i 's, hence writing each \hat{g}_i as a finite sum $\sum_k \alpha_{i,k} \hat{f}_k$ where $0 \leq \alpha_{i,k} \leq 1$ and $\sum_k \alpha_{i,k} = 1$, we get

$$\begin{aligned}
 \|g_{i|E_0}\|_{cb} &= \|q \otimes id_A(\hat{g}_i)\|_{E_0^* \otimes_{\min} A} \\
 &\leq \sum_k \alpha_{i,k} \|q \otimes id_A(\hat{f}_k)\|_{E_0^* \otimes_{\min} A} \\
 &\leq \sum_k \alpha_{i,k} \|\hat{f}_k\|_{E_1^* \otimes_{\min} A} \\
 &\leq \sum_k \alpha_{i,k} \|\hat{f}_k\|_{E_1^* \otimes A} \\
 &< \sum_k \alpha_{i,k} = 1.
 \end{aligned}$$

Note also that the finite dimension of E_0 guarantees that the spaces $E_0^* \hat{\otimes} A$ (the injective tensor product in the Banach space category) and $E_0^* \otimes_{\min} A$ are isomorphic. Hence, we have $\|g_{i|E_0} - u\|_{cb} \rightarrow 0$. And now, we are done when E_1^* lives in $C^*(F_\infty)$. If it is not the case, choose F_1 in $C^*(F_\infty)$ f.d., $T : F_1 \rightarrow E_1^*$ such that $\|T\|_{cb} \|T^{-1}\|_{cb} \leq (1 + \epsilon)d_f(E_1)$ and let $F_0 = T^*(E_0) \subset F_1^*$. Then, consider $v = u|_{F_0} : F_0 \rightarrow A$. Now, v can be extended to an operator $\tilde{v} : F_1^* \rightarrow A$ with $\|\tilde{v}\|_{cb} \leq (1 + \epsilon)\|v\|_{cb}$. Set $\tilde{u} = \tilde{v}T^*$. This is an extension of u with norm

$$\begin{aligned}
 \|\tilde{u}\|_{cb} &\leq \|\tilde{v}\|_{cb} \|T^*\|_{cb} \\
 &\leq (1 + \epsilon)\|v\|_{cb} \|T\|_{cb} \\
 &\leq (1 + \epsilon)\|u\|_{cb} \|T\|_{cb} \|T^{-1}\|_{cb} \\
 &\leq (1 + \epsilon)^2 d_f(E_1) \|u\|_{cb}. \quad \square
 \end{aligned}$$

Corollary 3.3. ((17)) *Let $E_0 \subset E_1$ be f.d. operator systems with $d_f(E_1) = 1$, A be a WEP C^* -algebra. Then for any unital completely positive map $u : E_0 \rightarrow A$ and any $\epsilon > 0$, there exists a unital completely positive map $\hat{u} : E_1 \rightarrow A$ such that $\|\hat{u}|_{E_0} - u\| < \epsilon$.*

Proof. This is a direct consequence of Proposition 3.2 above and Theorem 2.5 of [10]. Hence, Proposition 3.2 appears as a generalization of Lemma 2.5 in [17]. □

Corollary 3.4. *Let X be an operator space with $d_f(X) = 1$, $E_0 \subset X$ a f.d. subspace and A a WEP C^* -algebra. Then, for any operator $u_0 : E_0 \rightarrow A$ and any $\epsilon > 0$, there exists an operator $u : X \rightarrow A$ extending u_0 with $\|u\|_{cb} \leq (1 + \epsilon)\|u_0\|_{cb}$.*

Proof. Assume X separable for simplicity. Let $E_0 \subset E_1 \subset E_2 \subset \dots$ be an increasing family of f.d. operator subspaces of X and let $(\epsilon_n)_n$ be a sequence of positive numbers satisfying $\prod_n (1 + \epsilon_n) < \infty$. By applying successively Proposition 3.2, there exists maps $u_{n+1} : E_{n+1} \rightarrow A$ extending u_n with $\|u_{n+1}\|_{cb} \leq (1 + \epsilon_n)\|u_n\|_{cb}$. Define $u : \bigcup_n E_n \rightarrow A$ by setting $u(x) = u_n(x)$ for $x \in E_n$. Clearly, u is an extension of u_0 with

$$\|u\|_{cb} \leq \prod_n (1 + \epsilon_n) \|u_0\|_{cb},$$

and we are done by a suitable choice of the families $(E_n)_n$ and $(\epsilon_n)_n$ namely, $X = \overline{\bigcup_n E_n}$ and $\prod_n (1 + \epsilon_n) < 1 + \epsilon$. \square

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