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LINEAR ALGEBRA OF CURVATURE TENSORS AND THEIR COVARIANT DERIVATIVES

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1. Introduction. Fix a point on a Riemannian manifold, and consider the tangent space V at the point equipped with inner product g. The Riemann curvature tensor R and its first covariant derivative ∇R at the point are tensors in $\mathscr{T}_4(V)$ and $\mathscr{T}_5(V)$. If we take all the known symmetries of these tensors we can define subspaces $\mathrm{Curv} \subseteq \mathscr{T}_4(V)$ and $\nabla\mathrm{Curv} \subseteq \mathscr{T}_5(V)$ such that $R \in \mathrm{Curv}$ and $\nabla R \in \nabla\mathrm{Curv}$. Also, the orthogonal group $O(g)$ acts naturally on all these spaces. The two fundamental problems of the linear algebra of the spaces Curv and ∇Curv are: (1) find the decomposition into irreducible representations of $O(g)$, with corresponding projection operators, (2) give a description of the structure of the $O(g)$ orbits, by means of orbit invariant functions and a canonical form for elements of each orbit.

The solution of the first problem for Curv is well-known; there are three irreducible pieces (in dimension $n \geqq 4$). See [**8**] for an especially elegant description. This result is perhaps not as well-known as it should be, since the standard reference works in Riemannian geometry either fail to mention it at all, or relegate it to fine print. But there are many applications, especially since one of the projection operators is the Weyl conformal curvature tensor. See [**1**] for a sampler of recent applications. In Section 2 we state the decomposition explicitly without proof, and we give a few variations of the result that are perhaps new. We also give an application involving orthogonal Radon transforms in Section 6 which generalizes a characterization of Einstein metrics of Singer and Thorpe [**10**].

The solution to the first problem for ∇Curv is given in Section 3; now there are four irreducible pieces ($n \geqq 4$). After finding the projection operators explicitly, we find all the relations between the decompositions of R and ∇R for a Riemannian curvature tensor. These relations may be thought of as quantitative generalizations of results such as isotropic curvature (resp. Ricci curvature) implies constant curvature (resp. Ricci curvature). All the results of Sections 2 and 3 hold for semi-Riemannian metrics with minor modification.

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In Section 4 we solve analogous problems for the curvature of a symmetric connection, with respect to the general linear group. In this case we find a component with multiplicity 2 occurring in the decomposition of the covariant derivative. In Section 5 we show that one of the projection operators may be interpreted as the Weyl projective curvature tensor. This is well-known for metric connections, but appears to be new in this generality.

In Section 7 we discuss the second problem for Curv. The results obtained are preliminary in nature. We find some orbit invariants, but they do not suffice to distinguish all orbits. We give a preliminary canonical form, but again it is likely that further refinements will be needed. The main thrust of this section is to suggest that this problem is neither trivial nor intractable. The significance of the problem should be obvious: the only coordinate independent information contained in the curvature tensor at a point is the orbit it belongs to. Isometries must preserve orbits, so in particular a homogeneous space has curvature tensor in the same orbit at every point. This leads to the interesting question: exactly which orbits can occur for a homogeneous space? The Ambrose-Singer characterization of homogeneous space [**14**] does not seem to shed any light on this problem. Similarly, we can ask which orbits correspond to other geometric properties.

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In the proofs that follow, a number of lengthy but routine computations are omitted. Often these involve cancellation of terms, and so are difficult to reproduce in the static medium of the printed page.

The reader is expected to be familiar with the basic facts about curvature tensors (as in [**3**] or [**7**]) and with the theory of representations of classical groups (as in [**2**] or [**9**]). It goes without saying that the spirit of Hermann Weyl permeates this work.

After this paper was written, the author became aware of the preprint of Gray and Vanhecke [**5**] which obtains many of the same results as Section 3 of this paper.

2. Decomposition of Riemann curvature tensors. Fix a vector space V of dimension $n \geqq 3$ over $\mathbf{R}$ with non-degenerate quadratic form g, and let $O(g)$ denote the orthogonal group for g. In the case that g is definite this is the compact group $O(n)$. More generally it is $O(p, q)$ where $p + q = n$. However, the theory of finite dimensional representations for all these groups is essentially the same. Let $\mathcal{T}_k$ denote the space of tensors of rank $(0, k)$ on V, or equivalently, of k-linear functions on V. Elements of $\mathcal{T}_k$ will be denoted R, S, T, etc., and sometimes R_k to indicate the rank.

In $\mathcal{T}_4$ there is a subspace that we will denote Curv, which is defined to be the tensors which are skew-symmetric in the 1-2 and 3-4 places and acyclic in the 1-2-3 places: $T \in \text{Curv}$ if and only if

(2.1) $\quad T(X_1, X_2, X_3, X_4) = -T(X_2, X_1, X_3, X_4)$

(2.2) $\quad T(X_1, X_2, X_3, X_4) = -T(X_1, X_2, X_4, X_3)$

(2.3) $\quad T(X_1, X_2, X_3, X_4) + T(X_2, X_3, X_1, X_4) + T(X_3, X_1, X_2, X_4) = 0.$

It is well-known that these conditions imply acyclicity in any three variables, and symmetry with respect to the interchange of (X_1, X_2) with (X_3, X_4):

(2.4) $\quad T(X_1, X_2, X_3, X_4) = T(X_3, X_4, X_1, X_2).$

It is well-known that if T is the Riemannian curvature tensor

$$T(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$$

at a point on a Riemannian (or even semi-Riemannian) manifold, then $T \in \text{Curv}$, and furthermore any element of Curv arises in this way. The decomposition of Curv into irreducible components with respect to the natural action of $O(g)$ is therefore a basic fact of Riemannian geometry. To state the result concisely we introduce the "big wedge" notation. Let R_2 and S_2 denote symmetric tensors in $\mathscr{T}_2$. We define $R_2 \wedge S_2$ in $\mathscr{T}_4$ by

(2.5)
$$\begin{aligned} &R_2 \wedge S_2(X_1, X_2, Y_1, Y_2) \\ &= R_2(X_1, Y_1)S_2(X_2, Y_2) + R_2(X_2, Y_2)S_2(X_1, Y_1) \\ &- R_2(X_1, Y_2)S_2(X_2, Y_1) - R_2(X_2, Y_1)S_2(X_1, Y_2). \end{aligned}$$

A simple computation shows that $R_2 \wedge S_2 \in \text{Curv}$ (a generalization of this is in **[8]**). We will denote tensor contraction in the $i - j$ place by

$$\text{con}(i, j): \mathscr{T}_k \to \mathscr{T}_{k-2}.$$

For $T_2 \in \mathscr{T}_2$ we write just con T_2, and for $T_4 \in \text{Curv}$ we write con T_4 for con(2, 4) T_4. Thus if T_4 is a Riemann curvature tensor then con T_4 is the Ricci curvature tensor and $\text{con}^2\, T_4$ is the scalar curvature.

Finite dimensional representations of $O(g)$ will be denoted by $\pi(m)$ where $m = (m_1, m_2, \ldots, m_\mu)$ is the highest weight, $\mu = [n/2]$. For simplicity of notation we delete terminal strings of zeroes, so that $\pi(2)$ means $\pi(2, 0, \ldots, 0)$. The weights m must satisfy $m_1 \geqq m_2 \geqq \ldots \geqq m_\mu \geqq 0$, m_j integers. (Strictly speaking, if n is odd and $m_\mu > 0$ then there are two distinct representations of $O(g)$ with the same highest weight, but it will be clear in our context which is meant.)

THEOREM 2.1. *Under the action of* $O(g)$, Curv *decomposes as* $\pi(0) \oplus \pi(2) \oplus \pi(2, 2)$ (*when* $n = 3$ *the third summand is deleted*) *with corresponding projection operators*

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$$P_{(0)}T = \left(\frac{1}{2n(n-1)}\operatorname{con}^2 T\right)g \wedge g$$

$$P_{(2)}T = \frac{1}{n-2}g \wedge \operatorname{con} T - \left(\frac{1}{n(n-2)}\operatorname{con}^2 T\right)g \wedge g$$

$$P_{(2,2)}T = T - \frac{1}{n-2}g \wedge \operatorname{con} T$$
$$+ \left(\frac{1}{2(n-1)(n-2)}\operatorname{con}^2 T\right)g \wedge g.$$

The $\pi(2, 2)$ *component is characterized as the kernel of* con:

$$\operatorname{con} P_{(2,2)}T = 0$$

$$\operatorname{con} T = \operatorname{con} P_{(0)}T + \operatorname{con} P_{(2)}T.$$

The dimensions and highest weight vectors are given as follows:

Component	Dimension	Highest Weight Vector(s)
$\pi(0)$	1	$\sum_{j<k} (e_j \wedge e_k) \otimes (e_j \wedge e_k)$
$\pi(2)$	$\frac{1}{2}(n-1)(n+2)$	$\sum_j (e_j \wedge a_1) \otimes (e_j \wedge a_1)$
$\pi(2, 2)$	$\frac{1}{12}n(n+1)(n+2)(n-3)$	$(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$
Curv	$\frac{1}{12}n^2(n^2-1)$	———

where $e_1, \ldots, e_n$ *is an orthonormal basis for* V^* *and* a_1 *and* a_2 *denote the first two root vectors* (*so* $a_1 = e_1 + ie_2$, $a_2 = e_3 + ie_4$ *in the case* g *is definite*).

Remarks. $P_{(2,2)}T$ is the Weyl conformal curvature tensor. The vanishing of various components of the curvature tensor of a semi-Riemannian manifold have the following geometric meanings:

(a) $T = 0$, zero curvature, flat
(b) $P_{(2)}T = 0$ and $P_{(2,2)}T = 0$, constant sectional curvature
(c) $P_{(0)}T = 0$ and $P_{(2)}T = 0$, zero Ricci curvature, Ricci flat
(d) $P_{(0)}T = 0$, zero scalar curvature
(e) $P_{(2)}T = 0$, constant Ricci curvature, Einstein metric
(f) $P_{(2,2)}T = 0$, conformally flat

(when $n \geqq 4$). When $n = 3$ it is always true that $P_{(2,2)}T = 0$ and this identity shows explicitly how to recover the full curvature tensor from the Ricci tensor.

It is interesting to reformulate the results of the theorem in terms of sectional curvature. If u and v are orthonormal vectors in V we set

$$\kappa(u, v) = T(u, v, u, v)$$

for any $T \in \mathrm{Curv}$. Then $\kappa(u, v)$ depends only on the plane spanned by u, v, and so we may refer to $\kappa(u, v)$ as the sectional curvature of this plane. Thus κ is a function on the Grassmannian manifold $G_{n,2}$ of 2-dimensional subspaces of V. It is well-known that κ determines T, so there exists an inverse to the mapping $T \rightarrow \kappa$. We will not need an explicit formula for this inverse. Of course both $T \rightarrow \kappa$ and its inverse intertwine the actions of $O(g)$, so we can carry over the decomposition of Curv to $\kappa(\mathrm{Curv})$. We denote the Ricci curvature by $\mathrm{ric}(\kappa)(u)$; this is defined for u any unit vector by

$$\mathrm{ric}(\kappa)(u) = \sum_{j=1}^{n-1} \kappa(u, v_j)$$

where $v_1, \ldots, v_{n-1}, u$ is an orthonormal basis for $V(\mathrm{ric}(\kappa)(u)$ is of course independent of the choice of basis). The Ricci curvature may also be given by an integral rather than a sum, and so is a kind of Radon transform of sectional curvature. In terms of the Ricci tensor con T we have

$$\mathrm{ric}(\kappa)(u) = \mathrm{con}\ T(u, u).$$

We also need the scalar curvature $\mathrm{scal}(\kappa)$ which is defined to be

$$\sum_{j=1}^{n} \mathrm{ric}(\kappa)(u_j)$$

where $u_1, \ldots, u_n$ is any orthonormal basis of V. Of course $\mathrm{scal}(\kappa) = \mathrm{con}^2\ T$.

COROLLARY 2.2. *Under the action of* $O(g)$, $\kappa(\mathrm{Curv})$ *decomposes as* $\pi(0) \oplus \pi(2) \oplus \pi(2, 2)$ *with projections*

$$P_{(0)}\kappa(u, v) = \frac{\mathrm{scal}(\kappa)}{n(n-1)}$$

$$P_{(2)}\kappa(u, v) = \frac{1}{(n-2)}\left(\mathrm{ric}(\kappa)(u) + \mathrm{ric}(\kappa)(v) - \frac{2\ \mathrm{scal}(\kappa)}{n}\right)$$

$$P_{(2,2)}\kappa(u, v) = \kappa(u, v) - \frac{1}{(n-2)}(\mathrm{ric}(\kappa)(u) + \mathrm{ric}(\kappa)(v)) + \frac{\mathrm{scal}(\kappa)}{(n+1)(n-2)}.$$

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Proof. Apply the definition of sectional curvature to the expressions for $P_{(0)}$, $P_{(2)}$ and $P_{(2,2)}$ given in the theorem and compute.

Remarks. What kinds of functions on $G_{n,2}$ appear in $\kappa(\mathrm{Curv})$? By comparing the decomposition of $\kappa(\mathrm{Curv})$ with the complete description of harmonic analysis of functions on $G_{n,2}$ given in **[12]** (see **[13]** for corrections) we see that the answer is: all functions $f(u, v)$ which are polynomials homogeneous of degree 2 in u and degree 2 in v and are invariant under the transformations $(u, v) \rightarrow (au + bv, cu + dv)$ for any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $O(2)$. The invariance under $O(2)$ is clearly needed to make f a function on $G_{n,2}$, and the polynomial form is clear from the definition of sectional curvature. What we have shown is that there are no other pointwise restrictions on the sectional curvature.

The highest weight vectors for the representations give rise to the following sectional curvature functions:

$$1 \text{ for } \pi(0),$$

$$(a_1 \cdot u)^2 v \cdot v + (a_1 \cdot v)^2 u \cdot u \quad \text{for } \pi(2),$$

$$(\,(a_1 \cdot u)(a_2 \cdot v) - (a_2 \cdot u)(a_1 \cdot v)\,)^2 \quad \text{for } \pi(2, 2).$$

The functions in the $\pi(0)$ component are just the constants, and the functions in the $\pi(2, 2)$ component are characterized by vanishing Ricci curvature. It is interesting to observe that the functions in the $\pi(2)$ and $\pi(0)$ components split into sums of functions of u and v separately, and this condition actually characterizes those components.

