

# 3

## The harmonic oscillator and the quantum field

### 3.1 Introduction

In this and the next chapter we will consider some properties of quantum fields. The examples taken will be mostly scalar fields and only when necessary will we invoke the complexities stemming from the vector nature of the interactions in QED and QCD; there are many good text-books devoted to a detailed treatment of the subject.

We need only intuition and a set of understood formulas for the investigations contained in this book. We start with a discussion of the quantum mechanical harmonic oscillator coupled to an external force. There are several reasons to dwell on this particular system. Firstly its sine and cosine behaviour in time is matched by the corresponding harmonic behaviour of the plane wave solutions for the quanta in a field theory.

It was noted even in the first papers on quantum field theory that a free or weakly interacting quantum field is in a rather precise way a superposition of an infinite, although enumerable, set of harmonic oscillators, one for each degree of freedom.

A real interacting-field theory does not behave in this way with respect to its excitations. There is always, however, at the basis of any experiment in high-energy particle physics the idea of a three-act scenario in time.

- 1 In the first phase, a long time before the interaction, the initial states are prepared with production setups in general arranged so that each state is isolated.
- 2 After that there will be a more or less violent encounter in the second phase.
- 3 In the final state the produced quanta are observed by means of detectors placed far apart, a long time after the interaction.

Therefore the descriptions of the initial and final states are expected to correspond to the states of free non-interacting fields. For a confining theory like QCD this particular asymptotic before-and-after scenario does not hold but there is instead another asymptotics, the *asymptotic freedom* of the theory in which the free-field theories are expected to be relevant.

The second reason for considering both the free and the interacting harmonic oscillator is that from a mathematical point of view they correspond to very well-behaved systems. This is not the case in general for interacting quantum fields, which contain many different mathematical complications. But it turns out that almost all the things which can be done in a simple and precise way for the single harmonic oscillator can also, albeit after a large amount of cumbersome mathematics, be done for infinite-dimensional quantum fields. It is therefore easier to present the methods in a well-behaved manner for those who are not particularly interested in the mathematical complexities but nevertheless would like to understand what they are doing inside a computable framework.

After we have rehearsed the properties of interacting harmonic oscillators from an elementary quantum mechanical point of view we will exhibit the corresponding properties for a scalar quantum field. We will in particular consider quantum states which correspond as closely as possible to classical fields (coherent states). At the same time we will introduce the  $S$ -operator, which connects the initial- and final-state free fields, mentioned above as phases 1 and 3 in the interaction.

After that we consider interacting fields. It is then necessary to provide a more precise definition of the  $S$ -operator. We introduce the Feynman-Dyson prescription of time-ordering and, for simple cases, show how to make calculations in this framework. We consider the Feynman propagator and show its significance with regard to Heisenberg's indeterminacy requirements. We also calculate the scattering cross section in a simple situation. Finally we exhibit some features of the lightcone formulation of a field theory, often referred to as 'a field theory in the infinite-momentum frame'.

### 3.2 The quantum field as a sum of harmonic oscillators

This section will firstly contain a few reminders of the properties of the one-dimensional harmonic oscillator. After we have shown how the harmonic oscillator reacts to a time-dependent external force we discuss the corresponding properties of a scalar quantum field coupled to an external current.

In both cases we obtain a set of states called *coherent states*. They are the closest correspondence to classical behaviour which can be found for

simple quantum systems. Therefore they are often used as models for more complex situations. When we go from the single harmonic oscillator to quantum fields it will be necessary to introduce some cutoff procedures, which are used repeatedly in connection with the calculation of observables such as cross sections later in the book.

1 The one-dimensional harmonic oscillator

I *The equation of motion.* The (classical) equation of motion of a one-dimensional harmonic oscillator in an external field,  $j(t)$  is

$$m\ddot{x} + m\omega^2x = j(t) \tag{3.1}$$

Here the dot(s) correspond to time derivative(s) and the harmonic oscillator frequency  $\omega$  has been explicitly introduced.

Equation (3.1) can be derived from Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \tag{3.2}$$

where

$$H = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} - xj \tag{3.3}$$

II *The commutation relations.* Quantum considerations are introduced by means of the Heisenberg commutator relations

$$[p, x] = -i, \quad [p, p] = [x, x] = 0 \tag{3.4}$$

For the harmonic oscillator it is useful to introduce two adjoint operators  $a^*$  and  $a$ , usually referred to as the *raising* and *lowering* (or in more colorful language *creation* and *annihilation*) operators:

$$x = \frac{a + a^*}{\sqrt{2m\omega}}, \quad p = \frac{i\sqrt{m\omega}(a^* - a)}{\sqrt{2}} \tag{3.5}$$

Their commutation relations are obtained from Eq. (3.4):

$$[a, a^*] = 1, \quad [a, a] = [a^*, a^*] = 0 \tag{3.6}$$

III *The case of no disturbance.* For the case when  $j = 0$  the hamiltonian  $H \equiv H_0$  can be written as

$$H_0 = \omega \left( a^*a + \frac{1}{2} \right), \quad \langle \alpha | H_0 | \alpha \rangle \geq \frac{\omega}{2} \equiv E_0 \tag{3.7}$$

for any state  $|\alpha\rangle$ . There is a lowest energy eigenstate  $|0\rangle$  with an  $x$ -space representation,  $\psi_0(x)$ , obtained from the requirement  $a\psi_0(x) = 0$ , i.e.

$$\left( \sqrt{\frac{m\omega}{2}}x + \frac{1}{\sqrt{2m\omega}} \frac{d}{dx} \right) \psi_0(x) = 0, \quad \psi_0 = \sqrt{\frac{m\omega}{2\pi}} \exp\left(-\frac{m\omega x^2}{2}\right) \tag{3.8}$$

with  $\psi_0$  normalised to 1:  $\int dx |\psi_0(x)|^2 = 1$ . It obviously fulfils  $H|0\rangle = E_0|0\rangle$ .

IV *The excited states.* All other eigenstates of the hamiltonian are given by  $N_n(a^*)^n|0\rangle \equiv |n\rangle$ ; in an  $x$ -space representation these are polynomials of  $n$ th degree in  $x$  multiplying  $\psi_0$ . Using

$$[a, (a^*)^n] = n(a^*)^{(n-1)}, \quad [a^*a, (a^*)^n] = n(a^*)^n \quad (3.9)$$

the normalisation constant  $N_n$  can by iteration be shown to be

$$N_n = \frac{1}{\sqrt{n!}} \quad \text{so that} \quad |n\rangle = \frac{(a^*)^n}{\sqrt{n!}}|0\rangle \quad (3.10)$$

The corresponding eigenvalue is  $E_n = (n + 1/2)\omega$ .

V *Normal-ordering.* It is useful to introduce the notion of *normal-ordering*. This means that in an operator expression  $O$  containing both  $a$  and  $a^*$  operators the normal-ordered  $O$ , denoted  $:O:$ , contains all the  $a$ -operators to the right of the  $a^*$ -operators. In particular this means that  $\langle 0|:O:|0\rangle = 0$  if  $O$  contains a nonzero number of operators.

VI *The time dependence.* The time dependence of the operators  $a$  and  $a^*$  is found, in the Heisenberg picture (for  $j = 0$ ), as follows:

$$\begin{aligned} \frac{da}{dt} &= i[H_0, a] = -i\omega a \Rightarrow a(t) = a \exp(-i\omega t) \\ \frac{da^*}{dt} &= i[H_0, a^*] = i\omega a^* \Rightarrow a^*(t) = a^* \exp(i\omega t) \end{aligned} \quad (3.11)$$

We also note that the identification of the canonical momentum  $p$  with  $m\dot{x}$  is consistent with the time development:

$$p = mi[H_0, x] \quad (3.12)$$

VII *Time-independent disturbance.* When  $j$  is nonvanishing but *independent of time* the hamiltonian can be rewritten as

$$H = \frac{p^2}{2m} + \frac{m\omega^2(x - x_0)^2}{2} - \frac{m\omega^2 x_0^2}{2}, \quad x_0 = \frac{j}{m\omega^2} \quad (3.13)$$

We can then choose to re-express everything using a new coordinate  $x' = x - x_0$  and a new hamiltonian  $H' = H + j^2/(2m\omega^2)$  (where we have introduced the expression for  $x_0$  in the energy change).

The new ground-state wave function,  $\psi_{0j}$ , obviously corresponds to a translation of the old one:

$$\psi_{0j}(x) = \psi_0(x - x_0) = \left\{ \exp[-j^2/(2m\omega^3)] \right\} \left\{ \exp(xj/\omega) \right\} \psi_0(x) \quad (3.14)$$

and can therefore by a suitable expansion be expressed in terms of the the old set  $|n\rangle$  (this applies, of course, to any other state as well).

The exponential of an operator should be interpreted in terms of a power series expansion and can be handled in almost the same way as an ordinary exponential.

- We will use two simple properties of the operator  $T = \exp(jx/\omega)$  expressed in terms of the original operators  $a$  and  $a^*$ . In general if  $A$  and  $B$  are operators and if  $[A, B] = c$ ,  $c$  being an operator-independent constant (conventionally called a *c-number*),

$$\begin{aligned} (\exp A)B \exp(-A) &= B + [A, B] \\ (\exp A)(\exp B) \exp(-[A, B]/2) &= \exp(A + B) \end{aligned} \tag{3.15}$$

The first relation in Eq. (3.15) can be obtained from a Taylor series expansion of the function  $f(\lambda)$  around  $\lambda = 0$ , where

$$f(\lambda) = \exp(\lambda A)B \exp(-\lambda A) \tag{3.16}$$

Consider the derivatives of  $f$  (note the careful ordering!)

$$\frac{df}{d\lambda} = Af(\lambda) - f(\lambda)A = [A, f(\lambda)], \quad \frac{d^n f}{d\lambda^n} = [A, [A, \dots, [A, f(\lambda)] \dots]] \tag{3.17}$$

As  $f(\lambda = 0) = B$  we obtain that all but the first of the derivatives of  $f$  vanish at  $\lambda = 0$  :

$$f(\lambda) = B + \lambda[A, B] \tag{3.18}$$

The result in the first line in Eq. (3.15) then corresponds to  $\lambda = 1$ .

