A REGULAR SINGULAR FUNCTIONAL

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1. Introduction. In a joint paper with Leighton **(2)**, the author considered quadratic functionals of the type

$$\int_a^b [r(x)\, y'^2 + p(x)\, y^2]\, dx \qquad (0 < a < b) \tag{1.1}$$

in which $x = 0$ is a singular point of the functional which is otherwise regular on $[0, b]$. The hypothesis on a regular functional includes the assumption that r is continuous and positive on a closed interval $[0, b]$. This assures the existence of extremals for (1.1). As a consequence, the Riccati equation

$$z' - \frac{z^2}{r(x)} + p(x) = 0 \tag{1.2}$$

has continuous solutions, at least locally. If the function r vanishes on an interval, the equation (1.2), and presumably its solutions, is annihilated. However the functional (1.1) is much less affected. This fact has suggested to the author a method of extending the meaning of (1.2). Such is the subject of this article.

In this paper we consider the functional

$$J(y)\Big|_a^b = \int_a^b [r(x)\, y'^2 + 2q(x)\, yy' + p(x)\, y^2]\, dx, \tag{1.3}$$

in order to generalize the Riccati equation

$$z' - \frac{(z + q(x))^2}{r(x)} + p(x) = 0. \tag{1.4}$$

The integral (1.3) is a Lebesgue integral. The functions r, p, and q are measurable functions which are defined on $(-\infty, \infty)$. The functions r and q are bounded on each bounded subinterval of $(-\infty, \infty)$ and p is integrable Lebesgue on each bounded interval. We come to a definition.

A function y is said to be *F_t-admissible* on $[a, b]$ if

(1) y is absolutely continuous on $[a, b]$ and y'^2 is integrable Lebesgue on $[a, b]$;

(2) $y(a) = t$.

We denote by $F_t[a, b]$ the class of all functions y which are F_t-admissible on

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$[a, b]$. We shall be principally concerned with the cases in which $t = 0$ or $t = 1$. By the assumptions on r, p, and q,

$$J(y)\Big|_a^b$$

exists and converges absolutely for every y which is F_t-admissible on $[a, b]$ for some t.

In order to outline the method which we employ, we recall the results for the regular functional and then indicate the points of departure from these. Suppose that r, p, and q are continuous on $(-\infty, \infty)$ and that $r(x) > 0$ there. Then every solution of the Euler equation of (1.2)

$$(r(x)\, y' + q(x)\, y)' - (q(x)\, y' + p(x)\, y) = 0 \tag{1.5}$$

has a continuous derivative on $(-\infty, \infty)$. If u is the unique solution of (1.5) such that $u(b) = 0$, $u'(b) = -1$ and if $u(x) \neq 0$ on (a, b) then the Riccati equation (1.4) has a solution

$$-r(x)\,\frac{u'(x)}{u(x)} - q(x) \tag{1.6}$$

which is continuous and which has a continuous derivative on (a, b). Further, we have by a well-known formula (**4**, p. 260) that for $a < t < b$,

$$J(y)\Big|_t^b = \int_t^b r(x)\left(y'(x) - \frac{u'(x)}{u(x)}\,y(x)\right)^2 dx - y^2(t)\left(r(t)\frac{u'(t)}{u(t)} + q(t)\right). \tag{1.7}$$

As a consequence, if

$$L(t, b) = \min_y J(y)\Big|_t^b, \tag{1.8}$$

where the minimum is taken over all y in $F_1[t, b]$, it follows by (1.7) that $L(t, b)$ exists for every $t \in (a, b)$ and that

$$L(t, b) = -r(t)\,\frac{u'(t)}{u(t)} - q(t).$$

Moreover, the minimum is attained by the extremal

$$y(x) = \frac{u(x)}{u(t)}.$$

However, once the restriction $r(x) \neq 0$ is removed, the Euler equation (1.5) ceases to exist and the minimum (1.8) will not, in general, be attained. Nevertheless if we define $L(t, b)$ by the relation

$$L(t, b) = \operatorname*{g.l.b.}_y J(y)\Big|_t^b$$

among all $y \in F_1[t, b]$ we shall see that the functions $L(t, b)$ behave in a manner

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which resembles the behavior of solutions of a Riccati equation. In the last section we discuss the details of the case when $r(x) = 0$ for x on $[a, b]$.

2. The conjugate points. In this section we develop a condition which ensures that the function $L(x, b)$ exists on (a, b).