LEMMA 2.3. *Let $\kappa(u, v) \in \kappa(\mathrm{Curv})$. Then there exist functions f_1, f_2 such that $\kappa(u, v) = f_1(u) + f_2(v)$ if and only if $\kappa \in \pi(0) \oplus \pi(2)$, or in other words $P_{(2,2)}\kappa = 0$.*

Proof. If $P_{(2,2)}\kappa = 0$ we may take

$$f_1(u) = f_2(u) = \frac{1}{(n - 2)} \operatorname{ric}(\kappa)(u) - \frac{\operatorname{scal}(\kappa)}{2(n - 1)(n - 2)}.$$

Conversely, if $\kappa(u, v) = f_1(u) + f_2(v)$ then by $\kappa(u, v) = \kappa(v, u)$ we see that $f_1(u) - f_2(u) = f_1(v) - f_2(v)$ if $u \perp v$. By varying u and v we obtain $f_1(u) - f_2(u) = \text{const}$ for all u, and so by absorbing the constant into the functions we may assume $f_1 = f_2$, say $\kappa(u, v) = f(u) + f(v)$. Let $u_1, \ldots, u_n$ be a fixed orthonormal basis. Computing $\operatorname{ric}(\kappa)(u_j)$ using this basis we find

$$\operatorname{ric}(\kappa)(u_j) = (n - 2)f(u_j) + \sum_{k=1}^{n} f(u_k)$$

and summing on j we see

$$\operatorname{scal}(\kappa) = 2(n - 1) \sum_{k=1}^{n} f(u_k)$$

hence

$$f(u_j) = \frac{1}{(n-2)} \operatorname{ric}(\kappa)(u_j) - \frac{\operatorname{scal}(\kappa)}{2(n-1)(n-2)}.$$

But u_j can be an arbitrary unit vector, so we have $P_{(2,2)}\kappa = 0$.

COROLLARY 2.4. *A semi-Riemannian manifold with* $n \geqq 4$ *is conformally flat if and only if its sectional curvature can be written* $\kappa(u, v) = f_1(u) + f_2(v)$ *at every point.*

Proof. Combine the lemma with the Weyl-Schouten theorem in (f) above.

When $n = 3$, $P_{(2,2)}\kappa = 0$ and the resulting formula shows explicitly how to recover sectional curvature from Ricci curvature.

We conclude this section with a brief discussion of what happens in the special case $n = 4$. If V is given an orientation we can look at the subgroup $S0(g)$ of orientation preserving maps. For all the components $\pi(m)$ we have considered, they remain irreducible under $S0(g)$, with the exception of $\pi(2, 2)$ for $n = 4$, which splits

$$\pi(2, 2) = \tilde{\pi}(2, 2) \oplus \tilde{\pi}(2, -2).$$

Each $\tilde{\pi}(2, \pm 2)$ has dimension 5, and the highest weight vectors are $(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$ and $(a_1 \wedge \bar{a}_2) \otimes (a_1 \wedge \bar{a}_2)$ (we can always arrange matters so that $a_2 = e_3 + ie_4$). To understand how $P_{(2,2)}$ splits as $\tilde{P}_{(2,2)} + \tilde{P}_{(2,-2)}$ we need to use the Hodge $*$ map, which is an involution on 2-forms when $n = 4$. Because of the skew-symmetry in the 1-2 and 3-4 places, we may regard $T \in \text{Curv}$ as a function $\tilde{T}$ of $X_1 \wedge X_2$ and $X_3 \wedge X_4$, so

$$T(X_1, X_2, X_3, X_4) = \tilde{T}(X_1 \wedge X_2, X_3 \wedge X_4).$$

Then we define $*T$

$$*T(X_1, X_2, X_3, X_4) = \tilde{T}(*(X_1 \wedge X_2), X_3 \wedge X_4).$$

In general $*T$ may not be in Curv; however tensors in Curv for which $*T = \pm T$ are called *self-dual* and *anti-self-dual*, respectively. It turns out that the splitting of $\pi(2, 2)$ is into self-dual and anti-self-dual tensors. This can be seen most easily by computing $*(a_1 \wedge a_2) = a_1 \wedge a_2$ and $*(a_1 \wedge \bar{a}_2) = -a_1 \wedge \bar{a}_2$, hence the highest weight vectors for $\tilde{\pi}(2, \pm 2)$ are self-dual and anti-self-dual, and these properties are preserved under the $S0(g)$ action. Thus

$$\tilde{P}_{(2,\pm 2)}T = \frac{1}{2}(P_{(2,2)}T \pm *P_{(2,2)}T).$$

It is possible to simplify this expression, using the identities

$$*(\mathrm{con}^2\, T)g \wedge g = \mathrm{con}^2\, Tg \wedge g$$

$$*(g \wedge \mathrm{con}\, T) = \frac{1}{2}\,\mathrm{con}^2\, Tg \wedge g - g \wedge \mathrm{con}\, T$$

(these are easiest to see in terms of sectional curvature, because $(\mathrm{con}^2\, T)g \wedge g$ corresponds to 2 $\mathrm{scal}(\kappa)$ and $g \wedge \mathrm{con}\, T$ to $\mathrm{ric}(\kappa)(u) + \mathrm{ric}(\kappa)(v)$, and under $*$ the scalar curvature is preserved and $\mathrm{ric}(\kappa)(u) + \mathrm{ric}(\kappa)(v)$ is sent to $\mathrm{scal}(\kappa) - (\mathrm{ric}(\kappa)(u) + \mathrm{ric}(\kappa)(v)))$. Thus

$$\widetilde{P}_{(2,2)}T = \frac{1}{2}(T + *T) - \frac{1}{24}(\mathrm{con}^2\, T)g \wedge g,$$

$$\widetilde{P}_{(2,-2)}T = \frac{1}{2}(T - *T) - \frac{1}{2}g \wedge \mathrm{con}\, T + \frac{1}{8}(\mathrm{con}^2\, T)g \wedge g.$$

3. Covariant derivative of Riemann curvature tensors. Let $\nabla\mathrm{Curv} \subseteq \mathscr{T}_5$ denote the tensors that are skew-symmetric in the 1-2 and 3-4 places, and acyclic in the 1-2-3 and 3-4-5 places, so that $T \in \nabla\mathrm{Curv}$ if and only if

$$T(X_1, X_2, X_3, X_4, X_5) = -T(X_2, X_1, X_3, X_4, X_5) \tag{3.1}$$

$$T(X_1, X_2, X_3, X_4, X_5) = -T(X_1, X_2, X_4, X_3, X_5) \tag{3.2}$$

$$T(X_1, X_2, X_3, X_4, X_5) + T(X_2, X_3, X_1, X_4, X_5) + T(X_3, X_1, X_2, X_4, X_5) = 0 \tag{3.3}$$

$$T(X_1, X_2, X_3, X_4, X_5) + T(X_1, X_2, X_4, X_5, X_3) + T(X_1, X_2, X_5, X_3, X_4) = 0. \tag{3.4}$$

It is well-known that the first covariant derivative of the curvature tensor of a semi-Riemannian metric belongs to $\nabla\mathrm{Curv}$, and all elements of $\nabla\mathrm{Curv}$ arise in this way. Similarly, the first covariant derivative of a Ricci tensor is known to satisfy

$$T(X_1, X_2, X_3) = T(X_2, X_1, X_3) \tag{3.5}$$

$$\mathrm{con}(1, 3)\, T = \frac{1}{2}\,\mathrm{con}(1, 2)\, T \tag{3.6}$$

and no other identities, so we define $\nabla\mathrm{Ric} \subseteq \mathscr{T}_3$ to be all tensors satisfying (3.5) and (3.6).

For $T \in \nabla\mathrm{Curv}$ we write $\mathrm{con}\, T$ for $\mathrm{con}(2, 4)\, T$, and observe

$$\mathrm{con}: \nabla\mathrm{Curv} \to \nabla\mathrm{Ric}$$

by the usual derivation of (3.5) and (3.6) for the covariant derivative of the Ricci tensor. For $T \in \nabla\mathrm{Ric}$ we write $\mathrm{con}\, T$ for $\mathrm{con}(1, 2)\, T$, and observe that this is just the derivative of the scalar curvature if T is the covariant derivative of the Ricci tensor.

We begin by obtaining the decomposition of ∇Ric under the natural action of $O(g)$. We denote by sym T the symmetrization of the tensor T, so that for $T \in \nabla\mathrm{Ric}$,

$$\mathrm{sym}\, T(X_1, X_2, X_3) = \frac{1}{3}(T(X_1, X_2, X_3) + T(X_2, X_3, X_1) + T(X_3, X_1, X_2))$$

since T already is symmetric in the 1-2 place.

THEOREM 3.1. *Under the action of* $O(g)$, *the space* ∇Ric *decomposes as* $\pi(1) \oplus \pi(2, 1) \oplus \pi(3)$ (*when* $n = 3$ *the second summand is* $\pi(2)$) *with corresponding projection operators*

$$Q_{(1)}T = \frac{1}{2(n-1)}\left(g \otimes \mathrm{con}\, T + \left(\frac{3(n-2)}{n+2}\right)\right) \mathrm{sym}(g \otimes \mathrm{con}\, T))$$

$$Q_{(2,1)}T = T - \mathrm{sym}\, T - \frac{1}{2(n-1)}(g \otimes \mathrm{con}\, T - \mathrm{sym}(g \otimes \mathrm{con}\, T))$$

$$Q_{(3)}T = \mathrm{sym}\, T - \frac{2}{n+2}\, \mathrm{sym}(g \otimes \mathrm{con}\, T).$$

We have

$$\mathrm{con}\, Q_{(1)}T = \mathrm{con}\, T, \quad \mathrm{con}\, Q_{(2,1)}T = 0 \quad \text{and } \mathrm{con}\, Q_{(3)}T = 0.$$

The summands $\pi(2, 1)$ *and* $\pi(3)$ *make up the kernel of* con *and are distinguished by the fact that tensors in* $\pi(3)$ *are symmetric and tensors in* $\pi(2, 1)$ *are acyclic. The dimensions and highest weight vectors are as follows*:

Component	Dimension	Highest Weight Vector(s)
$\pi(1)$	n	$\sum_k \left(e_k \otimes e_k \otimes a_1 + \frac{n-2}{2n} \times (e_k \otimes a_1 \otimes e_k + a_1 \otimes e_k \otimes e_k)\right)$
$\pi(3)$	$\frac{1}{6}n(n-1)(n+4)$	$a_1 \otimes a_1 \otimes a_1$
$\pi(2, 1)$	$\frac{1}{3}n(n^2-4)$	$2a_1 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_1 - a_2 \otimes a_1 \otimes a_1$
$\pi(2)(n = 3)$	5	$2a_1 \otimes a_1 \otimes e_3 - a_1 \otimes e_3 \otimes a_1 - e_3 \otimes a_1 \otimes a_1$
∇Ric	$\frac{1}{2}n(n-1)(n+2)$	———

Proof. The subspace of $\mathcal{T}_3$ of tensors symmetric in the 1-2 place is the tensor product of symmetric 2-tensors with V, or $(\pi(2) \oplus \pi(0)) \otimes \pi(1)$, which is equal to $\pi(3) \oplus \pi(2, 1) \oplus \pi(1) \oplus \pi(1)$ by the Clebsch-Gordon theorem ([**9**]). Thus we know a priori that ∇Ric must be a proper subspace of this. Now a straightforward computation shows that the given Q's are projections into ∇Ric and preserve the given highest weight vectors, hence they map onto the correct components. Since there are no larger proper subspaces we have the complete decomposition.

The $\pi(2, 1)$ component is related to a tensor (sometimes called the *Cotton* tensor) used by Schouten as a substitute for the Weyl conformal tensor when $n = 3$. For $T \in \nabla$Ric we define the Schouten map Sch:∇Ric $\rightarrow \mathscr{T}_3$ by

$$\begin{aligned}\mathrm{Sch}(T)(X_1, X_2, X_3) &= \frac{1}{n-2}(T(X_1, X_2, X_3) - T(X_1, X_3, X_2)) \\ &\quad + \frac{1}{2(n-1)(n-2)}(g(X_1, X_3) \operatorname{con} T(X_2) \\ &\qquad - g(X_1, X_2) \operatorname{con} T(X_3)).\end{aligned}$$

The relationship with the Weyl conformal tensor $C = P_{(2,2)}T_4$ (for T_4 the Riemann curvature tensor) is that

$$(n - 3) \,\mathrm{Sch}(\nabla \operatorname{con} T_4) = \operatorname{con}(1, 5) \nabla C \quad \text{for } n \geqq 4$$

(when $n = 3$ both sides vanish), so that $C \equiv 0$ implies

$$\mathrm{Sch}(\nabla \operatorname{con} T_4) \equiv 0.$$

When $n = 3$ Schouten showed that $\mathrm{Sch}(\nabla \operatorname{con} T_4) \equiv 0$ is necessary and sufficient for the manifold to be conformally flat.

COROLLARY 3.2. $\mathrm{Sch}(Q_{(2,1)}T) = \mathrm{Sch}(T)$ *while* $\mathrm{Sch}(Q_{(1)}T) = 0$ *and* $\mathrm{Sch}(Q_{(3)}T) = 0$. *Thus, when* $n = 3$, *a necessary and sufficient condition for a manifold to be conformally flat is that the covariant derivative of the Ricci tensor belong to the* $\pi(3) \oplus \pi(1)$ *components.*

Proof. This is a straightforward computation.