For the second equation in (3.15) consider the function  $g(\lambda)$ , where

$$g(\lambda) = \exp(\lambda A) \exp(\lambda B) \exp\{-\lambda(A + B)\} \tag{3.19}$$

Using the first equation in (3.15), we obtain for the derivative of  $g$  :

$$\frac{dg}{d\lambda} = \lambda[A, B]g(\lambda) \tag{3.20}$$

This is a differential equation with a plain number  $\lambda c$  in front of  $g$  on the right-hand side. We conclude that  $g$ , which is equal to 1 for  $\lambda = 0$  from its definition, is the following simple function:

$$g(\lambda) = \exp(c\lambda^2/2) \tag{3.21}$$

which again provides the expected result for  $\lambda = 1$ . Note that we have extensively used that the commutator of  $A$  and  $B$  is a plain number.

Setting  $A = ja^*/(\sqrt{2m\omega^3})$  and  $B = ja/(\sqrt{2m\omega^3})$  we obtain

$$[A, B] = -\frac{j^2}{2m\omega^3} \tag{3.22}$$

so that already expressed in a normal-ordered form the operator  $T$  becomes

$$T = \exp\left[j^2/(4m\omega^3)\right] \exp(ja^*/\sqrt{2m\omega^3}) \exp(ja/\sqrt{2m\omega^3}) \tag{3.23}$$

From Eq. (3.14) this means that

$$\psi_{0j} = \exp \left[ -j^2 / (4m\omega^3) \right] \exp(ja^* / \sqrt{2m\omega^3}) \psi_0 \quad (3.24)$$

or

$$|0j\rangle = \exp \left[ -j^2 / (4m\omega^3) \right] \sum_{n=0}^{\infty} \left( \frac{j}{\sqrt{2m\omega^3}} \right)^n \frac{1}{\sqrt{n!}} |n\rangle \quad (3.25)$$

Therefore the application of a constant force  $j$  to the harmonic oscillator will bring it into a new ground state with the property that the transition amplitudes will fulfil

$$|\langle n|0j\rangle|^2 = \frac{\bar{n}^n}{n!} \exp(-\bar{n}) \quad (3.26)$$

This corresponds to a Poisson distribution with the mean excitation  $\bar{n}$  given by

$$\bar{n} = \frac{j^2}{2m\omega^3} \quad (3.27)$$

This is, however, dynamically incorrect: there is no way to change the system unless we use a time-dependent scenario so that there is energy pumped in or out of the system.

VIII *A time-dependent scenario.* In order to describe an actual dynamical situation we assume that the force  $j$  introduced above is nonvanishing and changes in time,  $t$ , during a finite period  $t_1 < t < t_2$  so that we can talk about the situation ‘before’,  $t \leq t_1$ , and ‘after’,  $t \geq t_2$  (the ‘three-way-scenario’ mentioned before!). Then the hamiltonian will be

$$\begin{aligned} H &= \omega(a^*a + 1/2) - g(t)a - g^*(t)a^* \equiv H_0 + H_1 \\ H_1 &= -j(t)x = -g(t)a - g^*(t)a^* \end{aligned} \quad (3.28)$$

where we have written  $j \rightarrow g(t) = g^* = j(t)/\sqrt{2m\omega}$  in anticipation of a more general situation, when  $g$  is a complex function.

The equations of motion become

$$\frac{da}{dt} = i[H, a] = -i\omega a + ig^*(t), \quad \frac{da^*}{dt} = i[H, a^*] = i\omega a^* - ig(t) \quad (3.29)$$

We will assume that there are *initial-state* operators  $a_i(t)$ ,  $a_i^*(t)$ , which, like the operators in Eq. (3.11), describe the undisturbed system before  $t = t_1$  (when  $g(t) = g^*(t) = 0$  so that the equations of motion coincide) and likewise *final-state* operators  $a_f(t)$ ,  $a_f^*(t)$ , which describe the system after  $t = t_2$ .

Then the equations (3.29) can be solved in a general way by means of the Green’s function method. We define the functions  $G_R(t)$  and  $G_A(t)$  as

the solutions of the equation

$$\frac{dG}{dt} + i\omega G = \delta(t) \tag{3.30}$$

with boundary conditions

$$\begin{aligned} G_R(t) &= 0 & \text{if } t < 0 \\ G_A(t) &= 0 & \text{if } t > 0 \end{aligned} \tag{3.31}$$

They are called the retarded and the advanced *Green's function*, respectively, and are in this case rather easily constructed:

$$G_R(t) = \Theta(t) \exp(-i\omega t), \quad G_A(t) = -\Theta(-t) \exp(-i\omega t) \tag{3.32}$$

where  $\Theta$  is the Heaviside distribution, which is equal to 1 for a positive argument and vanishes elsewhere.

The fact that the solutions of Eq. (3.30) should correspond to step-functions at  $t = 0$  can be understood from an integration of the equation from  $t = -\epsilon$  to  $t = +\epsilon$  when  $\epsilon \rightarrow +0$ :

$$\lim_{\epsilon \rightarrow 0} \left[ G(\epsilon) - G(-\epsilon) + i\omega \int_{-\epsilon}^{\epsilon} dt G(t) \right] = 1 \tag{3.33}$$

Here we have used the following property of the  $\delta$ -distribution:  $\int dt \delta(t) = 1$ , if the integration region includes  $t = 0$ . The fact that the contribution from the integral in Eq. (3.33) vanishes as  $\epsilon$  is left for the reader to prove.

In this way we obtain the following solutions for  $a(t)$ :

$$\begin{aligned} a(t) &= a_i(t) + i \int_{-\infty}^t dt' g^*(t') \exp[-i\omega(t-t')] \\ a(t) &= a_f(t) - i \int_t^{\infty} dt' g^*(t') \exp[-i\omega(t-t')] \end{aligned} \tag{3.34}$$

Therefore the final-state operators can be expressed in terms of the initial-state ones by a translation (noting that they all have the trivial time dependence  $\exp(\pm i\omega t)$ , which can be divided away):

$$a_f = a_i + i \int_{-\infty}^{\infty} dt' g^*(t') \exp(i\omega t') \equiv a_i + i\tilde{g}^*(\omega) \tag{3.35}$$

Consequently the final-state operators,  $a_f, a_f^*$ , in a similar way to VII above have been translated with respect to the initial ones,  $a_i, a_i^*$ , this time, however, by the Fourier transform of the force!

IX *The S-operator.* It is possible to construct a *unitary operator S*, which transforms the initial states into the final states in this simple situation:

$$\begin{aligned} S^* S &= 1 \Leftrightarrow S^{-1} = S^*, & a_f &= S^* a_i S = a_i + i\tilde{g}^*(\omega) \\ S^* |0i\rangle &= |0f\rangle & \Leftrightarrow & S |0f\rangle = |0i\rangle \end{aligned} \tag{3.36}$$

(note that this also fixes the relation between  $a_j^*$  and  $a_i^*$ !). The operator  $S$  provides a complete mapping of the eigenstates of the final system onto the initial eigenstates:

$$|nf\rangle = S^* |ni\rangle \quad (3.37)$$

It is easy to find by means of the results we have obtained in VII and VIII:

$$S = \exp \{i[\tilde{g}^*(\omega)a_i^* + \tilde{g}(\omega)a_i]\} \quad (3.38)$$

The expression in the exponent can be neatly reformulated by noting that

$$\begin{aligned} \int_{-\infty}^{\infty} dt j(t)x(t) &= \int_{-\infty}^{\infty} dt [g(t)a_i(t) + g^*(t)a_i^*(t)] \\ &= \int_{-\infty}^{\infty} dt [a_i g(t) \exp(-i\omega t) + a_i^* g^*(t) \exp(i\omega t)] \\ &= \tilde{g}(\omega)a_i + \tilde{g}^*(\omega)a_i^* \end{aligned} \quad (3.39)$$

Then the  $S$ -operator can be expressed as

$$S = \exp \left\{ i \int_{-\infty}^{\infty} dt [g(t)a_i(t) + g^*(t)a_i^*(t)] \right\} = \exp i \int_{-\infty}^{\infty} dt j(t)x(t) \quad (3.40)$$

This is a general result in the perturbative treatments of quantum field theory, which holds also when  $j$  is an operator-valued function. We obtain the (negative) difference between the operator  $H$  in Eq. (3.28) and the ‘free’ harmonic oscillator hamiltonian  $H_0$  in Eq. (3.2), integrated over time, as the exponent in the expression for the  $S$ -operator.

In this more general case the exponential must be treated with care because operators for different times have complicated commutation relations. One cannot without a prescription for *ordering* use the ordinary exponential property that the exponent of a sum is equal to the product of the exponents of the terms in the sum.