Let $x = a$ be a point of $(-\infty, +\infty)$. If there exists a number b, $b > a$, such that

$$J(y)\Big|_a^b \geqslant 0 \tag{2.1}$$

for all y in $F_0[a, b]$, we define $c(a)$ to be the least upper bound of all such b. If no such b exists, we define $c(a)$ to be a. The point $x = c(a)$, which may be $+\infty$, is termed the first *conjugate point* of $x = a$. We remark that this definition is consistent with the definition of the conjugate point for regular functionals (**3**, p. 8). It will be noted that $c(x)$ is a nondecreasing and right continuous function of x. It is not, in general, continuous as trivial examples will show.

THEOREM 2.1. *If $c(a) \geqslant b$, then*

$$J(y)\Big|_a^b \geqslant 0$$

for every y in $F_0[a, b]$.

This theorem is trivial except in the case when $c(a) = b$. In this case it is disposed of in a manner similar to the proof of (**4**, Theorem 5.2).

We now proceed to consider the condition under which it is possible to define the functions L. We recall from the introduction that $L(x, b)$ is defined to be the number

$$\operatorname*{g.l.b.}_{y} J(y)\Big|_x^b \tag{2.2}$$

where the greatest lower bound is taken over all $y \in F_1[x, b]$. If $L(x, b)$ exists on an interval (a, b), we shall call it the *Riccati function* associated with J and the point $x = b$.

THEOREM 2.2. *In order that $L(x, b)$ be finite on (a, b) it is necessary that $c(a) \geqslant b$.*

The proof is by contradiction. Suppose that $c(a) < b$. Then since c is right continuous, there exists $x_0 > a$ such that $c(x_0) < b$. As a consequence there exists z in $F_0[x_0, b]$ such that

$$J(z)\Big|_{x_0}^b < 0. \tag{2.3}$$

Let $y \in F_1[x_0, b]$. Then for every t, the function $w = y + tz \in F_1[x_0, b]$ and

$$J(w)\Big|_{x_0}^b = J(y)\Big|_{x_0}^b + 2tJ(y, z)\Big|_{x_0}^b + t^2 J(z)\Big|_{x_0}^b \tag{2.4}$$

where

$$J(y, z)\Big|_{x_0}^{b} = \int_{x_0}^{b} [r(x)\, y'z' + q(x)(yz)' + p(x)\, yz]\, dx.$$

Now because of (2.3) it follows that

$$\lim J(w)\Big|_{x_0}^{b} = -\infty$$

and thus $L(x_0, b)$ is not finite.

We now establish the converse of Theorem 2.2.

THEOREM 2.3. *If $c(a) \geqslant b$, then $L(x, b)$ is finite for every x on (a, b).*

Let $a < x_0 < x_1 < b$ and let y be an arbitrary function in $F_1[x_1, b]$. Consider any function z which coincides with y on $[x_1, b]$ and which is in $F_0[x_0, b]$. We have then that

$$J(z)\Big|_{x_0}^{b} \geqslant 0.$$

Therefore

$$J(y)\Big|_{x_1}^{b} \geqslant -J(z)\Big|_{x_0}^{x_1} \tag{2.5}$$

for every y in $F_1[x_1, b]$. Since the right hand side of (2.5) may be independent of y it follows that $L(x_1, b)$ is finite for the arbitrary value x_1 in (a, b) and the theorem is proved.

It follows from the above theorem that, for $a < b_1 < b$ and $a < x < b_1$, $L(x, b_1)$ exists and

$$L(x, b) \leqslant L(x, b_1) \qquad (a < x < b_1). \tag{2.6}$$

3. The Riccati functions. In this section we develop a number of properties of the Riccati functions. As we shall see the function $L(x, b)$ is not, in general, continuous. However, we have the following theorem.

THEOREM 3.1. *If $c(a) \geqslant b$ then $L(x, b)$ is right continuous everywhere on (a, b).*