Now we indicate how we obtain the decomposition of ∇Curv. We have the map con:∇Curv $\rightarrow \nabla$Ric which is group equivariant and is known to be onto (this fact will emerge in the proof). Thus we have a preliminary decomposition

$$\nabla\mathrm{Curv} = \ker(\mathrm{con}) \oplus \pi(3) \oplus \pi(2, 1) \oplus \pi(1).$$

To get the projection operators onto the $\pi(3)$, $\pi(2, 1)$ and $\pi(1)$ factors we need to solve the following algebraic problem: for each Q projection in Theorem 3.1 find a corresponding operator P:∇Curv $\rightarrow \nabla$Curv which is group equivariant, annihilates ker(con) and such that the diagram

$$\begin{tikzcd}
\nabla\mathrm{Curv} \arrow[r, "P"] \arrow[d, "\mathrm{con}"] & \nabla\mathrm{Curv} \arrow[d, "\mathrm{con}"] \\
\nabla\mathrm{Ric} \arrow[r, "Q"] & \nabla\mathrm{Ric}
\end{tikzcd}$$

commutes. The condition that P annihilate ker(con) is satisfied automatically if PT is given linearly in terms of con T. The resulting algebraic problem has a unique solution which is given below (these solutions prove that con is onto). What about the ker(con) component? By a fortunate accident it is irreducible. Indeed it is easy to find a weight vector with weight (3, 2) in ker(con), and (3, 2) is a priori the highest weight vector that could occur in ∇Curv because of the two skew-symmetry conditions. Thus ker(con) must contain a $\pi(3, 2)$ component. But a dimension count shows that there can't be any other components.

In order to describe the projections we introduce some simplifying notation. If $R_2 \in \mathscr{T}_2$ is symmetric and $S_3 \in \mathscr{T}_3$ is symmetric in the 1-2 place, we define the big wedge by ignoring the last place in S_3:

$$\begin{aligned}
&R_2 \wedge S_3(X_1, X_2, Y_1, Y_2, Z) \\
&= R_2(X_1, Y_1)S_3(X_2, Y_2, Z) + R_2(X_2, Y_2)S_3(X_1, Y_1, Z) \\
&- R_2(X_1, Y_2)S_3(X_2, Y_1, Z) - R_2(X_2, Y_1)S_3(X_1, Y_2, Z).
\end{aligned}$$

If S_3 is fully symmetric then $R_2 \wedge S_3 \in \nabla$Curv by the same reasoning that shows $R_2 \wedge S_2 \in$ Curv if S_2 is symmetric. Aside from a constant multiple, $R_2 \wedge S_3$ is obtained from $R_2(X_1, Y_1)S_3(X_2, Y_2, Z)$ by skew-symmetrizing in (X_1, X_2) and (Y_1, Y_2). We will also need a more complicated product (denoted $\wedge'$) which in addition involves symmetrizing in (X_1, Y_1, Z). This would appear to involve 24 terms, but because R_2 is symmetric it only requires 12. In gory detail

$$\begin{aligned}
&R_2 \wedge' S_3(X_1, X_2, Y_1, Y_2, Z) \\
&= R_2 \wedge S_3 \\
&+ R_2(X_1, Z)(S_3(X_2, Y_2, Y_1) - S_3(X_2, Y_1, Y_2)) \\
&+ R_2(X_2, Z)(S_3(X_1, Y_1, Y_2) - S_3(X_1, Y_2, Y_1)) \\
&+ R_2(Z, Y_1)(S_3(X_2, Y_2, X_1) - S_3(X_1, Y_2, X_2)) \\
&+ R_2(Z, Y_2)(S_3(X_1, Y_1, X_2) - S_3(X_2, Y_1, X_1)).
\end{aligned}$$

A straightforward but lengthy computation shows that if S_3 is also acyclic then $R_2 \wedge' S_3 \in \nabla$Curv.

If $R_4 \in$ Curv and $S_1 \in \mathscr{T}_1$ we define $R_4 \otimes' S_1$ by

$$R_4 \otimes' S_1(X_1, X_2, X_3, X_4, Z)$$
$$= R_4(X_1, X_2, X_3, X_4)S_1(Z)$$
$$+ \frac{1}{2}R_4(Z, X_2, X_3, X_4)S_1(X_1) + \frac{1}{2}R_4(X_1, Z, X_3, X_4)S_1(X_2)$$
$$+ \frac{1}{2}R_4(X_1, X_2, Z, X_4)S(X_3) + \frac{1}{2}R_4(X_1, X_2, X_3, Z)S_1(X_4).$$

It is straightforward to verify that $R_4 \otimes' S_1 \in \nabla\mathrm{Curv}$.

THEOREM 3.3. *Under the action of* $O(g)$, *the space* $\nabla\mathrm{Curv}$ *decomposes as* $\pi(1) \oplus \pi(3) \oplus \pi(2, 1) \oplus \pi(3, 2)$ (*when* $n = 3$ *the last summand is deleted and* $\pi(2, 1)$ *is replaced by* $\pi(2)$) *with corresponding projections*

$$P_{(1)}T = \frac{1}{2(n+2)(n-1)}(g \wedge g) \otimes' \mathrm{con}^2 \, T$$
$$P_{(3)}T = \frac{1}{n-2} g \wedge Q_{(3)} \,\mathrm{con}\, T$$
$$P_{(2,1)}T = \frac{1}{n+1} g \wedge' Q_{(2,1)} \,\mathrm{con}\, T$$
$$P_{(3,2)}T = T - P_{(1)}T - P_{(3)}T - P_{(2,1)}T$$

where $\pi(3, 2)$ *is the kernel of* con *and*

$$\mathrm{con}\, P_{(m)}T = Q_{(m)} \,\mathrm{con}\, T$$

for $m = (1), (3)$ *or* $(2, 1)$. *The corresponding dimensions and highest weight vectors are as follows*:

Component	Dimension	Highest Weight Vector(s)
$\pi(1)$	n	$\sum_{j<k} ((a_1 \wedge e_j) \otimes (e_j \wedge e_k) + (e_j \wedge e_k) \otimes (a_1 \wedge e_j)) \otimes e_k - (e_j \wedge e_k) \otimes (e_j \wedge e_k) \otimes a_1)$
$\pi(3)$	$\frac{1}{6}n(n-1)(n+4)$	$\sum_k (a_1 \wedge e_k) \otimes (a_1 \wedge e_k) \otimes a_1$
$\pi(2, 1)$	$\frac{1}{3}n(n^2-4)$	$\sum_k (2(a_1 \wedge e_k) \otimes (a_1 \wedge e_k) \otimes a_2 - (a_1 \wedge e_k) \otimes (a_2 \wedge e_k) \otimes a_1 - (a_2 \wedge e_k) \otimes (a_1 \wedge e_k) \otimes a_1 + 3(a_1 \wedge e_k) \otimes (a_1 \wedge a_2) \otimes e_k + 3(a_1 \wedge a_2) \otimes (a_1 \wedge e_k) \otimes e_k)$
$\pi(3, 2)$	$\frac{1}{24}(n+4)(n+2) \times n(n-1)(n-3)$	$(a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1$
$\nabla\mathrm{Curv}$	$\frac{1}{24}n^2(n^2-1)(n+2)$	———

(*when* $n = 3$ *replace* a_2 *by* e_3 *in the highest weight vector for* $\pi(2)$).

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Proof. As remarked above,

$$P_{(m)}:\nabla\mathrm{Curv} \to \nabla\mathrm{Curv}$$

for $m = (1)$, (3) or $(2, 1)$ since $Q_{(3)}$ con T is symmetric and $Q_{(2,1)}$ con T is acyclic. Obviously $P_{(m)}$ annihilates the kernel of con and is group equivariant. A simple but lengthy computation shows

$$\text{con } P_{(m)}T = Q_{(m)} \text{ con } T$$

(in this computation the various factors in the definition are determined). This proves that $P_{(m)}$ is the projection onto the $\pi(m)$ component for these values of (m). Now $(a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1$ is clearly a weight vector with weight $(3, 2)$, and a simple computation shows it is in ∇Curv and in the kernel of con. Thus $P_{(3,2)}$, which is clearly the projection onto ker(con), must project onto a $\pi(3, 2)$ component if we can verify that the sum of the dimensions of $\pi(1)$, $\pi(3)$, $\pi(2, 1)$ and $\pi(3, 2)$ equals dim ∇Curv. Now the dimensions of the $\pi(m)$ components is given as above from the Weyl dimension formula, and elementary algebra gives the sum as

$$\frac{1}{24}n^2(n^2 - 1)(n + 2).$$

A lengthy but routine argument shows that this is dim ∇Curv.

We consider now the interplay between the decompositions of Curv and ∇Curv. Of course at an individual point we can arrange to achieve any prescribed value of $T \in$ Curv and $\nabla T \in \nabla$Curv independently. But on a global level there are interactions. On the simplest level, the passage from T to ∇T can be represented by taking the tensor product with the representation $\pi(1)$. Now by the generalized Glebsch-Gordon formula we know

$$\pi(0) \otimes \pi(1) = \pi(1)$$
$$\pi(2) \otimes \pi(1) = \pi(3) \oplus \pi(2, 1) \oplus \pi(1)$$
$$\pi(2, 2) \otimes \pi(1) = \pi(3, 2) \oplus \pi(2, 2, 1) \oplus \pi(2, 1).$$

Therefore the $\pi(3, 2)$ component in ∇Curv arises solely from the $\pi(2, 2)$ component in Curv, and similarly for the $\pi(3)$ component in ∇Curv and the $\pi(2)$ component of Curv. On the other hand, the $\pi(1)$ and $\pi(2, 1)$ components of ∇Curv have two possible sources of origin. We can summarize the relations in the following diagrams:

$$n \geqq 4 \qquad \begin{tikzcd}[row sep=small] \mathrm{Curv} & \nabla\mathrm{Curv} \\ & \pi(3,2) \\ \pi(2,2) \arrow[ur, dash] \arrow[r, dash] & \pi(2,1) \\ \pi(2) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \pi(3) \\ \pi(0) \arrow[r, dash] & \pi(1) \end{tikzcd}$$

$$\begin{tikzcd}
 & \text{Curv} & \nabla\text{Curv} \\
 & & \pi(3) \\
n = 3 & \pi(2) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \pi(2) \\
 & \pi(0) \arrow[r, dash] & \pi(1)
\end{tikzcd}$$

We expect that if $\pi(m)$ is a component of Curv and $\pi(m')$ any component of ∇Curv joined by a line to $\pi(m)$, then $P_{(m')} \nabla R$ should be given in terms of $\nabla P_{(m)}R$. Conversely, $\nabla P_{(m)}R$ should be expressible in terms of all the $P_{(m')} \nabla R$ for which $\pi(m')$ is joined by a line to $\pi(m)$. Here we are taking R to be the Riemann curvature tensor and ∇ the covariant derivative on a semi-Riemannian metric space, and we apply the projections on the tangent space at each point. Since $\nabla g = 0$ and ∇ commutes with con, we compute

(3.7) $$\nabla P_{(0)}R = \frac{1}{2n(n-1)}(g \wedge g) \otimes \text{con}^2 \nabla R$$

(3.8) $$\nabla P_{(2)}R = \frac{1}{n-2} g \wedge \text{con} \nabla R - \frac{1}{n(n-2)}(g \wedge g) \otimes \text{con}^2 \nabla R$$

(3.9) $$\nabla P_{(2,2)}R = \nabla R - \frac{1}{n-2} g \wedge \text{con} \nabla R + \frac{1}{2(n-1)(n-2)}(g \wedge g) \otimes \text{con}^2 \nabla R.$$

It then is an algebraic problem to solve for $P_{(m')} \nabla R$ in terms of these and to express these in terms of $P_{(m')} \nabla R$.

THEOREM 3.4. *The 6 expressions for $P_{(m')} \nabla R$ in terms of $\nabla P_{(m)}R$ are as follows*:

(a) $$\begin{aligned} &P_{(1)} \nabla R(X_1, X_2, X_3, X_4, Z) \\ &= \frac{2n}{n+2}\Big(\nabla P_{(0)}R(X_1, X_2, X_3, X_4, Z) \\ &+ \frac{1}{2} \nabla P_{(0)}R(Z, X_2, X_3, X_4, X_1) + \frac{1}{2} \nabla P_{(0)}R(X_1, Z, X_3, X_4, X_2) \\ &+ \frac{1}{2} \nabla P_{(0)}R(X_1, X_2, Z, X_4, X_3) + \frac{1}{2} \nabla P_{(0)}R(X_1, X_2, X_3, Z, X_4)\Big) \end{aligned}$$

(b) $$\begin{aligned} &P_{(1)} \nabla R = \frac{2n}{(n+1)(n-2)(n-1)} g \wedge g \\ &\otimes' \text{con}(1, 3)\, \text{con}\, \nabla P_{(2)}R \end{aligned}$$

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(c) $$P_{(3)} \nabla R = \frac{1}{n-2} g \wedge Q_3(\operatorname{con} \nabla P_{(2)}R) + \frac{2}{(n-2)^2} g \wedge Q_3(g \otimes \operatorname{con}(1, 3) \operatorname{con} \nabla P_{(2)}R)$$

(d_1) &

(e_1) $$P_{(2,1)} \nabla R = \frac{1}{n+1} g \wedge' Q_{(2,1)}(\operatorname{con} \nabla R) \quad \textit{where}$$

(d_2) $$Q_{(2,1)}(\operatorname{con} \nabla R) = Q_{(2,1)}(\operatorname{con} \nabla P_{(2)}R) + \frac{2}{n-2} Q_{(2,1)}(g \otimes \operatorname{con}(1, 3) \operatorname{con} \nabla P_{(2)}R) \quad \textit{or}$$

(e_2) &

(f_2) $$Q_{(2,1)}(\operatorname{con} \nabla R)(X, Y, Z) = \frac{n-2}{3(n-3)}(\operatorname{con}(1, 5) \nabla P_{(2,2)}R(X, Y, Z) - \operatorname{con}(1, 5) \nabla P_{(2,2)}R(Y, Z, X)) \quad \textit{if } n > 3,$$

(f_1) $$P_{(3,2)} \nabla R = \nabla P_{(2,2)}R - \frac{1}{(n+1)} g \wedge' Q_{(2,1)}(\operatorname{con} \nabla R) + \frac{1}{n-2} g \wedge Q_{(2,1)}(\operatorname{con} \nabla R).$$