X *The transition probabilities.* For the case when  $j$  is an external ‘nice’ function ‘real’ transitions are possible. An original state such as the initial ground state,  $|0i\rangle$ , will afterwards become some outgoing, possibly excited, state:

$$\langle nf|0i\rangle = \langle ni|S|0i\rangle = \left[ \exp \left( -\frac{|\tilde{g}|^2}{2} \right) \right] \frac{(\tilde{g}^*)^n}{\sqrt{n!}} \quad (3.41)$$

In VII we presented the transition probabilities  $|\langle nf|0i\rangle|^2$  as a Poisson distribution in the free harmonic oscillator states. This is evidently still true and the mean excitation level,  $\bar{n}$ , for the Poissonian will be for the general case:

$$\bar{n} = \frac{1}{2m\omega} \left| \int dt j(t) \exp(i\omega t) \right|^2 \quad (3.42)$$

The result in the case (3.27) is characteristic for a single sudden change in the force. A suitable force (corresponding to a limiting situation when  $\epsilon > 0$  approaches 0 after the integral has been performed) would be

$$j(t) = j \exp(-\epsilon t) \Theta(t). \tag{3.43}$$

Before we go over to quantum fields we note another property of the states. The state  $|0i\rangle$  is actually an *eigenstate of the operator  $a_f$* :

$$\begin{aligned} a_f |0i\rangle &= a_f S |0f\rangle = a_f \sum_{n=1}^{\infty} \left[ \exp\left(-\frac{|\tilde{g}|^2}{2}\right) \right] \frac{(\tilde{g}^*)^n}{\sqrt{n!}} |nf\rangle \\ &= \tilde{g}^* S |0f\rangle = \tilde{g}^* |0i\rangle \end{aligned} \tag{3.44}$$

This also implies that the expectation value in the initial ground state of the final-state operator  $x_f(t) = [a_f \exp(-i\omega t) + a_f^* \exp(i\omega t)]/(\sqrt{2m\omega})$  is

$$\begin{aligned} \hat{x}(t) &= \langle 0i | x_f(t) | 0i \rangle = \frac{\tilde{g}^* \exp(-i\omega t) + \tilde{g} \exp(i\omega t)}{\sqrt{2m\omega}} \\ &= \int_{-\infty}^{\infty} dt' \frac{1}{m\omega} j(t') \cos[\omega(t' - t)] \end{aligned} \tag{3.45}$$

This is the final-state harmonic motion in a classical mechanics situation when one starts out with a harmonic oscillator at rest and then applies the external force  $j(t)$  over a finite time interval  $t_1 < t' < t_2$ . Evidently the integrand in Eq. (3.45) is only nonvanishing over this time region and we consider  $t > t_2$ .

In order to prove (3.45) it should be noted that the equations of motion in Eqs. (3.2) and (3.29) also work classically for the quantities  $a, a^*$  defined in Eqs. (3.5). The whole formalism involving Green's functions that relate the initial-state and final-state quantities  $a_i, a_i^*$  and  $a_f, a_f^*$  is just as valid when the  $a$ 's and  $a^*$ 's are classical c-numbers!

## 2 A scalar quantum field coupled to an external current

We will now consider the corresponding situation for a scalar quantum field  $\phi(x)$ . We will firstly show that it has the same behaviour as a superposition of an infinite number of independent harmonic oscillators. It will then follow that we can take over everything we have done in I to X when we treat  $\phi(x)$ . Every time one introduces an infinity, however, it is necessary to worry a little about convergence problems. We will soon find that there are plenty of such things to worry about when we go to interacting quantum fields!

XI *The Klein–Gordon equation.* A scalar field,  $\phi(\mathbf{x}, t)$ , which fulfils the Klein–Gordon equation

$$(\square + M^2)\phi \equiv \ddot{\phi} - \Delta\phi + M^2\phi = j \tag{3.46}$$

where stated earlier the Laplacian  $\Delta = \nabla^2$  is given by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \quad (3.47)$$

will, in momentum space,  $\phi \rightarrow \tilde{\phi}(t) \exp(i\mathbf{x} \cdot \mathbf{k})$ , fulfil the equation

$$\ddot{\tilde{\phi}} + \omega^2(\mathbf{k})\tilde{\phi} = \tilde{j} \quad (3.48)$$

This essentially coincides with Eq. (3.1) for the single harmonic oscillator with frequency  $\omega \rightarrow \omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + M^2}$ .

In order to facilitate this transfer to momentum space we assume that the whole system is enclosed in a large box with three space dimensions and volume  $V$ , and that only those waves that fit into the box with periodic boundary conditions are included. This means that instead of a field  $\phi$  defined at every space point we obtain an *enumerable set of amplitude fields for the momentum-space waves*.

The allowed momenta, e.g. in the 1-direction with a large box-length  $L_1$ , are, for any integer  $n_1$ ,

$$k_{1,n_1} = \frac{n_1 2\pi}{L_1} \quad (3.49)$$

A sum over  $n_1$  can be made into an integral over  $dk_1$  by the formal exchange (which is valid when we sum and integrate over 'nice' functions)

$$\sum_{n_1} \rightarrow \int dn_1 = \frac{L_1}{2\pi} \int dk_1 \Rightarrow \sum_{n_1, n_2, n_3} \rightarrow \frac{V}{(2\pi)^3} \int d^3k \quad (3.50)$$

With this construction we have the following identities

$$\int_V d^3x \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{x}] = V \delta_{\mathbf{k}, \mathbf{k}'}$$

$$\sum_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x}) = V \delta(\mathbf{x}) \quad (3.51)$$

In the first equation the symbol on the right-hand side is equal to 1 when the two arguments coincide and vanishes elsewhere. The second equation contains the usual  $\delta$ -distribution in three dimensions.

The results in Eq. (3.51) stem directly from Fourier analysis and correspond to the orthonormality and completeness relations of Fourier waves. We will later see that in all formulas describing physical observables the volume  $V$  will disappear.

XII The *hamiltonian formulation*. The field equation can also be described by a variation of the hamiltonian  $H$  in which  $\phi(x)$ ,  $\Pi(x)$  are the canonical

coordinates at every space point  $\mathbf{x}$ :

$$\begin{aligned}
 H_0 &= \frac{1}{2} \int_V d^3x [\Pi^2 + (\nabla\phi)^2 + M^2\phi^2] \\
 H &= H_0 + H_1
 \end{aligned}
 \tag{3.52}$$

with

$$H_1 = - \int_V d^3x j(\mathbf{x}, t)\phi(\mathbf{x}) \equiv \int_V d^3x \mathcal{H}_1
 \tag{3.53}$$

The fields  $\Pi$  and  $\phi$  can be decomposed as sums over the different momentum components similar to the single harmonic oscillator in Eq. (3.5):

$$\begin{aligned}
 \phi &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega(\mathbf{k})}} (a(\mathbf{k}) \exp[i\mathbf{k} \cdot \mathbf{x}] + a^*(\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{x})) \\
 \Pi &= \sum_{\mathbf{k}} \frac{i\sqrt{\omega(\mathbf{k})}}{\sqrt{2V}} [-a(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) + a^*(\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{x})]
 \end{aligned}
 \tag{3.54}$$

We note that the field  $\phi$  in this way is written as a set of harmonic oscillators (cf. Eq. (3.5))  $\mathbf{x} = \sum_j (1/\sqrt{2m\omega})(a_j + a_j^*)\mathbf{e}_j$ , although this time the (euclidean) vectors  $\mathbf{e}_j$  (with  $\mathbf{e}_j \cdot \mathbf{e}_m = \delta_{jm}$ ) are exchanged for the normalised eigenfunctions  $\exp(\pm i\mathbf{k} \cdot \mathbf{x})/\sqrt{V}$ , which are vectors in a Hilbert space, i.e. an infinite-dimensional generalisation of a euclidean space. This also implies that the field  $\phi$  has energy dimension  $\dim \phi = 1$  (corresponding to a negative length dimension  $-1$ ). We will use similar dimensional arguments many times later in the book.

This dimensional assignment for  $\phi$  is necessary in order that the hamiltonian  $H_0$  in Eq. (3.52) should also have energy dimension 1 ( $\dim d^3x = -3$ ,  $\dim M^2 = 2$  and  $\dim \nabla = 1$ ). In the same way we conclude that for  $H_1$  to have energy dimension 1 the current  $j$  must have  $\dim j = 3$ .

It is straightforward to prove that the commutation relations

$$[a(\mathbf{k}), a^*(\mathbf{k}')] = \delta_{\mathbf{k}, \mathbf{k}'}, \quad [a(\mathbf{k}), a(\mathbf{k}')] = [a^*(\mathbf{k}), a^*(\mathbf{k}')] = 0
 \tag{3.55}$$

imply

$$[\Pi(\mathbf{x}), \phi(\mathbf{x}')] = -i\delta(\mathbf{x} - \mathbf{x}'), \quad [\phi(\mathbf{x}), \phi(\mathbf{x}')] = [\Pi(\mathbf{x}), \Pi(\mathbf{x}')] = 0
 \tag{3.56}$$

if we use Eqs. (3.55), (3.50) and (3.51). The sets of commutation relations in Eqs. (3.55) and (3.56) are thus equivalent and are obvious generalisations of the harmonic oscillator relations in Eqs. (3.6) and (3.4).

XIII *The ground-state energy.* For an undisturbed set of harmonic oscillators the hamiltonian in terms of operators is

$$H_0 = \sum_{\mathbf{k}} \omega(\mathbf{k}) [a^*(\mathbf{k})a(\mathbf{k})] + C
 \tag{3.57}$$

The constant  $C$  corresponds to the sum of the energies of all the zero-point modes of the oscillators, i.e.  $C = \sum_{\mathbf{k}} \omega(\mathbf{k})/2$ . In that way it is simply the energy of the vacuum and is consequently not an observable quantity.

There are, however, situations when *the difference in energy between the 'total' vacuum fluctuations in  $C$  and those from a particular boundary configuration can be measured*, [41]. This effect is outside the scope of this book. It is, nevertheless, of great interest because it exhibits *experimentally* the existence of quantum field fluctuations in the vacuum state.

XIV *The time dependence.* To obtain the time dependence we use the same relations as in Eqs. (3.11) and (3.12):

$$\begin{aligned} \frac{da_{\mathbf{k}}}{dt} &= i[H_0, a_{\mathbf{k}}] = -i\omega(\mathbf{k})a_{\mathbf{k}} \Rightarrow a_{\mathbf{k}}(t) = a_{\mathbf{k}} \exp[-i\omega(\mathbf{k})t] \\ \Pi(\mathbf{x}) &= \dot{\phi} = i[H_0, \phi(\mathbf{x})] \end{aligned} \quad (3.58)$$

In this way  $\phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}, t)$  by including the time dependence of the  $a$ - and  $a^*$ -operators. We note in passing that this will result in Lorentz-invariant exponential factors  $\exp \pm(i\mathbf{k} \cdot \mathbf{x} - \omega t) = \exp \mp(ik_{\mu}x^{\mu}) \equiv \exp(\mp ikx)$  multiplying the  $a$ - and  $a^*$ -operators.