By the previous theorem, $L(x, b)$ is finite for each x in (a, b). Let c be in (a, b) and y be in $F_1[x, b]$. Then y_c defined as follows is a member of $F_1[c, b]$:

$$y_c(t) = \begin{cases} 1, & c < t < x, \\ y(x), & x < t < b. \end{cases}$$

Thus

$$J(y_c)\Big|_{c}^{b} = J(1)\Big|_{c}^{x} + J(y)\Big|_{x}^{b},$$

from which it follows that

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$$L(c, b) \leqslant J(1)\Big|_c^x + J(y)\Big|_x^b$$

for every y in $F_1[x, b]$. Therefore

$$L(c, b) \leqslant J(1)\Big|_c^x + L(x, b) \qquad (a < x < b), \tag{3.1}$$

and

$$\liminf_{x=c^+} L(x, b) \geqslant L(c, b). \tag{3.2}$$

Now let y_c be in $F_1[c, b]$. Then

$$y(t) = \frac{y_c(t)}{y_c(x)} \qquad (x \leqslant t \leqslant b),$$

is F_1-admissible on $[x, b]$ if x is sufficiently close to $x = c$ and $x > c$. Now let $\epsilon > 0$ and let y_c in $F_1[c, b]$ be chosen such that

$$J(y_c)\Big|_c^b < L(c, b) + \epsilon. \tag{3.3}$$

Now

$$J(y)\Big|_x^b = \frac{J(y_c)\Big|_x^b}{y_c^2(x)}$$

and consequently

$$L(x, b) \leqslant \frac{J(y_c)\Big|_x^b}{y_c^2(x)} \tag{3.4}$$

for every y_c in $F_1[c, b]$. It follows then that

$$\limsup_{x=c^+} L(x, b) \leqslant J(y_c)\Big|_c^b < L(c, b) + \epsilon. \tag{3.5}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_{x=c^+} L(x, b) \leqslant L(c, b). \tag{3.6}$$

A comparison of the inequality (3.2) with (3.6) yields the theorem.

We now wish to obtain an extension of equation (1.7). Before stating the next theorem, we note that

$$\underset{y}{\text{g.l.b.}}\ J(y)\Big|_c^b = y^2(c)\, L(c, b) \tag{3.7}$$

where the greatest lower bound is taken over all y in $F_t[c, b]$ for a fixed t.

THEOREM 3.2. *If $c(a) \geqslant b$ and if*

$$J(y)\Big|_x^b = Q(y, b)\Big|_x^b + y^2(x)\, L(x, b) \tag{3.8}$$

for every y in $F_c[a, b]$ *then*

$$Q(y, b)\Big|_x^b$$

is for each x on (a, b), *a quadratic functional which is defined for every y in* $F_c[x, b]$. *For each y in* $F_c[a, b]$, *it is a positive nonincreasing function of x.*

Let y be in $F_c[a, b]$ for some c. We must show that if $a < s < t < b$, then

$$Q(y, b)\Big|_s^t = Q(y, b)\Big|_s^b - Q(y, b)\Big|_t^b \geqslant 0. \tag{3.9}$$

Now

$$Q(y, b)\Big|_s^t = \left[J(y)\Big|_s^b - y^2(s)\, L(s, b)\right] - \left[J(y)\Big|_t^b - y^2(t)\, L(t, b)\right] \tag{3.10}$$

or

$$Q(y, b)\Big|_s^t = J(y)\Big|_s^t + y^2(x)\, L(x, b)\Big|_s^t. \tag{3.11}$$

From equation (3.11), it follows that this depends only upon the values of $y(x)$ on the interval $[s, t]$. Now let $z(x)$ be any function of the class $F_c[t, b]$ where $c = y(t)$. We have by the remark above that if

$$\bar{y}(x) = \begin{cases} y(x), & s < x \leqslant t, \\ z(x), & t \leqslant x < b. \end{cases}$$

then

$$\begin{aligned} Q(y, b)\Big|_s^t &= Q(\bar{y}, b)\Big|_s^t \\ &= J(\bar{y})\Big|_s^b - \bar{y}^2(s)\, L(s, b) - J(z)\Big|_t^b + z^2(t)\, L(t, b). \end{aligned} \tag{3.12}$$

Thus by (3.7),

$$Q(y, b)\Big|_s^t \geqslant - J(z)\Big|_t^b + z^2(t)\, L(t, b)$$

for every z in $F_c[t, b]$ where $c = y(t)$. On referring again to (3.7) it follows that

$$Q(y, b)\Big|_s^t \geqslant 0.$$

THEOREM 3.3. *If* $c(a) \geqslant b$, *then* $L(x, b)$ *is of bounded variation on every subinterval of* (a, b).

Letting $y(x) = 1$ and $s = x$ in equation (3.11) we have

$$L(x, b) = - Q(1, b)\Big|_x^t + J(1)\Big|_x^t + L(t, b) \qquad (s < t < b)$$

and since the middle term is absolutely continuous on $[x, t]$ the theorem follows.