The 3 *expressions for* $\nabla P_{(m)}R$ *in terms of* $P_{(m')} \nabla R$ *are*

(i) $$\nabla P_{(0)}R = \frac{1}{2n(n-1)} g \wedge g \otimes \operatorname{con}^2 P_{(1)} \nabla R$$

(ii) $$\nabla P_{(2)}R = \frac{1}{n-2}(g \wedge \operatorname{con} P_{(1)} \nabla R + g \wedge \operatorname{con} P_{(3)} \nabla R + g \wedge \operatorname{con} P_{(2,1)} \nabla R) - \frac{1}{n(n-2)}(g \wedge g) \otimes \operatorname{con}^2 P_{(1)} \nabla R)$$

(iii) $$\nabla P_{(2,2)}R = P_{(3,2)} \nabla R + P_{(2,1)} \nabla R - \frac{1}{n-2} g \wedge \operatorname{con} P_{(2,1)} \nabla R.$$

Proof. From (3.7) and the definition of $P_{(1)}$ we obtain (a). From (3.8) we contract to obtain

$$\operatorname{con} \nabla P_{(2)}R = \operatorname{con} \nabla R - \frac{1}{n} g \otimes \operatorname{con}^2 \nabla R$$

and

$$\text{(3.10)} \quad \operatorname{con}(1, 3) \operatorname{con} \nabla P_{(2)}R = \frac{n-2}{2n} \operatorname{con}^2 \nabla R$$

in view of (3.6) which holds for $\operatorname{con} \nabla R$, so

$$\text{(3.11)} \quad \operatorname{con} \nabla R = \operatorname{con} \nabla P_{(2)}R + \frac{2}{n-2} g \otimes \operatorname{con}(1, 3) \operatorname{con} \nabla P_{(2)}R.$$

We obtain (b) and (c) from the definitions of $P_{(1)}$ and $P_{(3)}$ using (3.10) and (3.11) and similarly (d_2) from (3.11). Of course (d_1) & (e_1) is just the definition of $P_{(2,1)}$. We obtain (i) from (3.7) and the fact that

$$\operatorname{con}^2 \nabla R = \operatorname{con}^2 P_{(1)} \nabla R,$$

and similarly (ii) from (3.8) and the fact that

$$\operatorname{con} \nabla R = \operatorname{con} P_{(1)} \nabla R + \operatorname{con} P_{(3)} \nabla R + \operatorname{con} P_{(2,1)} \nabla R.$$

This completes the proof of all the identities that do not involve $P_{(2,2)}$.

Now we have already observed that

$$\begin{aligned}\operatorname{con}(1, 5) \nabla P_{(2,2)}R &= (n-3) \operatorname{Sch}(\operatorname{con} \nabla R) \\ &= (n-3) \operatorname{Sch}(Q_{(2,1)} \operatorname{con} \nabla R)\end{aligned}$$

by Corollary 3.2 and the remarks preceding it. But $\operatorname{con} Q_{(2,1)} = 0$ and $Q_{(2,1)} \operatorname{con} \nabla R$ is acyclic so

$$\begin{aligned}&\operatorname{con}(1, 5) \nabla P_{(2,2)}R(X, Y, Z) - \operatorname{con}(1, 5) \nabla P_{(2,2)}R(Y, Z, X) \\ &= \frac{n-3}{n-2}(Q_{(2,1)} \operatorname{con} \nabla R(X, Y, Z) - Q_{(2,1)} \operatorname{con} \nabla R(Z, X, Y) \\ &\qquad - Q_{(2,1)} \operatorname{con} \nabla R(Y, Z, X) + Q_{(2,1)} \operatorname{con} \nabla R(X, Y, Z)) \\ &= \frac{3(n-3)}{n-2} Q_{(2,1)} \operatorname{con} \nabla R(X, Y, Z)\end{aligned}$$

which is (e_2) & (f_2).

Next we write

$$Q_{(3)} \operatorname{con} \nabla R = \operatorname{con} \nabla R - Q_{(2,1)} \operatorname{con} \nabla R - Q_{(1)} \operatorname{con} \nabla R$$

and substitute into the definition of $P_{(3)}$ to obtain

$$\begin{aligned}&P_{(3)} \nabla R + P_{(1)} \nabla R \\ &= \frac{1}{n-2}(g \wedge \operatorname{con} \nabla R - g \wedge Q_{(2,1)}(\operatorname{con} \nabla R)) \\ &+ P_{(1)} \nabla R - \frac{1}{n-2} g \wedge Q_{(1)}(\operatorname{con} \nabla R).\end{aligned}$$

But a computation from the definitions of $P_{(1)}$ and $Q_{(1)}$ shows

$$P_{(1)} \nabla R - \frac{1}{n-2} g \wedge Q_{(1)}(\mathrm{con}\, \nabla R)$$

$$= \frac{-1}{2(n-1)(n-2)} (g \wedge g) \otimes \mathrm{con}^2 \nabla R$$

hence by (3.9) we have

$$P_{(3)} \nabla R + P_{(1)} \nabla R$$

$$= \nabla R - \nabla P_{(2,2)} R - \frac{1}{n-2} g \wedge Q_{(2,1)}(\mathrm{con}\, \nabla R)$$

hence

$$P_{(3,2)} \nabla R = -P_{(2,1)} \nabla R + \nabla P_{(2,2)} R + \frac{1}{n-2} g \wedge Q_{(2,1)}(\mathrm{con}\, \nabla R).$$

But this is (f_1) when combined with (d_1), and gives (iii) since

$$Q_{(2,1)}(\mathrm{con}\, \nabla R) = \mathrm{con}\, P_{(2,1)} \nabla R.$$

COROLLARY 3.5. *If* $P_{(m)} R \equiv 0$ *or even just* $\nabla P_{(m)} R \equiv 0$, *it follows that* $P_{(m')} \nabla R \equiv 0$ *for* $\pi(m')$ *connected to* $\pi(m)$. *Conversely, if* $P_{(m')} \nabla R \equiv 0$ *for every* $\pi(m')$ *connected to* $\pi(m)$, *then* $\nabla P_{(m)} R \equiv 0$. *In particular, if the Ricci tensor is parallel and the manifold is conformally flat then the manifold is locally symmetric.*

Proof. To verify the last assertion remember that conformally flat means $P_{(2,2)} R \equiv 0$ and Ricci parallel is equivalent to $\nabla P_{(2)} R \equiv 0$, hence by the diagram all $P_{(m')} \nabla R = 0$ hence $\nabla R \equiv 0$ which is equivalent to the manifold being locally symmetric. When $n = 3$ we can drop the conformally flat hypothesis.

When $n = 4$ and V is oriented, the $\pi(3, 2)$ and $\pi(2, 1)$ components of ∇Curv split if we restrict to the special orthogonal group $S0(g)$, into

$$\pi(3, 2) = \tilde{\pi}(3, 2) \oplus \tilde{\pi}(3, -2)$$

$$\pi(2, 1) = \tilde{\pi}(2, 1) \oplus \tilde{\pi}(2, -1).$$

We can describe the splitting of the projection

$$P_{(3,2)} = \tilde{P}_{(3,2)} + \tilde{P}_{(3,-2)}$$

in much the same way as we did for $P_{(2,2)}$ on Curv. We write

$$\tilde{T}(X_1 \wedge X_2, X_3 \wedge X_4, X_5) = T(X_1, X_2, X_3, X_4, X_5)$$

for $T \in \nabla$Curv and define

$$*T(X_1, X_2, X_3, X_4, X_5) = \tilde{T}(*(X_1 \wedge X_2), X_3 \wedge X_4, X_5).$$

Then

$$\tilde{P}_{(3,\pm 2)}T = \frac{1}{2}(P_{(3,2)}T \pm *P_{(3,2)}T)$$

as can be seen by inspection from the highest weight vectors. The splitting

$$P_{(2,1)} = \tilde{P}_{(2,1)} + \tilde{P}_{(2,-1)}$$

is more complicated, however, and is best seen by first splitting

$$Q_{(2,1)} = \tilde{Q}_{(2,1)} + \tilde{Q}_{(2,-1)}.$$

The idea is that the Schouten map on $Q_{(2,1)}T$ produces a tensor which is skew-symmetric in the last two places, and so we can apply the star operation and then invert. Altogether this amounts to defining a map $*'T$ for $T \in \nabla\mathrm{Ric}$ by

$$*'T(X, Y, Z) = \frac{1}{3}(\tilde{T}(X, *(Y \wedge Z)) - \tilde{T}(Y, *(Z \wedge X)))$$

where

$$\tilde{T}(X, Y \wedge Z) = T(X, Y, Z) - T(X, Z, Y)$$

and then

$$\tilde{Q}_{(2,\pm 1)}T = \frac{1}{2}(Q_{(2,1)}T \pm *'Q_{(2,1)}T).$$

Again we verify the splitting by considering highest weight vectors. Then

$$\tilde{P}_{(2,\pm 1)}T = \frac{1}{n+1} g \wedge' \tilde{Q}_{(2,\pm 1)} \text{ con } T.$$

In principle it is possible to extend the analysis of this section to higher covariant derivatives of curvature. Since there are presumably no more identities that must be satisfied, on a group theoretic level each additional covariant derivative amounts to taking a tensor product with $\pi(1)$. Now the Clebsch-Gordon theorem [**9**] gives the explicit decomposition for $m = (m_1, \ldots, m_k)$, $m_k > 0$,

$$\begin{aligned}
&\pi(m) \otimes \pi(1) \\
&= \pi(m_1 + 1, m_2, \ldots, m_k) \oplus \pi(m_1, m_2 + 1, m_3, \ldots, m_k) \\
&\oplus \ldots \oplus \pi(m_1, \ldots, m_{k-1}, m_k + 1) \oplus \pi(m_1, \ldots, m_k, 1) \\
&\oplus \pi(m_1 - 1, m_2, \ldots, m_k) \oplus \pi(m_1, m_2 - 1, m_3, \ldots, m_k) \\
&\oplus \ldots \oplus \pi(m_1, \ldots, m_{k-1}, m_k - 1)
\end{aligned}$$

(with the understanding that only summands that correspond to dominant weights are included). Thus the abstract decomposition into irreducibles becomes a completely routine, but rather complicated, computation. Presumably the corresponding projection operators can also be determined. It might perhaps be worthwhile to do this for second covariant derivatives, since there are some tensors, such as the Bach tensor [**4**], that involve these derivatives.

4. Decomposition of symmetric curvature tensors. In this section V denotes an n-dimensional ($n \geqq 2$) real vector space, V^* its dual space, and $\mathscr{T}_3^1$ the space of (1, 3) tensors $T(X_1, X_2, X_3, \omega)$ with $X_j \in V$ and $\omega \in V^*$. The group $GL(V)$ acts naturally on $\mathscr{T}_3^1$ by

$$\pi(g)T(X_1, X_2, X_3, \omega) = T(g^{-1}X, g^{-1}X_2, g^{-1}X_3, (g^{-1})^{\mathrm{tr}}\omega).$$

We denote representations of $GL(V)$ by $\pi(m)$ where $m = (m_1, \dots, m_n)$ is the highest weight of the representation, with $m_1 \geqq m_2 \geqq \dots \geqq m_n$, all m_k integers, and for simplicity of notation we delete strings of zeroes (so that $\pi(2, -1)$ stands for $\pi(2, 0, \dots, 0, -1)$).

In $\mathscr{T}_3^1$ we consider a subspace, denoted Sym Curv, of all tensors that are skew-symmetric in the 1-2 place and acyclic in the 1-2-3 place:

$$T(X_1, X_2, X_3, \omega) = -T(X_2, X_1, X_3, \omega) \tag{4.1}$$

$$T(X_1, X_2, X_3, \omega) + T(X_2, X_3, X_1, \omega) + T(X_3, X_1, X_2, \omega) = 0. \tag{4.2}$$

It is well-known that the curvature tensor

$$T(X_1, X_2, X_3, \omega) = \langle R(X_1, X_2)X_3, \omega\rangle$$

for a symmetric (torsion-free) connection at a point belongs to Sym Curv, and all tensors in Sym Curv arise in this way. We seek the decomposition of Sym Curv under the action of $GL(V)$ into irreducible components.

Let con T = con(1, 4) T denote the Ricci contraction. It maps Sym Curv onto $\mathscr{T}_2$. Now $\mathscr{T}_2$ splits easily as $\pi(2) \oplus \pi(1, 1)$, the symmetric and skew-symmetric tensors. We then have

$$\text{Sym Curv} = \pi(2) \oplus \pi(1, 1) \oplus \ker(\text{con}).$$

Again we are lucky that ker(con) turns out to be irreducible. To describe the corresponding projection operators we introduce some notation. We let δ denote the Kronecker delta tensor in $\mathscr{T}_1^1$. If $R \in \mathscr{T}_1^1$ and $S \in \mathscr{T}_2$ we define two special products $R \otimes_1 S$ and $R \otimes_2 S$ as follows:

$$\begin{aligned} R \otimes_1 S(X_1, X_2, X_3, \omega) = {} & R(X_2, \omega)(S(X_1, X_3) + S(X_3, X_1)) \\ & - R(X_1, \omega)(S(X_2, X_3) + S(X_3, X_2)) \end{aligned}$$

and

$$\begin{aligned} R \otimes_2 S(X_1, X_2, X_3, \omega) = {} & R(X_2, \omega)(S(X_1, X_3) - S(X_3, X_1)) \\ & - R(X_1, \omega)(S(X_2, X_3) - S(X_3, X_2)) \\ & + 2R(X_3, \omega)(S(X_1, X_2) - S(X_2, X_1)). \end{aligned}$$

It is a straightforward exercise to verify that both these products are in Sym Curv, and that $\mathrm{con}(R \otimes_1 S)$ is symmetric while $\mathrm{con}(R \otimes_2 S)$ is skew-symmetric.