When the current  $j$  is nonvanishing the time dependences will take on the form of Eqs. (3.29):

$$\begin{aligned} \frac{da_{\mathbf{k}}}{dt} &= i[H, a_{\mathbf{k}}] = -i\omega(\mathbf{k})a_{\mathbf{k}} + ig^*(\mathbf{k}, t) \\ \frac{da_{\mathbf{k}}^*}{dt} &= i[H, a_{\mathbf{k}}^*] = i\omega(\mathbf{k})a_{\mathbf{k}}^* - ig(\mathbf{k}, t) \end{aligned} \quad (3.59)$$

$$g(\mathbf{k}, t) = \int_V d^3x \frac{1}{\sqrt{2V\omega(\mathbf{k})}} j(\mathbf{x}, t) \exp(i\mathbf{k} \cdot \mathbf{x}) \quad (3.60)$$

Thus here  $g(t) \rightarrow g(\mathbf{k}, t)$ , the Fourier transform of the external current. This means that the numbers  $g(\mathbf{k}, t)$  are in general complex but for a real-valued current  $j(x)$  they fulfil  $g^*(\mathbf{k}, t) = g(-\mathbf{k}, t)$ .

All these steps from the definition of the Green's functions to the resulting equation for the  $S$ -operator in Eqs. (3.30) to (3.40) can then be performed separately for each wavenumber  $\mathbf{k}$ . The final  $S$ -operator is a product over all components and can be written as

$$S = \exp \left[ -i \int_{-\infty}^{\infty} dt H_{1i}(t) \right] = \exp \left[ i \int d^4x \phi_i(x) j(x) \right] \quad (3.61)$$

The index  $i$  is introduced in order to stress that we are using the initial-state fields, i.e. those that describe the state a long time before the interaction is turned on. The time dependence in  $H_{1i}(t)$  contains also the free-field time dependence of the oscillators so that  $a_i(\mathbf{k})$  is changed into  $a_i(\mathbf{k}) \exp(-i\omega(\mathbf{k})t)$ . The integration symbol  $\int d^4x = \int_V d^3x \int_{-\infty}^{\infty} dt$ .

An interesting observable is the probability that the vacuum before the

interaction is turned on (the no-quanta state) is still the vacuum after the interaction, i.e. the probability that there has been no excitation due to the onset of the current  $j$

$$|\langle 0f|0i\rangle|^2 = |\langle 0i|S|0i\rangle|^2 = \exp(-U) \tag{3.62}$$

$$U = \sum_{\mathbf{k}} \frac{1}{2\omega(\mathbf{k})V} \left| \int_V d^3x dt j(\mathbf{x}, t) \exp[i\omega(\mathbf{k})t - \mathbf{k}\mathbf{x}] \right|^2$$

The quantity  $U$  is the sum over all the mean excitations for the Poisson-distributed oscillators (cf. Eq. (3.42)). It can be rearranged by changing the sum over  $\mathbf{k}$  to an integral, see Eq. (3.50); we then arrive at (with the vector  $\delta x = (t - t', \mathbf{x} - \mathbf{x}')$ )

$$U = \int \frac{d^3k}{2(2\pi)^3\omega(\mathbf{k})} \int d^4x d^4x' j(x)j(x') \exp\{i[\omega(\mathbf{k})(\delta t) - \mathbf{k}(\delta\mathbf{x})]\}$$

$$= \int dx dx' j(x)\Delta_+(\delta x)j(x') \tag{3.63}$$

$$\Delta_{\pm}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(\mathbf{k})} \exp\{i[\pm\omega(\mathbf{k})t - \mathbf{k} \cdot \mathbf{x}]\} \tag{3.64}$$

We firstly note that the volume  $V$  has vanished from these expressions (when we have taken the limit  $V \rightarrow \infty$  we use the symbol  $dx$  instead of  $d^4x$ ). Secondly we note that the functions  $\Delta_{\pm}$  defined in the last line of Eq. (3.63) are Lorentz-invariant. In order to show that we use the following property of the  $\delta$ -distribution:

$$\int dadb\Theta(\pm a)\delta(a^2 - b^2)f(a, b)$$

$$= \int dadb\Theta(\pm a)\frac{[\delta(a - |b|) + \delta(a + |b|)]f(a, b)}{2|b|}$$

$$= \int \frac{dbf(\pm|b|, b)}{2|b|} \tag{3.65}$$

For Eq. (3.64) we have

$$\int \frac{d^3k}{2\omega(\mathbf{k})} f(\mathbf{k}, \pm\omega(\mathbf{k})) = \int dk\delta^{\pm}(k^2 - M^2)f(\mathbf{k}, k_0) \tag{3.66}$$

where the symbols  $dk \equiv d^3kdk_0$  and  $\Theta(\pm k_0)\delta(k_0^2 - \mathbf{k}^2 - M^2) \equiv \delta^{\pm}(k^2 - M^2)$  will be used from now on. (Note that the prescription  $k_0 > 0$  is Lorentz-invariant together with the  $\delta$ -distribution!)

Thus the functions  $\Delta_{\pm}$  become (changing  $\mathbf{k}$  to  $-\mathbf{k}$  for  $\Delta_-$ ):

$$\Delta_{\pm}(x) = \frac{1}{(2\pi)^3} \int dk\delta^{\pm}(k^2 - M^2) \exp(ikx) \tag{3.67}$$

The distribution  $\Delta_+(x)$  actually corresponds to the matrix element

$$\begin{aligned} \langle 0i|\phi_i(x_1)\phi_i(x_2)|0i\rangle &= \langle 0i|\phi_i^-(x_1)\phi_i^+(x_2)|0i\rangle \\ &= \sum_{\mathbf{k}} \frac{\exp[i\mathbf{k}(x_2 - x_1)]}{2V\omega(k)} = \Delta_+(x_2 - x_1) \end{aligned} \quad (3.68)$$

We have here introduced the notation  $\phi = \phi^- + \phi^+$  where we include the sum of all the  $a$ -operators ( $a^*$ -operators) in  $\phi^-$  ( $\phi^+$ ). The second line stems from the fact that the only (nonvanishing) intermediate state is a single quantum, which can be created by  $\phi_i^+$  and annihilated by  $\phi_i^-$ . For the third and fourth lines we have used Eqs. (3.63) and (3.64).

We also note that the (in-)vacuum expectation value of the field  $\phi_f(x)$  is

$$\langle 0i|\phi_f(x)|0i\rangle = \int dx [\Delta^+(x - x') + \Delta^-(x - x')] j(x') \quad (3.69)$$

which in the same way as for Eq. (3.45) is the classical solution to the field equation in Eq. (3.46) after the interaction.

In conclusion we have shown the following:

- quantum fields, including that of the single harmonic oscillator, which are coupled to an external current contain excitations of a Poissonian nature, the mean number of quanta being determined from the Fourier components of the current;
- they also have vacuum expectation values that coincide with the classical c-number solutions for the interaction;
- the phases of the states, called *coherent states*, are well defined by the Fourier components of the external current.

### 3.3 Feynman's time-ordering prescription

In this section we will generalise the expression we have derived for the  $S$ -operator in Eqs. (3.40) and (3.61) from the simple case when the current  $j$  is an external c-number function to the general case when  $j$  is operator-valued. This will lead us to ways to calculate high-energy multiparticle production amplitudes in perturbation theory.

It is necessary to provide an ordering prescription for the  $S$ -operator in Eq. (3.61) when the current  $j$  is operator-valued. The right prescription (first introduced by Feynman and Dyson) is that *all expressions should be time-ordered*. If we would like to express the  $S$ -operator solely in the

initial-state fields then

$$S = \mathcal{T} \left\{ \exp \left( i \int d^4x \mathcal{H}_{1i} \right) \right\} \equiv 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \mathcal{T} \left\{ \prod_{j=1}^n dt_j H_{1i}(t_j) \right\} \quad (3.70)$$

with the *time-ordering* symbol  $\mathcal{T}$  implying that all operators should be written so that those with a later time are to the *left* of those with an earlier time.

Intuitively the prescription is rather easy to understand. The free initial quantum fields get distorted as time goes by. Each new distortion evidently follows the earlier ones and must therefore be applied after one has applied the previous interactions. (If we would like for some reason to write everything in terms of the final-state fields then we must *anti-time-order* everything, i.e. all operators should be arranged so that those with a later time are to the *right* of the others.)

As an example of the time-ordering procedure consider the second-order term in Eq. (3.70):

$$\mathcal{T} \{ H_{1i}(t_1) H_{1i}(t_2) \} = \Theta(t_1 - t_2) H_{1i}(t_1) H_{1i}(t_2) + \Theta(t_2 - t_1) H_{1i}(t_2) H_{1i}(t_1) \quad (3.71)$$

We have now defined two different ordering prescriptions, *normal-ordering* where all annihilation operators  $a$  are to the right of all creation operators  $a^*$  and *time-ordering* where all earlier-time operators are to the right of all the later-time operators. There is a mathematical manipulation theorem, Wick's theorem, which provides a connection between these orderings; you will find it described in great detail in many text-books.

### 1 Time-ordered products and the Feynman propagator, causality and locality

In order to understand some features of quantum fields we will show how Wick's theorem works in connection with the time-ordered product of a free field  $\phi$  at two different space-time points. Again using the notation  $\phi^\pm$  from Eq. (3.68) we obtain

$$\begin{aligned} \mathcal{T} \{ \phi_i(x_1) \phi_i(x_2) \} &= \phi_i^+(x_1) \phi_i^+(x_2) + \phi_i^-(x_1) \phi_i^-(x_2) \\ &\quad + \Theta(t_1 - t_2) \{ ([\phi_i^-(x_1), \phi_i^+(x_2)] \\ &\quad \quad \quad + \phi_i^+(x_2) \phi_i^-(x_1) + \phi_i^+(x_1) \phi_i^-(x_2) \} \\ &\quad + \Theta(t_2 - t_1) \{ ([\phi_i^-(x_2), \phi_i^+(x_1)] \\ &\quad \quad \quad + \phi_i^+(x_1) \phi_i^-(x_2) + \phi_i^+(x_2) \phi_i^-(x_1) \} \quad (3.72) \end{aligned}$$

We have thus item by item brought the time-ordered operators into normalordering. The result is evidently

$$\begin{aligned} \mathcal{T} \{ \phi_i(x_1) \phi_i(x_2) \} &= : \phi_i(x_1) \phi_i(x_2) : + \Theta(t_1 - t_2) [ \phi_i^-(x_1), \phi_i^+(x_2) ] \\ &\quad + \Theta(t_2 - t_1) [ \phi_i^-(x_2), \phi_i^+(x_1) ] \\ &\equiv : \phi_i(x_1) \phi_i(x_2) : + \Delta_F(x_2 - x_1, M) \end{aligned} \quad (3.73)$$

The function  $\Delta_F$  ( $F$  stands for Feynman) could have been constructed directly from the fact that the normal-ordered product  $: \phi_i(x_1) \phi_i(x_2) :$  has a vanishing vacuum expectation value. We then obtain

$$\Delta_F(x_2 - x_1, M) = \langle 0i | \mathcal{T} \{ \phi_i(x_1) \phi_i(x_2) \} | 0i \rangle \quad (3.74)$$

Using the result from Eq. (3.68) in Eq. (3.74) we may write the following expression for  $\Delta_F$  :

$$\Delta_F(x_2 - x_1) = \Theta(x_1 - x_2) \Delta_+(x_2 - x_1) + \Theta(x_2 - x_1) \Delta_+(x_1 - x_2) \quad (3.75)$$

(Note the order of the arguments in the  $\Delta_+$  distributions. For each this is related to the time dependence of the creation and annihilation operators.)

Before we construct an expression for  $\Delta_F$  we note from the result in Eq. (3.68) the following result for the general commutator:

$$[ \phi_i(x_1), \phi_i(x_2) ] = \Delta_+(x_2 - x_1) - \Delta_-(x_2 - x_1) \equiv -i\Delta(x_2 - x_1) \quad (3.76)$$

The notation is conventional and the factor  $i$  introduced to make  $\Delta$  real.

The general commutator  $\Delta$ , just like the  $\Delta_{\pm}$ -distributions, can be computed by straightforward means. We will give  $\Delta$  in detail because it has two properties of direct interest for what follows:

$$\Delta(x) = -\frac{\epsilon(x)}{2\pi} \left[ \delta(x^2) - \frac{M}{2\sqrt{x^2}} J_1(M\sqrt{x^2}) \Theta(x^2) \right]. \quad (3.77)$$

We have used the conventional sign-distribution  $\epsilon(x) \equiv \epsilon(x_0) = \Theta(x_0) - \Theta(-x_0)$  and the Bessel function of the first rank  $J_1$  in Eq. (3.77).

Firstly note that the *commutator distribution*  $\Delta$  *vanishes for spacelike vectors*  $x$ . This is our first encounter with practical *causality*. There is no possible signal connecting two space-time points with a spacelike difference. Therefore two *local* field operators taken at two such points commute. *They are independent and a measurement of the observable corresponding to one of the operators at one point cannot influence a measurement of the observable corresponding to the other operator at another point separated from the first by a spacelike difference.*

The word *local* is essential, however. All the field operators are singular from a strict function-definition point of view (note the occurrence of the  $\delta$ - and  $\epsilon$ -distributions in Eq. (3.77)). Mathematically such expressions

should be defined by means of a *test function*  $f$ , [31]:

$$\phi(f) = \int dx \phi(x) f(x) \quad (3.78)$$

A local operator is such that if we choose the test function  $f$  to be strongly localised around a point  $x$  (i.e. vanishing outside a suitably small region around  $x$ ) then also all the matrix elements of the operator  $\phi(f)$  should have this property.

If we consider the definition of  $\Delta_F$  from Eq. (3.73) we find that this function can also be defined by means of commutators. But these are commutators of field operators which are not local. None of the  $\phi^\pm$  is local because they contain only positive or negative frequencies, respectively. There is no way to localise anything in time by means of a function containing only frequencies of a definite sign.

The distribution  $\Delta_F$  can instead, according to the result in Eq. (3.75), be written e.g. as

$$\Delta_F(x) = -i\Theta(-x)\Delta(x) + \Delta_+(x) \quad (3.79)$$

and only the first term on the right-hand side is local in the sense used above.

Secondly we note from Eq. (3.77) that the commutator is highly singular along the lightcones. Although the quanta have mass  $M$  and therefore always move with a velocity below  $c = 1$  the corresponding quantum fields can influence each other in principle at infinite distances along the lightcones. It is also worthwhile to note that the *principal singularity* (the second term inside the large parentheses of Eq. (3.77) approaches a constant for  $x^2 \rightarrow 0$ ) is independent of the mass-value  $M$ .

## 2 The formula for the Feynman propagator, the lightcone singularities

We will next provide a formula for  $\Delta_F$  using a distribution-valued integral we have referred to in Eq. (3.43):

$$\Theta(x) \equiv \Theta(x_0) = \lim_{\epsilon \rightarrow 0} \int \frac{-idk'_0}{2\pi(k'_0 - i\epsilon)} \exp(ik'_0 x_0) \quad (3.80)$$

From Eq. (3.75) we may then use the result in Eq. (3.80) to obtain an integral representation for  $\Delta_F$ . We will subsequently not write out the limit sign but we will keep  $\epsilon$  as a small but arbitrary number.

$$\Delta_F(x) = \frac{i}{(2\pi)^4} \int \frac{d^3k}{2\omega(\mathbf{k})} \left( \frac{dk'_0}{k'_0 + i\epsilon} \exp(i(\omega + k'_0)x_0 - \mathbf{k} \cdot \mathbf{x}) - \frac{dk'_0}{k'_0 - i\epsilon} \exp[i(-\omega + k'_0)x_0 + \mathbf{k} \cdot \mathbf{x}] \right)$$

$$\begin{aligned}
&= \frac{i}{(2\pi)^4} \int \frac{d^3k}{2\omega} \exp(ikx) \left( \frac{1}{k_0 - \omega + i\epsilon} - \frac{1}{k_0 + \omega - i\epsilon} \right) \\
&= \frac{i}{(2\pi)^4} \int \frac{dk \exp(ikx)}{k^2 - M^2 + i\epsilon} \tag{3.81}
\end{aligned}$$

Here we have introduced the result of Eq. (3.80) together with the corresponding result for  $\Theta(-x)$  and then changed the integration variable  $k'_0$  to  $k_0 = k'_0 \pm \omega$  (as well as replacing  $\mathbf{k}$  by  $-\mathbf{k}$  in the second term). In the last line we have gathered the two denominators into one.

The final result corresponds to the limiting situation when the number  $\epsilon$  approaches 0. This means that  $\Delta_F$  is actually singular for all values of the vector  $\mathbf{k}$  which correspond to a 'real' particle with mass  $M$ .

When we want to consider a physical observable that is sensitive to the limit then it is necessary to be more precise in the definition of the size of  $\epsilon$ . An example of this is provided in Chapter 14.

From a mathematical point of view  $\Delta_F$  is a *distribution*, which must be defined by means of integration over suitable test functions, as mentioned above. It is also the Fourier transform of the boundary value  $\epsilon \rightarrow 0$  of an *analytic function* defined on complex-valued vectors  $k$  with  $\text{Im}k^2 > 0$ . In that case it can be described as analytic and Lorentz-invariant with poles whenever  $k^2 = M^2$ .

In Chapter 6 we will provide a formula for the behaviour of the Feynman propagator for spacelike arguments. That formula will be based upon the property that  $\Delta_F$  satisfies the Klein-Gordon equation

$$(\square + M^2)\Delta_F(x, M^2) = 0 \tag{3.82}$$

everywhere outside the origin,  $x = 0$ .

For the investigations in Chapter 19 it is also of interest to know the space-time singularities of both the Feynman propagator  $\Delta_F$  and the function  $\Delta_+$ . We will not give the formulas for the general case but only for the case when the mass  $M = 0$  because *just as for the function  $\Delta$  in Eq. (3.77) the main singularities of all the functions are independent of the mass.*

The following formal development may be used in such a derivation. We firstly note that

$$\frac{i}{k^2 + i\epsilon} = \int_0^\infty d\alpha \exp(i\alpha k^2) \tag{3.83}$$

(the integral on the right-hand side converges when we add a small positive imaginary part to  $k^2$ ). If we introduce this result into the formula for the Feynman propagator given in Eq. (3.81) we obtain gaussian integrals

(which due to the imaginary parts are called Fresnel integrals):

$$\begin{aligned}\Delta_F(x, M^2 = 0) &= \frac{1}{(2\pi)^4} \int d\alpha \int dk \exp(i\alpha k^2 + ikx) \\ &= \frac{i}{4(2\pi)^2} \int \frac{d\alpha}{\alpha^2} \exp\left(\frac{-ix^2}{4\alpha}\right) = \frac{1}{(2\pi)^2(x^2 - i\epsilon)}\end{aligned}\quad (3.84)$$

In the second line we have made the change of integration variable  $1/\alpha \rightarrow \alpha$ ; performing the integral shows that  $x^2$  must contain a small *negative* imaginary part, which ensures convergence.

If we perform the integrals for the function  $\Delta_+(x)$  with the mass  $M = 0$  (which is straightforward) we obtain the same result as in Eq. (3.84) but with the boundary value  $x^2 \rightarrow x^2 + i\epsilon x_0$ . This means that the imaginary part depends upon the sign of the time-component of the vector  $x$ .