THEOREM 4.1. *Under the action of* $GL(V)$, *the space* Sym Curv *decomposes as*

$$\pi(2) \oplus \pi(1, 1) \oplus \pi(2, 1, -1)$$

(*when* $n = 2$ *the third component is deleted*) *with corresponding projections*

$$P_{(2)}T = \frac{1}{2(n-1)}\delta \otimes_1 \mathrm{con}\, T$$

$$P_{(1,1)}T = \frac{1}{2(n+1)}\delta \otimes_2 \mathrm{con}\, T$$

$$P_{(2,1,-1)}T = T - \frac{1}{2(n-1)}\delta \otimes_1 \mathrm{con}\, T - \frac{1}{2(n+1)}\delta \otimes_2 \mathrm{con}\, T.$$

The $\pi(2, 1, -1)$ *component is the kernel of* con, *while*

$$\mathrm{con}\, P_{(2)}T(X_1, X_2) = \frac{1}{2}(\mathrm{con}\, T(X_1, X_2) + \mathrm{con}\, T(X_2, X_1))$$

while

$$\mathrm{con}\, P_{(1,1)}T(X_1, X_2) = \frac{1}{2}(\mathrm{con}\, T(X_1, X_2) - \mathrm{con}\, T(X_2, X_1)).$$

The corresponding dimensions and highest weight vectors are as follows:

Component	Dimension	Highest Weight Vector(s)
$\pi(2)$	$\frac{1}{2}n(n+1)$	$\sum_k (e_1 \wedge e_k) \otimes e_1 \otimes e_k^*$
$\pi(1, 1)$	$\frac{1}{2}n(n-1)$	$\sum_k (2(e_1 \wedge e_2) \otimes e_k \otimes e_k^* + (e_1 \wedge e_k) \otimes e_2 \otimes e_k^* + (e_k \wedge e_2) \otimes e_1 \otimes e_k^*)$
$\pi(2, 1, -1)$	$\frac{1}{3}n^2(n^2 - 4)$	$(e_l \wedge e_2) \otimes e_1 \otimes e_n^*$
Sym Curv	$\frac{1}{3}n^2(n^2 - 1)$	————

Proof. It is clear that $P_{(2)}$ and $P_{(1,1)}$ are group equivariant and annihilate ker(con). A simple computation shows $\mathrm{con}\, P_{(2)}T$ and $\mathrm{con}\, P_{(1,1)}T$ are the symmetrization and skew-symmetrization of con T, and these properties characterize them as the projections onto the $\pi(2)$ and $\pi(1, 1)$ components. Now $(e_1 \wedge e_2) \otimes e_1 \otimes e_n^*$ is clearly a weight vector of weight $(2, 1, -1)$, which is a priori the highest weight that can appear in Sym Curv, and a direct computation shows it is in the kernel of con($n \geqq 3$ here). Thus ker(con) must contain a $\pi(2, 1, -1)$ component, and the proof is completed by the dimension count

$$\dim \text{Sym Curv} = \frac{1}{3}n^2(n^2 - 1).$$

We will give an interpretation of $P_{(2,1,-1)}T$ as the Weyl projective curvature tensor in the next section.

Next we consider a subspace ∇Sym Curv of $\mathscr{T}_4^1$ of tensors that are skew-symmetric in the 1-2 place and acyclic in the 1-2-3 and 1-2-4 places:

(4.3) $$T(X_1, X_2, X_3, X_4, \omega) = -T(X_2, X_1, X_3, X_4, \omega)$$

(4.4) $$T(X_1, X_2, X_3, X_4, \omega) + T(X_2, X_3, X_1, X_4, \omega) + T(X_3, X_1, X_2, X_4, \omega) = 0$$

(4.5) $$T(X_1, X_2, X_3, X_4, \omega) + T(X_2, X_4, X_3, X_1, \omega) + T(X_4, X_1, X_3, X_2, \omega) = 0.$$

These are exactly the tensors that arise as

$$\langle \nabla_{X_4} R(X_1, X_2)X_3, \omega \rangle$$

for a symmetric connection. We write con T = con(2, 5) T for the Ricci contraction. It is easy to verify that con maps ∇Sym Curv onto ∇Sym Ric $\subseteq \mathscr{T}_3$ which is defined by the condition

(4.6) $$\begin{aligned} T(X_1, X_2, X_3) + T(X_2, X_3, X_1) + T(X_3, X_1, X_2) \\ - T(X_2, X_1, X_3) - T(X_1, X_3, X_2) \\ - T(X_3, X_1, X_2) = 0. \end{aligned}$$

We begin by obtaining the decomposition of ∇Sym Ric. Let

$$\text{cycl } T(X_1, X_2, X_3) = \frac{1}{3}(T(X_1, X_2, X_3) + T(X_2, X_3, X_1) + T(X_3, X_1, X_2)).$$

THEOREM 4.2. *Under the action of $GL(V)$, the space* ∇Sym Ric *decomposes as $\pi(3) \oplus \pi(2, 1) \oplus \pi(2, 1)$ with corresponding projections*:

$$Q_{(3)}T = \text{cycl } T$$

$$Q_{(2,1)}T = T - \text{cycl } T$$

(this is the projection onto $\pi(2, 1) \oplus \pi(2, 1)$). The $\pi(3)$ component consists of fully symmetric tensors, while the $\pi(2, 1)$ components consist of acyclic tensors. The corresponding dimensions and highest weight vectors are as follows:

Component	Dimension	Highest Weight Vector(s)
$\pi(3)$	$\frac{1}{6}n(n+1)(n+2)$	$e_1 \otimes e_1 \otimes e_1$
$\pi(2, 1)$	$\frac{1}{3}n(n^2-1)$	$(e_1 \wedge e_2) \otimes e_1$ *and* $e_1 \otimes (e_1 \wedge e_2)$
∇Sym Ric	$\frac{1}{6}n(5n-2)(n+1)$	————

Proof. An easy computation shows that $\mathcal{T}_3$ decomposes as $\pi(3) \oplus \pi(2, 1) \oplus \pi(2, 1) \oplus \pi(1, 1, 1)$ where the $\pi(1, 1, 1)$ component consists of fully skew-symmetric tensors. But (4.6) clearly eliminates this component, and the rest of the theorem is straightforward.

Now we know that ∇Sym Curv must decompose as $\pi(3) \oplus \pi(2, 1) \oplus \pi(2, 1) \oplus \ker(\text{con})$. To obtain the projections $P_{(3)}$ and $P_{(2,1)}$ we need only solve the algebraic problem of finding group equivariant maps from ∇Sym Curv to ∇Sym Curv defined linearly in terms of con T such that

$$\text{con } P_{(3)}T = Q_{(3)} \text{ con } T \text{ and con } P_{(2,1)}T = Q_{(2,1)} \text{ con } T.$$

In this case ker(con) turns out to be reducible, but it splits easily into two components characterized by symmetry and skew-symmetry in the 3-4 place.

To describe the projection operators we introduce two special products $R \otimes_3 S$ and $R \otimes_4 S$ for $R \in \mathcal{T}_1^1$ and $S \in \nabla$Sym Ric. We define

$$\begin{aligned} &R \otimes_3 S(X_1, X_2, X_3, X_4, \omega) \\ &= R(X_2, \omega)S(X_1, X_3, X_4) - R(X_1, \omega)S(X_2, X_3, X_4) \\ &+ R(X_3, \omega)(S(X_1, X_2, X_4) - S(X_2, X_1, X_4)) \\ &+ R(X_4, \omega)(S(X_1, X_3, X_2) - S(X_2, X_3, X_1)). \end{aligned}$$

Notice that if S is fully symmetric this simplifies to

$$R(X_2, \omega)S(X_1, X_3, X_4) - R(X_1, \omega)S(X_2, X_3, X_4).$$

In any case, $R \otimes_3 S$ belongs to ∇Sym Curv. Also

$$\begin{aligned} &R \otimes_4 S(X_1, X_2, X_3, X_4, \omega) \\ &= R(X_1, \omega)(S(X_2, X_4, X_3) - S(X_2, X_3, X_4)) \\ &+ R(X_2, \omega)(S(X_1, X_3, X_4) - S(X_1, X_4, X_3)) \\ &+ R(X_3, \omega)(S(X_4, X_2, X_1) - S(X_4, X_1, X_2)) \\ &+ R(X_4, \omega)(S(X_3, X_1, X_2) - S(X_3, X_2, X_1)) \end{aligned}$$

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and again $R \otimes_4 S \in \nabla\mathrm{Sym\ Curv}$.

THEOREM 4.3. *Under the action of* $GL(V)$, *the space* $\nabla\mathrm{Sym\ Curv}$ *decomposes as*

$$\pi(3) \oplus \pi(2, 1) \oplus \pi(2, 1) \oplus \pi(2, 2, -1) \oplus \pi(3, 1, -1)$$

(*when* $n = 2$ *the last two components are deleted*) *with corresponding projections*

$$P_{(3)}T = \frac{1}{n-1}\delta \otimes_3 Q_{(3)} \operatorname{con} T$$

$$P_{(2,1)}T = \frac{1}{n(n+2)}(n\delta \otimes_3 Q_{(2,1)} \operatorname{con} T + \delta \otimes_4 Q_{(2,1)} \operatorname{con} T)$$

$$\begin{aligned} &P_{(2,2,-1)}T(X_1, X_2, X_3, X_4, \omega) \\ &= \frac{1}{2}(\,(T - P_{(3)}T - P_{(2,1)}T)(X_1, X_2, X_3, X_4, \omega) \\ &\qquad - (T - P_{(3)}T - P_{(2,1)}T)(X_1, X_2, X_4, X_3, \omega)\,) \\ &P_{(3,1,-1)}T(X_1, X_2, X_3, X_4, \omega) \\ &= \frac{1}{2}(\,(T - P_{(3)}T - P_{(2,1)}T)(X_1, X_2, X_3, X_4, \omega) \\ &\qquad + (T - P_{(3)}T - P_{(2,1)}T)(X_1, X_2, X_4, X_3, \omega)\,). \end{aligned}$$

The kernel of con *consists of* $\pi(2, 2, -1) \oplus \pi(3, 1, -1)$, *and these are distinguished by skew-symmetry and symmetry in the* 3-4 *place, while*

$$\operatorname{con} P_{(3)}T = Q_{(3)} \operatorname{con} T \text{ and } \operatorname{con} P_{(2,1)}T = Q_{(2,1)} \operatorname{con} T.$$

The corresponding dimensions and highest weight vectors are as follows:

Component	Dimension	Highest Weight Vector(s)
$\pi(3)$	$\frac{1}{6}n(n+1)(n+2)$	$\sum_k (e_1 \wedge e_k) \otimes e_1 \otimes e_1 \otimes e_k^*$
$\pi(2, 1)$	$\frac{1}{3}n(n^2-1)$	$\sum_k ((e_1 \wedge e_k) \otimes (e_1 \wedge e_2) \otimes e_k^* + (e_1 \wedge e_2) \otimes (e_1 \wedge e_k) \otimes e_k^*)$ *and* $\sum_k (3(e_1 \wedge e_2) \otimes (e_1 \otimes e_k + e_k \otimes e_1) \otimes e_k^* + (e_1 \wedge e_k) \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_k^* - 2(e_2 \wedge e_k) \otimes e_1 \otimes e_1 \otimes e_k^*)$
$\pi(2, 2, -1)$	$\frac{1}{12}n(n^2-1)(n^2-4)$	$(e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \oplus e_n^*$
$\pi(3, 1, -1)$	$\frac{1}{8}(n+3)(n+1)n^2(n-2)$	$(e_1 \wedge e_2) \otimes e_1 \otimes e_1 \otimes e_n^*$
$\nabla\mathrm{Sym\ Curv}$	$\frac{1}{24}n^2(n+1)^2(5n-4)$	———

Proof. We give a direct abstract argument for the given decomposition. For T in ∇Sym Curv, if we fix X_4 and ω and look at the resulting 3-tensor

$$S(X_1, X_2, X_3) = T(X_1, X_2, X_3, X_4, \omega),$$

it is acyclic and skew-symmetric in the 1-2 place. It is straightforward to verify the space of such 3-tensors is an irreducible $\pi(2, 1)$. Next we keep ω fixed and allow X_4 to vary. Before imposing (4.5), the space of such 4-tensors

$$R(X_1, X_2, X_3, X_4) = T(X_1, X_2, X_3, X_4, \omega)$$

is $\pi(2, 1) \otimes \pi(1)$, and this product is easily computed to decompose as $\pi(3, 1) \oplus \pi(2, 2) \oplus \pi(2, 1, 1)$. Now (4.5) eliminates the last component, leaving $\pi(3, 1)$ as the tensors symmetric in the 3-4 place and $\pi(2, 2)$ as the tensors skew-symmetric in the 3-4 place, with highest weight vector

$$(e_1 \wedge e_2) \otimes e_1 \otimes e_1 \quad \text{and} \quad (e_1 \wedge e_2) \otimes (e_1 \wedge e_2).$$

Finally we allow ω to vary, which results in

$$\nabla\text{Sym Curv} = (\pi(3, 1) \oplus \pi(2, 2)) \otimes \pi(-1).$$

Again a direct computation shows

$$\pi(3, 1) \otimes \pi(-1) = \pi(3, 1, -1) \oplus \pi(3) \oplus \pi(2, 1)$$

and

$$\pi(2, 2) \otimes \pi(-1) = \pi(2, 2, -1) \oplus \pi(2, 1).$$

To see that the given projections yield this decomposition is fairly straightforward. We have already observed that $P_{(3)}T$ and $P_{(2,1)}T$ map into ∇Sym Curv, so it is only a matter of computing their contractions.

We consider next the relationships between the decompositions of Sym Curv and ∇Sym Curv applied to the curvature tensor R and its covariant derivative ∇R for a symmetric connection. They are summarized in the diagrams:

$$\begin{tikzcd}
 & \text{Sym Curv} & \nabla\text{Sym Curv} \\
 & & \pi(3, 1, -1) \\
 & \pi(2, 1, -1) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \pi(2, 2, -1) \\
n \geqq 3 & \pi(1, 1) \arrow[r, dash] & \pi(2, 1) \oplus \pi(2, 1) \\
 & \pi(2) \arrow[ur, dash] \arrow[r, dash] & \pi(3)
\end{tikzcd}$$

$$\begin{tikzcd}
 & \pi(1, 1) \arrow[r, dash] & \pi(2, 1) \oplus \pi(2, 1) \\
n = 2 & \pi(2) \arrow[ur, dash] \arrow[r, dash] & \pi(3)
\end{tikzcd}$$

We expect to express $P_{(m')} \nabla R$ in terms of $\nabla P_{(m)} R$ for each $\pi(m)$ connected to $\pi(m')$, except that since $\pi(2, 1)$ appears with multiplicity two it turns out that we need two $\nabla P_{(m)} R$'s. Conversely, each $\nabla P_{(m)} R$ is expressible in terms of all the $P_{(m')} \nabla R$'s for $\pi(m')$ connected to $\pi(m)$.