At this point we will consider a particular distribution-valued boundary value. Suppose that we have a (test)function,  $f(x)$ , of a single real variable  $x$  and that we consider the result of integrating it together with the boundary value  $1/(x - i\epsilon)$ . We may then start by using the following formal manipulation:

$$\frac{1}{x - i\epsilon} = \frac{x + i\epsilon}{x^2 + \epsilon^2} = R + iI \quad (3.85)$$

If we start with the imaginary part then we obtain the result for  $I$ :

$$\int dx f(x) \frac{\epsilon}{x^2 + \epsilon^2} = \int dy f(\epsilon y) \frac{1}{y^2 + 1} \rightarrow f(0)\pi \equiv \int dx f(x) \pi \delta(x) \quad (3.86)$$

We have assumed that the function  $f$  vanishes sufficiently fast that we may take the limit  $f(y\epsilon) \rightarrow f(0)$  outside the integral; then as is well known,  $\int dy/(y^2 + 1) = \pi$ .

We have in this way obtained a representation of the  $\delta$ -distribution which is very useful. It is the difference between the boundary values:

$$\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} = 2i\pi\delta(x) \quad (3.87)$$

For the real part,  $R$ , in Eq. (3.85) we may use the trick of adding and subtracting the quantity

$$\int_{-\alpha}^{\alpha} f(0) \frac{xdx}{x^2 + \epsilon^2} = 0 \quad (3.88)$$

This result is obviously valid for any (finite) positive number  $\alpha$  because the integrand is an odd function. For values outside  $-\alpha < x < \alpha$  we now have no problem in taking the limit  $\epsilon \rightarrow 0$  for  $R$  in Eq. (3.88) for a

well-behaved function  $f$  (we again use the Heaviside function  $\Theta$ ):

$$R(f) = \int \left\{ \Theta(x^2 - \alpha^2)f(x) + \Theta(\alpha^2 - x^2)[f(x) - f(0)] \right\} \frac{dx}{x} \quad (3.89)$$

If afterwards we let  $\alpha \rightarrow 0$  we find that we always have a well-defined integral, called the *principal part of  $f$*  and defined so that in the neighborhood of the singular point  $x = 0$  we make the change  $f(x) \rightarrow f(x) - f(0)$ .

As a simple example for this limiting situation consider the relationship between the commutator distribution  $\Delta$  and  $\Delta_+$ . If we take the indicated difference in Eq. (3.76) we obtain just the lightcone  $\delta$ -distribution in Eq. (3.77) from the result in Eq. (3.87) and the limiting behaviour of  $\Delta_+$  we mentioned above.

We have in this section stressed the following facts:

- a local quantum field must contain both positive and negative frequencies;
- the  $S$ -operator must be defined by means of time-ordering.

These are the origins of the Feynman propagator distribution.

It is, of course, possible to interpret the two parts of the time-ordering process in Eq. (3.75) as respectively ‘forwards’ and ‘backwards’ transmission in time for the quanta involved (the former would be ‘particles’ and the latter ‘antiparticles’). There is, however, no reason to inflict nonsense upon one’s physical intuition and we prefer to consider the propagator as a unity.

In the last section of this chapter we will show that in a lightcone dynamical scenario it makes sense to talk about the propagator in terms of old-fashioned energy denominators.

In the next subsection we will discuss the Fierz [61] interpretation of the Feynman propagator, which is how the physicists working with Stückelberg thought about it. This is done in order to convince the reader that the way in which it works is not only in accordance with the Heisenberg indeterminacy principle. *The Feynman propagator is actually as causal as it can be when the principle is fulfilled.*

### 3 An interpretation of the Feynman propagator

For a simple and intuitively useful example we will consider the case when  $H_1 = g\phi(x):\psi^2(x):$  (with  $\phi$  and  $\psi$  free independent scalar quantum fields), an interaction which we will discuss later in the book. This is meant to be a simplified version of the current-vector-potential interaction in a gauge theory.

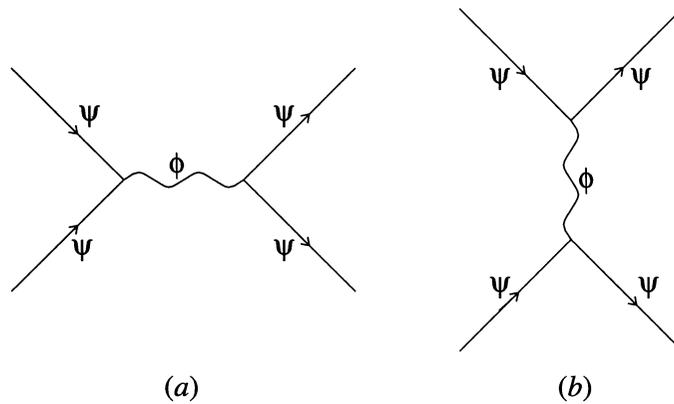


Fig. 3.1. Two simple examples of Feynman graphs. The situation in (a) corresponds to the annihilation of two  $\psi$ -particles; the state then propagates as a virtual  $\phi$  and finally two outgoing particles appear. In (b) there is scattering with the exchange of energy-momentum. The straight lines symbolise  $\psi$ -quanta and the wavy lines the  $\phi$ -propagator.

For this case we will need the fact that  $R_2 = (1/2)\mathcal{T} \{ \mathcal{H}_{1i}(x_1)\mathcal{H}_{1i}(x_2) \}$  (cf. Eq. (3.71)) contains among many others the term

$$R'_2 = \frac{g^2}{2} : \psi_i^2(x_1)\psi_i^2(x_2) : \Delta_F(x_2 - x_1, M_\phi) \quad (3.90)$$

The result in Eq. (3.90) corresponds to the scattering of two  $\psi$ -particles which come in, interact at the point  $x_1$  and are either annihilated into a virtual  $\phi$  (Fig. 3.1(a)) and afterwards reappear as outgoing  $\psi$ -particles at  $x_2$  or exchange energy-momentum between points  $x_1$  and  $x_2$  through a virtual  $\phi$  (Fig. 3.1(b)).

In this subsection we will simplify the working by assuming that there are two kinds of  $\psi$ -particle, which we call  $p$ - and  $e$ -flavored, which may interact via the common  $\phi$ -field. This assumption does not change the argument in the least but makes it easier to discuss.

Any kind of interpretation of a physical quantity is always defined by means of a measurement that is at least theoretically possible. We will show that a measurement made in accordance with quantum mechanical requirements will preserve all causality and energy-momentum conservation properties and that this is due to the properties of the Feynman propagator.

In order to further simplify the problem we will assume that there are regions of space-time  $R_j$  within which we can measure what is going on in connection with the scattering. As always in a measurement process

we expect these regions to be determined by some size parameters. We will solely be interested in the *time slices* of the regions, i.e. the time intervals they span; these we will call  $T_j$ . Thus we assume that there is in any one of the space-time regions an ideal detector (but working in accordance with quantum mechanics, of course!) recording what is going on as time passes.

We then consider the case when an  $e_1$ -particle scatters against a  $p_2$ -particle and goes out after the process as an  $e_3$ -particle while the  $p_4$ -particle recoils. This corresponds to the situation described diagrammatically in Fig. 3.1(b). We assume that their energy-momenta are  $k_j$   $j = 1, \dots, 4$  and we will now write the transition matrix element as

$$\mathcal{M} = \int dx_1 dx_2 g^2 \langle k_4 | : \psi_p^2(x_1) : | k_2 \rangle \Delta_F(x_2 - x_1) \langle k_3 | : \psi_e^2(x_2) | k_1 \rangle. \quad (3.91)$$

We then change the integral over all space-time into an integral over the regions where we have the detectors:

$$\int dx_1 dx_2 = \sum_{j,k} \int_{R_j} dx_1 \int_{R_k} dx_2 \quad (3.92)$$

The only argument of which we are going to make use is related to the energies so it is not necessary that we expand the  $\Delta_F$ -function in plane waves; energy harmonics  $\exp(\pm i\omega x_0)$  are sufficient. The next thing is to go back to the definition of  $\Delta_F$ , Eq. (3.75), and rewrite  $\mathcal{M}$  in Eq. (3.91) as (note that we must include both time-orderings!)

$$g^2 \sum_{j,k} \int_{R_j} dx_1 \int_{R_k} dx_2 \langle k_4 | : \psi_p^2(x_1) : | k_2 \rangle \langle k_3 | : \psi_e^2(x_2) | k_1 \rangle \\ \times [\Theta(x_1 - x_2) \Delta_+(x_2 - x_1) + \Theta(x_2 - x_1) \Delta_+(x_1 - x_2)] \quad (3.93)$$

If we write out the time dependence of the first term we will find for regions  $R_1$  and  $R_2$  (spreading over the times  $T_j$ ,  $j = 1, 2$ ; note that  $k_0$  must be positive as it corresponds to the argument in the  $\Delta_+$ -distribution)

$$\exp[-i(\omega_2 - \omega_4)x_{01} - i(\omega_1 - \omega_3)x_{02}] \Theta(x_{01} - x_{02}) dk_0 \exp[ik_0(x_{02} - x_{01})] \quad (3.94)$$

Now we gather the terms containing  $x_{01}$  and  $x_{02}$ , respectively, and assume that the time slices  $T_j$  for the detector configuration are such that

$$T_1 |\omega_2 - \omega_4| \gg 1 \quad \text{and} \quad T_2 |\omega_1 - \omega_3| \gg 1 \quad (3.95)$$

This is what Heisenberg would require in order that we should be able to measure the energies in each of the detectors so precisely that we can distinguish between the energies of  $p_2$  and  $p_4$  and between those of  $p_1$  and  $p_3$ . It is necessary to have sufficiently long times available for such measurements, at least several frequency periods. But we note that there

is then little to work on if we are to obtain a nonvanishing value for the integrals. The only possibility is to choose the value of  $k_0$  such that

$$k_0 \simeq \omega_4 - \omega_2 \simeq \omega_1 - \omega_3 \quad (3.96)$$

This requirement is a direct result of the properties of the Fourier integrals, for which it is necessary not to have strongly fluctuating integrands if we want nonvanishing results.