THEOREM 4.4. *Let R be the curvature tensor of a symmetric connection, and ∇R its covariant derivative, and fix a point in the manifold. Then the following* 6 *expressions give $P_{(m')} \nabla R$ in terms of $\nabla P_{(m)} R$:*

(a) $$P_{(3)} \nabla R = \frac{1}{n-1} \delta \otimes_3 Q_{(3)}(\mathrm{con} \nabla P_{(2)} R)$$

(b) $$P_{(2,1)} \nabla R = \frac{1}{n(n+2)}(n\delta \otimes_3 Q_{(2,1)}(\mathrm{con} \nabla P_{(2)} R + \mathrm{con} \nabla P_{(1,1)} R) + \delta \otimes_4 Q_{(2,1)}(\mathrm{con} \nabla P_{(2)} R + \mathrm{con} \nabla P_{(1,1)} R))$$

(c_1) &

(d_1) $$P_{(2,1)} \nabla R = \frac{1}{n(n+2)}(n\delta \otimes_3 Q_{(2,1)}(\mathrm{con} \nabla R) + \delta \otimes_4 Q_{(2,1)}(\mathrm{con} \nabla R))$$

(c_2) $$\begin{aligned} Q_{(2,1)}&(\mathrm{con} \nabla R)(X, Y, Z) \\ &= \frac{n-1}{n-2}(\mathrm{con}\ \mathrm{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(Z, X, Y) \\ &\qquad - \mathrm{con}\ \mathrm{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X, Y, Z)) \\ &\quad + \frac{2}{3} \mathrm{con} \nabla P_{(1,1)} R(X, Y, Z) - \frac{2}{3}\frac{n}{n+1} \mathrm{con} \nabla P_{(1,1)} R(Y, Z, X) \\ &\quad - \frac{2}{3(n+1)} \mathrm{con} \nabla P_{(1,1)} R(Z, X, Y) \end{aligned}$$

(d_2) $$\begin{aligned} Q_{(2,1)}(\mathrm{con} \nabla R)(X, Y, Z) &= \frac{n+1}{n-2}(2\ \mathrm{con}\ \mathrm{cycl}(1, 2, 4) \\ \nabla P_{(2,1,-1)} R(Y, Z, X) &- \mathrm{con}\ \mathrm{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X, Y, Z) \\ &- \mathrm{con}\ \mathrm{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(Z, X, Y)) \\ &+ \frac{2}{3} \mathrm{con} \nabla P_{(2)} R(X, Y, Z) \\ &+ \frac{2}{3}\frac{n+2}{n-1} \mathrm{con} \nabla P_{(2)} R(Y, Z, X) \end{aligned}$$

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$$- \frac{2}{3}\frac{2n+1}{n-1} \operatorname{con} \nabla P_{(2)} R(Z, X, Y)$$

(e$_1$) &

(f$_1$)
$$\begin{aligned}
&P_{(2,2,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&+ P_{(3,1,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&= \nabla P_{(2,1,-1)} R(X_1, X_2, X_3, X_4, \omega) \\
&- \frac{1}{n(n+2)} \delta(X_4, \omega) \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_1, X_3, X_2) \\
&+ \frac{1}{n(n+1)(n+2)} \delta(X_3, \omega) \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_1, X_4, X_2) \\
&+ \frac{1}{n(n^2-1)(n+2)} \delta(X_2, \omega) \\
&\times ((n+1) \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_1, X_3, X_4) \\
&\qquad + \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_3, X_4, X_1)) \\
&- \frac{1}{n(n^2+1)(n+2)} \delta(X_1, \omega) \\
&\times ((n+1) \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_2, X_3, X_4) \\
&\qquad + \operatorname{con} \operatorname{cycl}(1, 2, 4) \nabla P_{(2,1,-1)} R(X_3, X_4, X_2))
\end{aligned}$$

(e$_2$)
$$\begin{aligned}
&P_{(2,2,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&= \frac{1}{2}(P_{(2,2,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&\qquad + P_{(3,1,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega)) \\
&- \frac{1}{2}(P_{(2,2,-1)} \nabla R(X_1, X_2, X_4, X_3, \omega) \\
&\qquad + P_{(3,1,-1)} \nabla R(X_1, X_2, X_4, X_3, \omega))
\end{aligned}$$

(f$_2$)
$$\begin{aligned}
&P_{(3,1,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&= \frac{1}{2}(P_{(2,2,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega) \\
&\qquad + P_{(3,1,-1)} \nabla R(X_1, X_2, X_3, X_4, \omega)) \\
&+ \frac{1}{2}(P_{(2,2,-1)} \nabla R(X_1, X_2, X_4, X_3, \omega) \\
&\qquad + P_{(3,1,-1)} \nabla R(X_1, X_2, X_4, X_3, \omega)).
\end{aligned}$$

The following 3 *expressions give* $\nabla P_{(m)}R$ *in terms of* $P_{(m')}\nabla R$:

(i) $$\nabla P_{(2)}R = \frac{1}{2(n-1)}\delta \otimes_1 \operatorname{con} P_{(2,1)} \nabla R + \frac{1}{2(n-1)}\delta \otimes_1 \operatorname{con} P_{(3)} \nabla R$$

(ii) $$\nabla P_{(1,1)}R = \frac{1}{2(n+1)}\delta \otimes_2 \operatorname{con} P_{(2,1)} \nabla R$$

(iii) $$\nabla P_{(2,1,-1)}R = P_{(3,1,-1)} \nabla R + P_{(2,2,-1)} \nabla R + P_{(2,1)} \nabla R + \delta \otimes_5 \operatorname{con} P_{(2,1)} \nabla R$$

where the products $\otimes_1$ *and* $\otimes_2$ *are taken with respect to the non-differentiated variables, and* $\otimes_5$ *is defined by*

$$\begin{aligned} S \otimes_5 T(X_1, X_2, X_3, X_4, \omega) &= \frac{1}{n^2-1}(S(X_2, \omega)(T(X_1, X_3, X_4) - T(X_3, X_1, X_4)) \\ &\qquad - S(X_1, \omega)(T(X_2, X_3, X_4) - T(X_3, X_2, X_4))) \\ &\quad - \frac{1}{n+1} S(X_3, \omega)(T(X_1, X_2, X_4) - T(X_2, X_1, X_4)). \end{aligned}$$

Proof. We have

$$\operatorname{con} \nabla R = \operatorname{con} P_{(2,1)} \nabla R + \operatorname{con} P_{(3)} \nabla R,$$

and this implies (i). It is easy to see that

$$\delta \otimes_2 \operatorname{con} P_{(3)} \nabla R = 0$$

because $P_{(3)} \nabla R$ is fully symmetric, and this implies (ii). A direct computation yields

$$\begin{aligned} &\nabla P_{(2,1,-1)}R \\ &= P_{(3,1,-1)} \nabla R + P_{(2,2,-1)} \nabla R + P_{(2,1)} \nabla R \\ &- \frac{1}{2(n+1)}\delta \otimes_2 \operatorname{con} \nabla R - \frac{1}{2(n-1)}\delta \otimes_1 \operatorname{con} \nabla R \\ &+ \frac{1}{n-1}\delta \otimes_3 \operatorname{con} P_{(3)} \nabla R \end{aligned}$$

and we also compute

$$-\frac{1}{2(n-1)}\delta \otimes_1 \operatorname{con} \nabla R + \frac{1}{n-1}\delta \otimes_3 \operatorname{con} P_{(3)} \nabla R$$

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$$= \frac{1}{2(n-1)}[\delta(X_2, \omega)(\text{con } \nabla R(X_1, X_3, X_4) - \text{con } \nabla R(X_3, X_1, X_4)) - \delta(X_1, \omega)(\text{con } \nabla R(X_2, X_3, X_4) - \text{con } \nabla R(X_3, X_2, X_4))].$$

Because of the skew-symmetry of this last expression we can replace con ∇R with con $P_{(2,1)} \nabla R$, and then (iii) follows after simplification.

In the other direction, (a), (b), (c_1) and (d_1) follow immediately from the definition of $P_{(3)}$ and $P_{(2,1)}$ and the observation

$$Q_{(3)} \text{ con } \nabla P_{(1,1)}R = 0$$

because con $P_{(1,1)}R$ is skew-symmetric. A lengthy but straightforward computation shows

$$\text{(4.7)} \quad \text{con cycl}(1, 2, 4) \nabla P_{(2,1,-1)}R(X, Y, Z) = \frac{2-n}{3(n^2-1)}(n(\text{con } \nabla R(X, Y, Z) - \text{con } \nabla R(Z, Y, X)) + \text{con } \nabla R(Y, X, Z) - \text{con } \nabla R(Y, Z, X))$$

and (c_2) and (d_2) follow from this by direct computation.

Now (e_2) and (f_2) are immediate from the definitions of $P_{(2,2,-1)}$ and $P_{(3,1,-1)}$, and it is easy to see that

$$\text{(4.8)} \quad P_{(2,2,-1)} \nabla R + P_{(3,1,-1)} \nabla R = \nabla P_{(2,1,-1)}R + (\nabla P_{(1,1)}R + \nabla P_{(2)}R - P_{(3)} \nabla R - P_{(2,1)} \nabla R).$$

Finally (e_1) and (f_1) follow from (4.6), (4.7) and (4.8) by lengthy but routine computations.

COROLLARY 4.5. *If* $\nabla P_{(m)}R \equiv 0$ *then* $P_{(m')} \nabla R \equiv 0$ *provided* $\pi(m)$ *is connected to* $\pi(m')$ *and* $m' \neq (2, 1)$. *If* $\nabla P_{(m)}R \equiv 0$ *and* $\nabla P_{(\tilde{m})}R \equiv 0$ *for two distinct* $\pi(m)$ *and* $\pi(\tilde{m})$ *connected to* $\pi(2, 1)$, *then* $P_{(2,1)} \nabla R \equiv 0$. *Conversely, if* $P_{(m')} \nabla R \equiv 0$ *for every* $\pi(m')$ *connected to* $\pi(m)$, *then* $\nabla P_{(m)}R \equiv 0$. *In particular, an affine manifold that is projectively flat and has skew-symmetric Ricci curvature, must be locally affine symmetric.*

5. The Weyl projective curvature tensor. Recall that two symmetric connections Γ and $\tilde{\Gamma}$ are called *projectively equivalent* if every geodesic for Γ can be reparametrized to be a geodesic for $\tilde{\Gamma}$. (Also, without reparametrizing, an arbitrary connection can be replaced by a symmetric one without changing the geodesics.) A connection is said to be *projectively flat* if it is projectively equivalent to a flat connection (curvature and torsion zero). Of course, it is only fair to point out that there are still some open questions concerning a description of all flat connections.

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For connections associated with Riemannian or semi-Riemannian metrics, Weyl introduced the projective curvature tensor

$$\begin{aligned} &W(X_1, X_2, X_3, \omega) \\ &= R(X_1, X_2, X_3, \omega) \\ &- \frac{1}{n-1}(\delta(X_2, \omega) \operatorname{con} R(X_1, X_3) - \delta(X_1, \omega) \operatorname{con} R(X_2, X_3)) \end{aligned}$$

and proved that projectively equivalent metric connections have the same projective curvature tensors, and that a metric connection is projectively flat if and only if the projective curvature tensor vanishes (this is easily seen to be equivalent to the space having constant curvature, which is the usual way the result is stated). See, e.g., [**3**] Sections 40-41.

Now observe that if the Ricci tensor con R is symmetric, then $W = P_{(2,1,-1)}R$. We will see that in the context of symmetric connections, $P_{(2,1,-1)}R$ plays the role of the Weyl projective curvature tensor. Since this is the natural context for the concept of projectivity, we propose to call $P_{(2,1,-1)}R$ the *Weyl projective curvature tensor* for the symmetric connection. The proofs are very similar to those in the metric connection context, so we will be brief.

LEMMA 5.1. *Let $\Gamma^k_{ij}(x)$ and $\tilde{\Gamma}^k_{ij}(x)$ be expressions for symmetric connections Γ and $\tilde{\Gamma}$ in the same coordinate system. Then Γ is projectively equivalent to $\tilde{\Gamma}$ if and only if there exists a one-form $\sigma_j(x)$ such that*

$$\tilde{\Gamma}^k_{ij}(x) = \Gamma^k_{ij}(x) + \delta^k_i\sigma_j(x) + \delta^k_j\sigma_i(x). \tag{5.1}$$

Proof. Let y be a $\tilde{\Gamma}$-geodesic and let $y(h(t)) = x(t)$. If (5.1) holds then by solving

$$h''(t) = 2\sigma_j(y(h(t)))\dot{y}^j(h(t))h'(t)^2$$

we find that x is a Γ-geodesic since

$$\begin{aligned} &\ddot{x}^k(t) + \Gamma^k_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t) \\ &= h'(t)^2(\ddot{y}_k(h(t)) + \Gamma^k_{ij}(y(h(t)))\dot{y}^2(t)\dot{y}^j(t) \\ &+ \frac{h''(t)}{h'(t)^2}\dot{y}^k(h(t))). \end{aligned} \tag{5.2}$$

Conversely, if for every $\tilde{\Gamma}$-geodesic y there exists $h(t)$ such that x is Γ-geodesic, then from (5.2) we see

$$\tilde{\Gamma}^k_{ij}(y)u^iu^j = \Gamma^k_{ij}(y)u^iu^j + c(y, u)u^k$$

for some function $c(y, u)$, for every point y and every tangent vector u at y. By substituting $u = \lambda e_i + \mu e_j$ and simple algebra we obtain (5.1) with $\sigma_j(y) = (1/2)c(y, e_j)$.

THEOREM 5.2. *Let Γ and $\widetilde{\Gamma}$ be projectively equivalent symmetric connections with curvatures R and $\widetilde{R}$. Then*

$$P_{(2,1,-1)}R \equiv P_{(2,1,-1)}\widetilde{R}.$$

Proof. Computing in local coordinates and using the lemma we find

$$(5.3) \qquad \widetilde{R}^i_{jkl} = R^i_{jkl} + \delta^i_j\left(\frac{\partial \sigma_l}{\partial x^k} - \frac{\partial \sigma_k}{\partial x^l}\right) + \delta^i_l\frac{\partial \sigma_j}{\partial x^k} - \delta^i_k\frac{\partial \sigma_j}{\partial x^l}$$
$$- \delta^i_l\sigma_k\sigma_j + \delta^i_k\sigma_l\sigma_j + \Gamma^\mu_{lj}\sigma_\mu\delta^i_k - \Gamma^\mu_{kl}\sigma_\mu\delta^i_l$$

and then contracting

$$(5.4) \qquad \widetilde{R}_{jk} = R_{kj} + n\frac{\partial \sigma_j}{\partial x^k} - \frac{\partial \sigma_k}{\partial x^j} + (1-n)(\sigma_k\sigma_j + \Gamma^\mu_{kj}\sigma_\mu).$$

It is then a straightforward but lengthy computation to verify

$$P_{(2,1,-1)}\widetilde{R} = P_{(2,1,-1)}R.$$

THEOREM 5.3. *Suppose $n \geqq 3$. A symmetric connection is projectively flat if and only if the Weyl projective curvature tensor $P_{(2,1,-1)}R$ vanishes identically.*

Proof. In view of the previous theorem we need only show that $P_{(2,1,-1)}R \equiv 0$ implies projectively flat, and by the lemma this means we need to find

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i\sigma_j + \delta^k_j\sigma_i$$

for σ such that $\widetilde{R} \equiv 0$. By the previous theorem

$$P_{(2,1,-1)}\widetilde{R} \equiv 0$$

so it suffices to show $\widetilde{R}_{jk} \equiv 0$.

By (5.4) this is just

$$(5.5) \qquad R_{jk} + n\,\nabla_j\,\sigma_k - \nabla_k\sigma_j + (1-n)\sigma_k\sigma_j = 0$$

and by simple algebra this is equivalent to

$$(5.6) \qquad \nabla_j\sigma_k + \frac{1}{n^2-1}R_{jk} + \frac{n}{n^2-1}R_{kj} - \sigma_j\sigma_k = 0.$$

To complete the proof we need to solve (5.6). This system is solvable if and only if the Ricci identities

$$(5.7) \qquad \nabla_l\nabla_j\sigma_k - \nabla_j\nabla_l\sigma_k - R^\mu_{kjl}\sigma_\mu = 0$$

are consistent. But we claim this is a consequence of the hypothesis $P_{(2,1,-1)}R \equiv 0$. Indeed, a direct computation of the left side of (5.7) using (5.6) yields

$$(5.8)\qquad -\frac{1}{n^2-1}[(\nabla_l R_{jk} - \nabla_j R_{lk}) + n(\nabla_l R_{kj} - \nabla_j R_{kl}) - \sigma_j(R_{lk} + nR_{kl}) + \sigma_l(R_{jk} + nR_{kj}) - (n-1)\sigma_k(R_{jl} - R_{lj})] - \sigma_\mu R^\mu_{kjl}.$$

On the other hand $P_{(2,1,-1)}R \equiv 0$ says

$$(5.9)\qquad R^h_{ijk} = \frac{1}{n^2-1}[\delta^h_k(nR_{ij} + R_{ji}) - \delta^h_j(nR_{ik} + R_{ki}) + (n-1)\,\delta^h_i(R_{kj} - R_{jk})]$$

and after covariant differentiation and contraction in the 1-5 place (using (4.5) and (4.6)) we obtain

$$(5.10)\qquad \frac{n-2}{n^2-1}(\nabla_l R_{jk} - \nabla_j R_{lk} + n(\nabla_l R_{kj} - \nabla_j R_{kl})) = 0.$$

Together, (5.9) and (5.10) imply the vanishing of (5.8) hence the consistency of the Ricci identities (5.7).

THEOREM 5.4. *Suppose $n = 2$. A symmetric connection is projectively flat if and only if*

$$(5.11)\qquad \mathrm{con}\,\nabla\, R(X, Y, Z) - \mathrm{con}\,\nabla\, R(Z, Y, X) + 2\,\mathrm{con}\,\nabla\, R(Y, X, Z) - 2\,\mathrm{con}\,\nabla\, R(Y, Z, X) \equiv 0,$$

or in local coordinates

$$(5.11')\qquad \nabla_l R_{jk} - \nabla_j R_{lk} + 2(\nabla_l R_{kj} - \nabla_j R_{kl}) \equiv 0.$$

Proof. If (5.11) holds we can use it in place of (5.10) and repeat the argument of the previous theorem since (5.9) is always true when $n = 2$. Thus the connection is projectively flat. Conversely, if the connection is projectively flat then by the lemma there exists σ with $\widetilde{R} \equiv 0$. By (5.4) we have

$$R_{jk} + 2R_{kj} = -3\,\nabla_j\,\sigma_k + 3\sigma_j\sigma_k.$$

Using the Ricci identity (5.7) we compute

$$\nabla_l(R_{jk} + 2R_{kj}) - \nabla_j(R_{lk} + 2R_{kl}) = 3(-R^\mu_{kjl}\sigma_\mu + \nabla_l(\sigma_j\sigma_k) - \nabla_j(\sigma_l\sigma_k)).$$

But then a simple computation using (5.3) shows this vanishes, establishing (5.11′).

6. Radon transforms of sectional curvature. Singer and Thorpe [**10**] characterize Einstein metrics (curvature in $\pi(0) \oplus \pi(2, 2)$) on 4-

dimensional Riemannian manifolds by the condition that the sectional curvature of each tangent plane is equal to the sectional curvature of the orthogonal plane (see also **[6]**). Clearly such a condition makes sense only in dimension 4. To obtain an analogous condition in other dimensions involves taking averages over families of orthogonal planes. Such averages may be either discrete or continuous, as is the case with Ricci and scalar curvature.

In **[13]** we introduced a general notion of orthogonal Radon transforms (most of the discussion in **[13]** involves affine subspaces, but here it is linear subspaces that are relevant). If V is an n-dimensional vector space with inner product g then for suitable integers k, k', j we define a Radon transform

$$R(k, k', j): C^{\infty}(G_{n,k}) \rightarrow C^{\infty}(G_{n,k'})$$

where $G_{n,k}$ denotes the Grassmannian manifold of k-dimensional subspaces of V, as follows: for a fixed $\sigma' \in G_{n,k'}$ we define $R(k, k', j)f(\sigma')$ to be the average of $f(\sigma)$ where σ runs over all elements of $G_{n,k}$ such that σ intersects σ' orthogonally in a j-dimensional subspace. The set of such σ carries a unique $O(n)$-invariant measure normalized to have total mass equal to one. The cases of interest here are $R(2, 2, 0)$ (when $n \geqq 4$) and $R(2, 2, 1)$ (when $n \geqq 3$). When $n = 4$,

$$R(2, 2, 0)f(\sigma) = f(\sigma^{\perp})$$

since $\sigma^{\perp}$ is the only orthogonal plane intersecting σ at the origin.

The point is that these Radon transforms obviously intertwine the group action, and so must act as scalars on each irreducible representation (the Grassmannians are symmetric spaces and so all irreducible representations occur with multiplicity one). In particular, they preserve $\kappa(\mathrm{Curv})$ and we must have ($j = 0$ or 1)

$$R(2, 2, j)\kappa = a_j P_{(0)}\kappa + b_j P_{(2)}\kappa + c_j P_{(2,2)}\kappa$$

for some constants a_j, b_j, c_j for every $\kappa \in \kappa(\mathrm{Curv})$. We now compute the constants.

THEOREM 6.1. *When $n \geqq 4$ we have*

$$R(2, 2, 0)\kappa = P_{(0)}\kappa - \frac{2}{n-2}P_{(2)}\kappa + \frac{2}{(n-2)(n-3)}P_{(2,2)}\kappa.$$

When $n \geqq 3$ we have

$$R(2, 2, 1)\kappa = P_{(0)}\kappa + \left(\frac{1}{2} - \frac{1}{n-2}\right)P_{(2)}\kappa - \frac{1}{n-2}P_{(2,2)}\kappa$$

(*delete the last summand when $n = 3$*).

Proof. To compute the constants it suffices to evaluate $R(2, 2, 1)\kappa$ at a single point where κ does not vanish for a single function κ in each of the three components. We choose for κ the highest weight vector given in the remarks following Corollary 2.2. For $\pi(0)$ the choice is $\kappa \equiv 1$ and it is obvious that $a_0 = 1$ and $a_1 = 1$. For $\pi(2)$ the choice is

$$\kappa_{(2)}(\sigma) = (u_1 + iu_2)^2 + (v_1 + iv_2)^2$$

where u, v is any orthonormal basis for σ, while for $\pi(2, 2)$ the choice is

$$\kappa_{(2,2)}(\sigma) = (\,(u_1 + iu_2)(v_3 + iv_4) - (u_3 + iu_4)(v_1 + iv_2)\,)^2.$$

We choose $\sigma = \widetilde{\sigma}$ to be the $e_1 - e_3$ plane, say $u = e_1$, $v = e_3$. Notice that $\kappa(\widetilde{\sigma}) = 1$ in both cases.

Now to compute $R(2, 2, 0)$ we need to average $\kappa(\sigma)$ over all planes in the span of $e_2, e_4, e_5, \dots, e_n$. Such planes clearly have $u_1 = u_3 = v_1 = v_3 = 0$ and so

$$\kappa_{(2)}(\sigma) = -u_2^2 - v_2^2$$

and

$$\kappa_{(2,2)}(\sigma) = (u_1v_3 - u_3v_1)^2.$$

It is easiest to compute the averages using symmetry considerations, rather than integral formulas. Thus the average of $-u_j^2 - v_j^2$ is going to be the same for any $j = 2, 4, 5, \dots, n$. But the sum of all $-u_j^2 - v_j^2$ is -2, so

$$R_{(2,2,0)}\kappa(\widetilde{\sigma}) = -\frac{2}{n-2}.$$

Similarly, the average of $(u_jv_k - u_kv_j)^2$ is going to be the same for any distinct j and k in $(2, 4, 5, \dots, n)$. But

$$\begin{aligned}\sum \sum_{j \neq k} (u_jv_k - u_kv_j)^2 &= \sum_j \sum_k (u_jv_k - u_kv_j)^2 \\ &= 2 \sum_j u_j^2 \sum_k u_k^2 - 2(u \cdot v)^2 = 2\end{aligned}$$

because u and v are orthonormal. Thus

$$R_{(2,2,0)}\kappa_{(2,2)}(\widetilde{\sigma}) = \frac{2}{(n-2)(n-3)}.$$

To compute $R(2, 2, 1)$ we have to take the average over planes with say u in the span of e_1, e_3 and v in the span of $e_2, e_4, e_5, \dots, e_n$, and we have

$$\kappa_{(2)}(\sigma) = u_1^2 - v_2^2$$

$$\kappa_{(2,2)}(\sigma) = -(u_1v_4 - u_3v_2)^2.$$

Reasoning by symmetry, the average of u_1^2 is $1/2$ and the average of $v_2^2 = 1/(n-2)$ which yields

$$R(2, 2, 1)\kappa_{(2)}(\sigma) = \frac{1}{2} - \frac{1}{n-2}.$$

Similarly, the average of $2u_1u_3v_1v_3$ is zero and the average of $-u_1^2v_4^2 - u_3^2v_2^2$ is $-(1/(n-2))$ so

$$R_{(2,2,1)}\kappa_{(2,2)}(\sigma) = -\frac{1}{n-2}.$$

The average defining $R(2, 2, 0)$ for functions in $\kappa(\mathrm{Curv})$ can be given discretely as follows:

$$R(2, 2, 0)\kappa(\sigma) = \frac{2}{(n-2)(n-3)} \sum_{j \leqq k} \kappa(\varphi_j \wedge \varphi_k)$$

where $(\varphi_1, \ldots, \varphi_{n-2})$ is any orthonormal basis for $\sigma^{\perp}$. The reason for this is the same as the reason that scalar curvature can be given discretely or as a multiple of a continuous average. Notice that if we chose the discrete sum without the factor $2/(n-2)(n-3)$ to define $R(2, 2, 0)$ then the factor in front of $P_{(2,2)}\kappa$ would be one. It is not possible, however, to replace $R(2, 2, 1)$ by a discrete average.

Using the theorem, it is possible to state a number of analogues of the Singer-Thorpe characterization. For example, a metric is Einstein ($n \geqq 4$) if and only if

$$\frac{(n-2)(n-3)}{2} R(2, 2, 0)\kappa - \kappa = \text{constant},$$

or if and only if

$$(n-3)R(2, 2, 0)\kappa - (n-4)R(2, 2, 1)\kappa = \kappa.$$

7. Orbit structure of Riemannian curvature. We return to the space Curv and attempt to describe the structure of the $O(g)$ orbits. In a sense, this is the crucial question, because the only information about curvature at a point that is coordinate-independent is the orbit it belongs to. Given the group representation description of Curv as $\pi(0) \oplus \pi(2) \oplus \pi(2, 2)$, the description of the orbit structure could be posed entirely in the abstract setting of group representations, but as far as we can tell, nothing is gained by this. In particular, it is not clear how the orbit structure of the direct sum is related to the orbit structure of the summands. For simplicity we assume the metric is definite.

To study the orbit structure of Curv we construct numerical functions of curvature tensors that are constant on orbits. Such functions we will call *orbit invariants*. When we have enough orbit invariants to distinguish all orbits we may stop; we call such a set a *complete* set of orbit invariants. Then we need to describe the possible values of the orbit invariants as the tensor varies over all orbits. That would constitute a complete description of the orbit structure. In addition, we would like to have a *canonical form*, a representative tensor from each orbit. As a slight variation, we may do the same for $S0(g)$ orbits.