We conclude that, as the time in region  $R_2$  is earlier than the time in region  $R_1$ , according to the  $\Theta$ -distribution, and as  $k_0$  is positive:

- the energy of the  $e$ -flavor particle decreases from  $\omega_1$  to  $\omega_3$  by emission of the (virtual)  $\phi$ -quantum in the region  $R_2$ ;
- then the  $p$ -flavor particle absorbs the  $\phi$ -quantum in the region  $R_1$  and so increases its energy from  $\omega_2$  to  $\omega_4$ ;
- in both cases it is necessary to have time slices  $T_j$  large enough to measure the energy loss and energy increase, respectively, with sufficient precision.

In the other term in Eq. (3.93) the region  $R_1$  is before the region  $R_2$  in time; this corresponds to the opposite process. *The basic point is that the Feynman propagator describes emission and absorption (within the requirements of Heisenberg) in a causal way.*

### 3.4 The method for calculating the scattering cross sections

Here we consider the steps that are necessary to get from the transition amplitude to the scattering cross section for a multiparticle interaction. The reasons for doing this are two-fold. On the one hand we have introduced a cutoff procedure with the box  $V$  and we want to show why it does not appear in our final formulas. On the other hand, in the last section, at Eq. (3.90) and Fig. 3.1, we considered a particular scattering process. To understand the physics of that process we will calculate its properties in some detail. The result will serve as an example of other formulas that we will meet later on.

We will consider the matrix element  $\mathcal{M}$  between two incoming  $\psi$ -particles (energy-momentum  $k_1, k_2$ ) and two outgoing  $\psi$ -particles ( $k_3$  and  $k_4$ ) interacting via the field  $\phi$  according to the interaction term

$$\int H_1(t)dt = \int dxg : \psi^2(x) : \phi(x) \quad (3.97)$$

From Eq. (3.90) we know the term responsible for the transition and so we obtain for the matrix element  $\mathcal{M}(k_3, k_4; k_1, k_2)$

$$\begin{aligned}
 & \int d^4x d^4x' \langle k_3, k_4 | \frac{1}{2} g^2 : \psi^2(x) :: \psi^2(x') : \Delta_F(x' - x) | k_1, k_2 \rangle \\
 &= \int d^4x d^4x' \frac{2g^2}{4V^2 \sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} \times \\
 & \quad \{ \exp[ix(k_3 - k_1) + ix'(k_4 - k_2)] + \exp[ix(k_4 - k_1) + ix'(k_3 - k_2)] \\
 & \quad + \exp[-ix(k_1 + k_2) + ix'(k_3 + k_4)] \} \Delta_F(x' - x) \\
 &= \frac{2g^2}{4V^2 \sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} (2\pi)^8 \delta(k_1 + k_2 - k_3 - k_4) \frac{i}{(2\pi)^4} \\
 & \quad \times \left[ \frac{1}{(k_1 - k_3)^2 - M_\phi^2} + \frac{1}{(k_1 - k_4)^2 - M_\phi^2} + \frac{1}{(k_1 + k_2)^2 - M_\phi^2} \right] \\
 &\equiv AB \tag{3.98}
 \end{aligned}$$

We have here introduced in the second line of the equation the wave functions for the incoming and outgoing particles, i.e. the factors multiplying the necessary annihilation and creation operators in the representation of the operators  $\psi$ . In the third line we have, after the introduction of the Fourier representation of the Feynman propagator, performed the space-time integrals. In the last line we re-express the three terms inside the square bracket as  $B$  and the remaining factors as  $A$ . We note in particular that the energy-momentum conserving  $\delta$ -distribution appears in  $A$ .

The cross section, according to Fermi's Golden Rule, is obtained by multiplying the transition rate per unit time by the inverse of the incoming particle flux and by the final-state density. We are going to introduce and discuss these factors in turn.

The *transition rate* is obtained from the square of the matrix element  $\mathcal{M}$  and we immediately encounter the difficulty of squaring a  $\delta$ -distribution in the factor  $A$ . If we go back to Eq. (3.51) we note that the distribution for a finite box  $V$  is, for the momentum part,

$$(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \rightarrow V \delta_{\mathbf{k}, \mathbf{k}'} \tag{3.99}$$

Consequently the square of the space-momentum part is, formally,

$$[\delta(\mathbf{k} - \mathbf{k}')]^2 \rightarrow \frac{V}{(2\pi)^3} \delta(\mathbf{k} - \mathbf{k}') \tag{3.100}$$

For the energy part we note that the  $\delta$ -distribution stems from an integral

$$\delta(\Delta E) = \frac{1}{2\pi} \left[ \lim_{t_0 \rightarrow \infty} \int_{-t_0}^{t_0} dt \exp(it\Delta E) \right] = \lim_{t_0 \rightarrow \infty} \left[ \frac{\sin(t_0 \Delta E)}{\pi \Delta E} \right] \tag{3.101}$$

The last expression is a well-known representation of the  $\delta$ -distribution. We always have in mind the physical picture that there should be a finite

time overlap for the interaction; this corresponds to a finite ‘effective’ value of  $t_0$ . Therefore this representation is in accordance with our intuition. If we formally square the last line and note the well-known relation

$$\lim_{x \rightarrow 0} \left[ \frac{\sin(xy)}{x} \right] = y \tag{3.102}$$

we find the following formal definition of the square of the energy part of the  $\delta$ -distribution (with  $\Delta t = 2t_0$  the ‘interaction time’):

$$[\delta(\Delta E)]^2 \rightarrow \frac{\Delta t}{2\pi} \delta(\Delta E) \tag{3.103}$$

Thus the transition rate per unit time is

$$\frac{w}{\Delta t} = \frac{(2g^2)^2}{(4V^2)^2 \omega_1 \omega_2 \omega_3 \omega_4} (2\pi)^8 \frac{V}{(2\pi)^4} \delta(k_1 + k_2 - k_3 - k_4) |B|^2 \tag{3.104}$$

The *incoming flux*, i.e. the number of states interacting per unit time and unit transverse area, is  $v_r/V$ , where  $v_r$  is the relative velocity of the particles. If we divide the formula in Eq. (3.104) by this flux factor we notice that we obtain two factors  $V$  in the numerator, one from the (space-momentum)  $\delta$ -distribution and one from the flux. These two compensate the two factors  $V$  in the denominator stemming from the *two incoming particle wave functions*.

The remaining factors from the incoming wave functions,  $4\omega_1\omega_2$ , combine in the denominator with the velocity  $v_r$  so that we have

$$\begin{aligned} 4\omega_1\omega_2v_r &= 4\omega_1\omega_2|v_1 - v_2| = 4||\mathbf{k}_1|\omega_2 - |\mathbf{k}_2|\omega_1| \\ &= 4M_1M_2 |\sinh(y_1 - y_2)| = 4M_1M_2 \sqrt{\cosh^2(y_1 - y_2) - 1} \end{aligned} \tag{3.105}$$

$$2\sqrt{(s - M_1^2 - M_2^2)^2 - 4M_1^2M_2^2} \equiv 2\sqrt{\lambda(s, M_1^2, M_2^2)} \rightarrow 2s$$

with  $s$  the squared cms energy  $s = (k_1 + k_2)^2$ . Here we have first introduced the relative velocity and used that each particle velocity is  $v_j = |\mathbf{k}_j|/\omega_j$  and that energies and momenta can be written in terms of rapidities  $\omega_j = M_j \cosh y_j$ ,  $|\mathbf{k}_j| = M_j \sinh y_j$ . The rest is simple manipulation and we note that the function  $\lambda(a, b, c)$  is totally symmetric:

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \tag{3.106}$$

The quantity  $\lambda$  is very useful for quick calculations of Lorentz boosts. Thus the cms momenta of two particles (indexed 1 and 2) with a common cms energy  $\sqrt{s}$  has the common cms momentum

$$|\mathbf{k}_{j,cms}| = \frac{\sqrt{\lambda(s, M_1^2, M_2^2)}}{2\sqrt{s}} \tag{3.107}$$

while in the rest frame of particle 1, particle 2 has momentum

$$|\mathbf{k}_{2,lab}| = \frac{\sqrt{\lambda(s, M_1^2, M_2^2)}}{2M_1} \quad (3.108)$$

In the rest frame of 2 we simply exchange the indices.

The third factor in the cross section, the *final-state density* is the number of momentum states available and is given by Eq. (3.50). We note that it will contain in the numerator as many  $V$ -factors as particles. This will compensate the corresponding denominator  $V$ -factors from the *final state particle wave functions*. All in all this final-state density therefore combines with the wave function factors into

$$\prod_{j_f} \frac{d^3 k_{j_f}}{2\omega_{j_f} (2\pi)^3} = \prod_{j_f} \frac{dk_{j_f}}{(2\pi)^3} \delta^+(k_{j_f}^2 - M_{j_f}^2) \quad (3.109)$$

where we have used Eq. (3.66).

The full cross section then will appear as (for  $n_f$  final-state particles)

$$d\sigma = \frac{2g^4}{(2\pi)^{(3n_f-4)} \sqrt{\lambda(s, M_\psi^2, M_\psi^2)}} |B|^2 \times \prod_{j_f} dk_{j_f} \delta^+(k_{j_f}^2 - M_{j_f}^2) \delta(k_1 + k_2 - \sum_{j_f} k_{j_f}) \quad (3.110)$$

The general phase-space factors in Eq. (3.110) will always occur in two-body to many-body processes but the factor  $2g^4 |B|^2$  (with the *matrix element*  $B$  defined in Eq. (3.98)) is specific to the particular process we have considered. We will meet the result repeatedly later in the book and we note that it is manifestly Lorentz-invariant.

### 3.5 The propagators in lightcone physics in the infinite-momentum frame

#### 1 The formalism

We will in this section provide a different picture of the the Feynman rules by exhibiting the properties of perturbation theory when lightcone coordinates are used. The propagator in energy-momentum space will then have strong similarities to the old-fashioned energy denominators occurring in time-dependent perturbation theory in nonrelativistic dynamics.

Basically the scenario describes a two-dimensional field theory in transverse dimensions with a varying mass parameter which corresponds to one of the lightcone components. The whole idea stems from early investigations by Weinberg, [111], into the possibility of simplifying the

Feynman rules by performing all the integrals in a frame moving very fast in some direction. This has been called the ‘infinite-momentum frame’. The discussion is based upon the development in [87].

The formalism is useful to understand intuitively some of the features of the parton model which is discussed in Chapter 5. We will use some of the results in connection with heavy quark fragmentation in Chapter 13.

We begin by defining the *lightcone components*  $\eta, H$  and  $\tau, \zeta$  of the energy-momentum and space-time operators:

$$\begin{aligned} \eta &= \frac{P_0 + P_3}{\sqrt{2}}, & \tau &= \frac{t + x_3}{\sqrt{2}} \\ H &= \frac{P_0 - P_3}{\sqrt{2}}, & \zeta &= \frac{t - x_3}{\sqrt{2}} \end{aligned} \tag{3.111}$$

We will call the 1- and 2-components the *transverse components* of the corresponding four-vector and denote these by  $\mathbf{p}_\perp$  and  $\mathbf{x}_\perp$ .

According to the ordinary commutation relations we have

$$\begin{aligned} [\eta, \tau] &= [\eta, H] = [H, \zeta] = [\tau, \zeta] = 0 \\ [\eta, \zeta] &= [H, \tau] = i \end{aligned} \tag{3.112}$$

and all these components commute with the transverse ones.

The mass-shell condition for a free particle means that

$$m^2 = P_0^2 - P_3^2 - \mathbf{p}_\perp^2 \quad \Rightarrow \quad H = \frac{\mathbf{p}_\perp^2}{2\eta} + V_0 \tag{3.113}$$

where  $V_0 = m^2/2\eta$  is similar to a potential term. This is evidently a reduction of the problem to the two transverse dimensions using the variable ‘mass’-parameter  $\eta$ .

We next consider the Feynman propagator and rewrite it in terms of the variables given above:

$$\begin{aligned} \Delta_F(x) &= \frac{i}{(2\pi)^4} \int \frac{dk \exp(ikx)}{k^2 - M^2 + i\epsilon} \\ &= \frac{i}{(2\pi)^4} \int d^2p_\perp \int d\eta \exp i(\eta\zeta - \mathbf{p}_\perp \cdot \mathbf{x}_\perp) \\ &\quad \times \int dH \exp(iH\tau)(2\eta H - \mathbf{p}_\perp^2 - M^2 + i\epsilon)^{-1} \end{aligned} \tag{3.114}$$

We note that by use of the results in Eq. (3.80) we may now write the following formula for the Feynman propagator:

$$\Delta_F(x) = \frac{1}{2(2\pi)^3} \int d^2p_\perp \int_0^\infty \frac{d\eta}{\eta} [\Theta(\tau) \exp(-ipx) + \Theta(-\tau) \exp(ipx)] \tag{3.115}$$

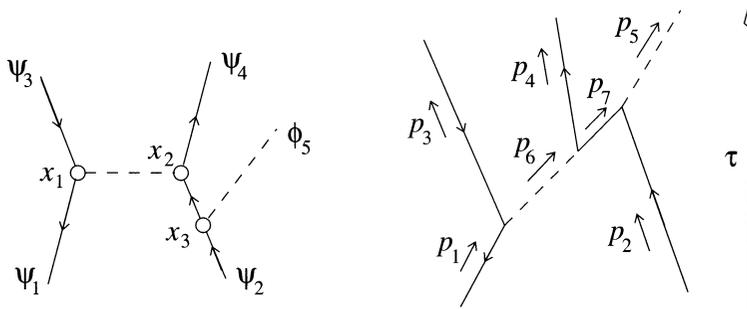


Fig. 3.2. One of the possible Feynman diagrams in the process  $\psi_1 + \psi_2 \rightarrow \psi_3 + \psi_4 + \phi_5$  and the same diagram ordered according to one of the possible orderings along the lightcone.

where  $px = H\tau + \eta\zeta - \mathbf{p}_\perp \cdot \mathbf{x}_\perp$  and  $H$  is defined by the mass-shell condition above.

In order to obtain the result in Eq. (3.115) we have divided the integration region of  $\eta$  into positive and negative parts to obtain the sign of the limiting imaginary part and then changed sign for the negative part. This provides the signs in the complex exponents.

We have thus come back to expressions with the properties described before. The ‘effective’ energy  $H$  is like a nonrelativistic kinetic energy term related to the generalised ‘time’, i.e. the lightcone coordinate  $\tau$ .

### 2 An example

We will next provide an example of how the Feynman rules work when lightcone coordinates are used; we consider the Feynman diagram in Fig. 3.2(a). This corresponds to the  $g\phi:\psi^2$ -theory we have discussed before and contains the scattering of two  $\psi$ -particles together with the emission of a  $\phi$ -particle in a bremsstrahlung process. We note that there are several more diagrams which will contribute to the process.

In Fig. 3.2(b) we have drawn a version of the diagram in which there is a particular ordering of the  $\tau$ -variables. A little thought will convince us that if we have  $n$  vertices in the primary Feynman diagram then there are  $n!$  such ordered diagrams possible. That means six in this case and we have considered the one corresponding to the ordering  $\tau_1 \leq \tau_2 \leq \tau_3$ .

In the ordered diagram we must perform the  $\tau$ -integrals with this ordering requirement, which means that only one of the  $\Theta$ -terms in the representation of Eq. (3.115) survives the requirement.

There are two propagator terms and three  $\tau$ -integrals. Note that all the

transverse integrals and the  $\zeta$ -integrals can easily be performed to give at each vertex a  $\delta$ -distribution contribution

$$(2\pi)^3 \delta(\mathbf{p}_{\perp i} - \mathbf{p}_{\perp f}) \delta(\eta_i - \eta_f) \tag{3.116}$$

where the indices  $i, f$  correspond to the ‘in’- and ‘out’-contributions at that vertex. Note that we have directed the vectors in Fig. 3.2(b).

The  $\tau$ -integrals are given by

$$I = \int d\tau_1 d\tau_2 d\tau_3 \Theta(\tau_3 - \tau_2) \Theta(\tau_2 - \tau_1) \exp\{-i[(H_1 - H_3 - H_6)\tau_1 + (H_6 - H_4 - H_7)\tau_2 + (H_7 + H_2 - H_5)\tau_3]\} \tag{3.117}$$

If we introduce the natural variables  $T_0 = \tau_1, T_1 = \tau_2 - \tau_1, T_2 = \tau_3 - \tau_2$  then the integrals are transformed to give

$$I = \int dT_0 \exp[-i(\mathcal{H}_i - \mathcal{H}_f)T_0] \int_0^\infty dT_1 \exp[-i(\mathcal{H}_1 - \mathcal{H}_f)T_1] \times \int_0^\infty dT_2 \exp[-i(\mathcal{H}_2 - \mathcal{H}_f)T_2] \tag{3.118}$$

where we have introduced the notation

$$\begin{aligned} \mathcal{H}_i &= H_1 + H_2, & \mathcal{H}_f &= H_3 + H_4 + H_5 \\ \mathcal{H}_1 &= H_3 + H_6 + H_2, & \mathcal{H}_2 &= H_2 + H_7 + H_4 \end{aligned} \tag{3.119}$$

Again the indices  $i, f$  correspond to the energies of the incoming and outgoing states (this time for the whole diagram, with signs) and the two indices 1 and 2 correspond to the *intermediate states*. If we consider Fig. 3.2(b) it is obvious what is meant by the intermediate states. They refer to those particles which exist at a particular  $\tau$ -slice, for the index 1 the slice between  $\tau_1$  and  $\tau_2$ , for the index 2 the slice between  $\tau_2$  and  $\tau_3$ .

The  $T_0$ -integral, which is taken over the whole lightcone time, provides a  $\delta$ -distribution for overall energy conservation. The  $T_1$ - and  $T_2$ -integrals only cover the positive regions and each give

$$\int_0^\infty dT \exp(i\mathcal{H}T) = \frac{i}{\mathcal{H} + i\epsilon} \tag{3.120}$$

This means that the total result will contain, besides an overall energy-momentum-conserving  $\delta$ -distribution, ‘mass’-conserving, i.e.  $\eta$ -conserving, and transverse-momentum-conserving  $\delta$ -distributions at each vertex, something very similar to old-fashioned energy denominators:

$$(\mathcal{H}_1 - \mathcal{H}_f)^{-1} (\mathcal{H}_2 - \mathcal{H}_f)^{-1} \tag{3.121}$$

one for each intermediate state. It is not difficult to see that this structure survives for all the different contributions. Further, as one may guess, it is possible to do the same for any kind of field theory, although there

are often more singular parts of the propagators (for QED cf. [87]) than those we encountered in the simple scalar theory.

It is worthwhile to note that that the  $\eta$ -terms we find everywhere are nothing other than the quantities

$$\frac{dp_l}{2e} = \frac{dy}{2} = \frac{d\eta}{2\eta} \quad (3.122)$$

which we met before in the method of virtual quanta in Chapter 2, and also will meet later as Feynman's 'wee spectrum' of partons.

In this way each  $n$ -vertex Feynman graph can be reduced to  $n!$  old-fashioned energy denominator integrals. This might not seem to be a major achievement. But this formalism often makes it easier to perform reasonable approximations among the many diagrammatic contributions to a particular scattering situation or bound-state configuration. It also provides an intuitively appealing picture of the difference between the longitudinal and the transverse dynamics.