It seems likely that such a program can be achieved, although the complete description might turn out to be somewhat complicated. Here we will make a start on the problem by defining a number of orbit invariants, some of them quite well known.

We begin with the well-known observation that tensors in Curv may be regarded as symmetric bilinear forms on $\Lambda^2(V)$. The term *curvature operator* is often used. We thus define Curv Op $\subseteq \mathcal{T}_4$ to be the space of tensors $T(X_1, X_2, X_3, X_4)$ which are skew-symmetric in the 1-2 and 3-4 places, and symmetric under the interchange of (X_1, X_2) with (X_3, X_4). Thus Curv $\subseteq$ Curv Op. We will define orbit invariants for Curv Op, and there are clearly orbit invariants for Curv. The first Bianchi identity, which characterizes Curv as a subspace of Curv Op, will then put some restrictions on the values of the orbit invariants if the orbit lies in Curv.

Now the canonical form for symmetric operators on vector space produces the first set of orbit invariants, the eigenvalues of the curvature operator. We denote these by $\lambda_1, \lambda_2, \ldots, \lambda_N$ where

$$N = \frac{1}{2}n(n - 1) = \dim \Lambda^2(V),$$

and we order them by increasing size: $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_N$. We refer to them as *first stage eigenvalues*. If we were interested in the orbit structure of Curv Op under the full orthogonal group of $\Lambda^2(V)$ we would be done. But the group $O(g)$ is only a small subgroup, so we are far from having a complete set of orbit invariants. In other words, the set of tensors in Curv Op with given first stage eigenvalues splits up into a union of many $O(g)$-orbits.

Clearly we need to look at the eigenvectors associated to the eigenvalues, say $\omega_j \in \Lambda^2(V)$ is associated to λ_j. We say tensor in Curv Op is *generic at the first stage* if all the first stage eigenvalues are distinct. In that case the eigenvectors are unique up to a scalar multiple, and even up to a sign if we require them to have norm one. (Also, the isotropy subgroup of the orbit is the two element group $\pm I$.) Our original problem is then reduced to describing the orbits under $O(g)$ of the space of orthonormal bases of $\Lambda^2(V)$. Here, of course, the inner product on $\Lambda^2(V)$ is that induced by the inner product g on V, and may be succinctly given by the formula

$$\langle \omega, \tilde{\omega} \rangle = *((^*\omega) \wedge \tilde{\omega})$$

(an orientation is required to define the star operator, but since it appears twice the inner product does not depend on the choice of orientation). The tensor in Curv Op is given by the formula

$$(7.1)\qquad T(X_1, X_2, X_3, X_4) = \sum_{j=1}^{N} \lambda_j \omega_j(X_1, X_2)\omega_j(X_3, X_4).$$

Now each ω_j as an element of $\Lambda^2(V)$ has itself a canonical form as a linear combination of planes. We identify the plane with orthonormal basis u_1, u_2 with the 2-form $u_1 \wedge u_2$, with an ambiguity of $\pm$ sign. This is the natural embedding $G_{n,2} \subseteq \Lambda^2/\pm$. If we deal with oriented planes we do not have the $\pm$ ambiguity. Then

$$(7.2)\qquad \omega_j = \sum_{k=1}^{v} a_{jk}\sigma_{jk} \quad \text{where } v = [n/2],$$

$\sigma_{jk} = u_{jk} \wedge v_{jk} \in G_{n,2}$ are fully orthogonal oriented planes (orthogonal planes intersecting at the origin) and a_{jk} are non-negative real scalars arranged in increasing order. We say ω_j is *generic* if all the a_{jk} are distinct, $k = 1, \dots, v$, and non-zero. In that case the decomposition is unique. If the tensor is generic at the first stage and all the ω_j are generic, we say the tensor is *generic at the second stage*. For such tensors we have Nv orbit invariants a_{jk}, which we call *second stage eigenvalues*. Because $|\omega_j| = 1$ we have the identities

$$(7.3)\qquad \sum_{k=1}^{v} |a_{jk}|^2 = 1 \quad \text{for } j = 1, \dots, N.$$

The condition that σ_{jp} be fully orthogonal to σ_{jq} for $p \neq q$ can be expressed succinctly by the equation

$$(7.4)\qquad |\sigma_{jp} \wedge \sigma_{jq}| = 1, \quad p \neq q.$$

The condition that ω_j be orthogonal to ω_k for $j \neq k$ is

$$(7.5)\qquad \sum_{p=1}^{v}\sum_{q=1}^{v} a_{jp}a_{kq}\langle \sigma_{jp}, \sigma_{kq}\rangle = 0, \quad j \neq k.$$

To summarize, given distinct first stage eigenvalues $\lambda_1 < \lambda_2 \dots < \lambda_N$ and distinct non-zero second stage eigenvalues $0 < a_{j1} < a_{j2} < \dots < a_{jv}$ for $j = 1, \dots, N$ satisfying (7.3), if we can find oriented planes σ_{jk} satisfying (7.4) and (7.5), then there is a tensor T in Curv Op given by

$$(7.6)\qquad T(X_1, X_2, X_3, X_4) = \sum_{j=1}^{N}\sum_{p=1}^{v}\sum_{q=1}^{v} \lambda_j a_{jp} a_{jq} \sigma_{jp}(X_1, X_2)\sigma_{jq}(X_3, X_4)$$

with eigenvalues $\{\lambda_j\}$ and second stage eigenvalues $\{a_{jp}\}$. The solvability of (7.4) and (7.5) involves some complicated algebraic conditions on the second stage eigenvalues that we are unable to make explicit. At this point it is easy to compute that the first Bianchi identity requires

$$(7.7) \qquad \sum_{j=1}^{N} \sum_{p=1}^{v} \sum_{q=1}^{v} \lambda_j a_{jp} a_{jq} \sigma_{jp} \wedge \sigma_{jq} = 0$$

in order for T to belong to Curv. The solvability of (7.4), (7.5) and (7.7) puts further algebraic restrictions on the first and second stage eigenvalues.

To distinguish generic orbits we need more invariants that describe the relative positions of the planes σ_{jp}. We will refer to these as *angular invariants*. There is a simple recipe for manufacturing angular invariants: take a finite number of oriented planes σ_{jp}, and combine them using wedge products and the star operation in a specified order (e.g. $*(\sigma_{12} \wedge \sigma_{23}) \wedge \sigma_{37}$), and then take the norm of the resulting form. This gives an $O(g)$ invariant. If the resulting form is a zero form it is not necessary to take the norm; this will give an $O(g)$ invariant if the number of star operations is even, and an $S0(g)$ invariant if the number of star operations is odd (this can only happen if n is even). It will be necessary to introduce still further angular invariants in order to distinguish generic orbits, but it is not clear how to do this in a systematic fashion.

Note that conditions (7.4) and (7.5) are already expressed entirely in terms of orbit invariants, but (7.7) is not. To remedy this we remark that since $\{\omega_j\}$ is a basis for $\Lambda^2(V)$, we certainly have $\{\sigma_{jp}\}$ spanning $\Lambda^2(V)$, hence $\{\sigma_{jp} \wedge \sigma_{kq}\}$ spans $\Lambda^4(V)$. Thus we can replace (7.7) by the system

$$(7.7') \qquad \sum_{j=1}^{N} \sum_{p=1}^{v} \sum_{q=1}^{v} \lambda_j a_{jp} a_{jq} *(\sigma_{jp} \wedge \sigma_{jq} \wedge *(\sigma_{kr} \wedge \sigma_{ls})) = 0$$

for all k, r, l, s, and this involves only orbit invariants.

In terms of the representation (7.6), it is possible to describe all the important geometric quantities. It is immediate that the sectional curvature is

$$(7.8) \qquad \kappa(\sigma) = \sum_j \lambda_j \left(\sum_p a_{jp} \langle \sigma_{jp}, \sigma \rangle \right)^2$$

and a computation shows that the Ricci curvature is

$$(7.9) \qquad \operatorname{ric}(\kappa)(u) = \sum_j \lambda_j \sum_p a_{jp}^2 |\Pr(\sigma_{jp})u|^2$$

where $\Pr(\sigma)$ denotes the orthogonal projection onto the σ plane. Similarly, the Ricci tensor is

$$(7.10) \qquad \operatorname{con} R(X, Y) = \sum_j \lambda_j \sum_p a_{jp}^2 \langle \Pr(\sigma_{jp})X, Y \rangle.$$

The scalar curvature is

(7.11) $$\operatorname{scal}(\kappa) = \operatorname{con}^2 R = 2 \sum_j \lambda_j.$$

In principle, the eigenvalues of the Ricci tensor should be expressible in terms of orbit invariants, but we see no direct way to do this. When n is even we have

$$\sum_p |\operatorname{Pr}(\sigma_{jp})u|^2 = 1$$

so we obtain from (7.9) the estimate

$$\sum_{\lambda_j<0} \lambda_j a_{jv}^2 + \sum_{\lambda_j>0} \lambda_j a_{j1}^2 \leqq \operatorname{ric}(\kappa)(u) \leqq \sum_{\lambda_j<0} \lambda_j a_{j1}^2 + \sum_{\lambda_j>0} \lambda_j a_{jv}^2.$$

When $n = 4$ we can simplify the description considerably. We choose an orientation for V. Once σ_{j1} is chosen, σ_{j2} is determined up to a $\pm$ multiple, $\sigma_{j2} = \pm * \sigma_{j1}$. Therefore we change conventions and write

(7.12) $$\omega_j = \cos \theta_j \sigma_j + \sin \theta_j (*\sigma_j)$$

where $-\pi/4 \leqq \theta_j \leqq \pi/4$ (strict inequality in the generic case). We have 30 angular $S0(g)$ invariants, namely

(7.13) $$\langle *\sigma_j, \sigma_k \rangle = *(\sigma_j \wedge \sigma_k) \quad \text{and} \quad \langle \sigma_j, \sigma_k \rangle = *(\sigma_j \wedge (*\sigma_k))$$

for $1 \leqq j < k \leqq 6$. Conditions (7.3) and (7.4) are automatic, and (7.5) is replaced by

(7.5′) $$\cos(\theta_j - \theta_k)\langle \sigma_j, \sigma_k \rangle + \sin(\theta_j + \theta_k)\langle *\sigma_j, \sigma_k \rangle = 0$$

for $j \neq k$, while the Bianchi identity is

(7.7″) $$\sum_{j=1}^{6} \lambda_j \sin 2\theta_j = 0.$$

In place of (7.8-10) we have

(7.8′) $$\kappa(\sigma) = \sum_{j=1}^{6} \lambda_j (\cos \theta_j \langle \sigma, \sigma_j \rangle + \sin \theta_j \langle \sigma, *\sigma_j \rangle)^2$$

(7.9′) $$\operatorname{ric}(\kappa)(u) = \sum_{j=1}^{6} \lambda_j (\sin^2 \theta_j + \cos 2\theta_j |\operatorname{Pr}(\sigma_j)u|^2)$$

(7.10) $$\operatorname{con} R(X, Y) = \sum_{j=1}^{6} \lambda_j (\sin^2 \theta_j \langle X, Y \rangle + \cos 2\theta_j \langle \operatorname{Pr}(\sigma_j)X, Y \rangle)$$

while (7.11) remains unchanged.

In his famous address "On the hypotheses that lie at the foundations of geometry", Riemann discusses the theorem that vanishing of sectional curvature implies flatness (see **[11]** for an English translation and commentary). He gives a heuristic argument why the vanishing of $N = (1/2)n(n - 1)$ quantities should suffice, without specifically mentioning which quantities (it is not correct to take these to be the sectional curvatures in the planes determined by pairs of elements from a fixed basis, since these Ricci quantities can vanish without forcing all sectional curvatures to vanish). It is clear from (7.1) that the first stage eigenvalues $\lambda_1, \ldots, \lambda_N$ are N quantities whose vanishing implies flatness. Of course this result is misleading, since the orbit of $R \equiv 0$ is so far from generic; to characterize a generic orbit requires many more than N invariants.

References

1. A. Besse, *Géométrie Riemannienne et dimension* 4 (Cedic, Paris, 1981).
2. H. Boerner, *Representations of groups*, 2nd edition (North Holland, 1970).
3. L. P. Eisenhart, *Riemannian geometry* (Princeton U. Press, 1950).
4. C. Fefferman and C. R. Graham, *Conformal invariants*, Proc. of the Symposium: Elic Cartan et les mathématiques d'aujourd'hui, Asterisque, to appear.
5. A. Gray and L. Vanhecke, *Decomposition of the space of covariant derivatives of curvature operators*, preprint.
6. G. Herglotz, *Zur Einsteinschen Gravitationstheorie*, *Gesammelte Schriften* (Vandenhoerk and Ruprecht, Gottingen, 1979), 356-360.
7. S. Kobayashi and K. Nomizu, *Foundations of differential geometry* I and II (Interscience, New York, 1963 and 1969).
8. R. S. Kulkarni, *On the Bianchi identities*, Math. Ann. *199* (1972), 175-204.
9. F. D. Murnaghan, *The theory of group representations* (Johns Hopkins Press, Baltimore, 1938).
10. I. M. Singer and J. A. Thorpe, *The curvature of 4-dimensional Einstein manifolds*, in *Global analysis papers in honor of K. Kodaira* (Princeton University Press, 1969), 355-365.
11. M. Spivak, *A comprehensive introduction to differential geometry*, Vol. II, 2nd edition (Publish or Perish, Berkeley, 1979).
12. R. S. Strichartz, *The explicit Fourier decomposition of* $L^2(SO(n)/SO(n - \mathrm{m}))$, Can. J. Math. *24* (1972), 915-925.
13. ——— *Harmonic analysis on Grassmannian bundles*, Trans. Amer. Math. Soc. *296* (1986), 387-409.
14. F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series *83* (Cambridge U. Press, 1983).
15. H. Weyl, *Classical groups, their invariants and representations* (Princeton U. Press, 1946).

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Ithaca, New York