THEOREM 3.4. *If $c(a) \geqslant b$ then*

$$r(x)[L'(x, b) + p(x)] - [L(x, b) + q(x)]^2 \geqslant 0 \tag{3.13}$$

for almost all x on (a, b).

A trivial integration by parts provides a proof of the following lemma:

LEMMA 3.1. *If $f(x)$ is integrable and $g(x)$ is in C' on $[a, b]$ and if c is a point at which $f(c)$ is the derivative of*

$$\int_a^x f(t)\,dt$$

then c is also a point at which $f(c)\,g(c)$ is the derivative of

$$\int_a^x f(t)\,g(t)\,dt.$$

We continue with a proof of the theorem. Let E be a measurable subset of (a, b) such that

$$m(E) = b - a$$

and such that for every c in E, $L'(c, b)$ is finite and the derivatives of the functions

$$\int_x^b r(t)\,dt, \quad \int_x^b q(t)\,dt, \quad \int_x^b p(t)\,dt \tag{3.14}$$

exist and equal $-r(c)$, $-q(c)$ and $-p(c)$ respectively. (Henceforth, if A is a measurable subset of $(-\infty, \infty)$, we will denote its one dimensional Lebesgue measure by $m(A)$.) Now by (3.8) we have for $y = m(x - t) + n$ that

$$Q(y, b)\Big|_x^c = J(y)\Big|_x^c - y^2(t)\,L(t, b)\Big|_x^c \qquad (a < x < c < b) \tag{3.15}$$

and

$$Q(y, b)\Big|_x^b$$

is nonincreasing on (a, b). Further, since $m(x - c) + n$ is of class C^1 on $[a, b]$ we have that for $x = c$ in E, there is a derivative

$$\frac{d}{dx}\,Q(y, b)\Big|_x^b \leqslant 0. \tag{3.16}$$

An application of the Lemma to (3.16) gives the result

$$r(c)m^2 + 2(q(c) + L(c, b))\,mn + (p(c) + L'(c, b))\,n^2 \geqslant 0$$

for every m and n and every c in E.

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But from the theory of quadratic forms it follows that

(3.17) $$r(c) \geqslant 0, \quad p(c) + L'(c, b) \geqslant 0$$

and

(3.18) $$r(c)[L'(c, b) + p(c)] - [L(c, b) + q(c)]^2 \geqslant 0,$$

for all c in E. The theorem is proved.

The relation (3.17) gives at once the following theorem:

THEOREM 3.5. *In order that*

$$J(y)\Big|_a^b \geqslant 0$$

for every y in $F_0[a, b]$ it is necessary that the set of x of $[a, b]$ for which $r(x) < 0$ be of Lebesgue measure zero.

If $r(x)$ is zero on a set T of positive measure then except for a subset of T of measure zero the inequality (3.18) implies that

$$L(x, b) = -q(x).$$

We thus have the theorem:

THEOREM 3.6. *If*

(3.19) $$R[x, z] = \begin{cases} z' - \dfrac{(z + q(x))^2}{r(x)} + p(x), & r(x) \neq 0, \\ -(z + q(x))^2, & r(x) = 0, \end{cases}$$

then

(3.20) $$R[x, L(x, b)] \geqslant 0$$

almost everywhere on (a, b).

One may show by example that the inequality in relation (3.20) cannot be removed without further assumptions. We have, however, the following result.

THEOREM 3.7. *If $r(x)$ is continuous on $[a, b]$ and if $c(a) \geqslant b$ then $L(x, b)$ satisfies the equation*

(3.21) $$R[x, z] = 0$$

almost everywhere in (a, b).

Before attending to the proof of Theorem 3.7 we consider some extensions of the theory of regular functionals.

If $r(x)$ is continuous and positive on $[s, t]$ then $1/r(x)$ is continuous on $[s, t]$ and

(3.22) $$\int_s^t \frac{dx}{r(x)} \neq \infty.$$

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It follows that the system of equations

$$u' = a_{11}u + a_{12}v, \qquad (3.23)$$
$$v' = a_{21}u + a_{22}v,$$

where

$$a_{11} = -\,q/r, \qquad a_{12} = 1/r, \qquad (3.24)$$
$$a_{21} = p - q^2/r, \qquad a_{22} = q/r$$

has a unique solution (u,v), where u and v are absolutely continuous functions on $[s, t]$ which almost everywhere satisfy equations (3.23) and which at c in $[s, t]$ satisfy the relations

$$u(c) = u_0, \qquad v(c) = v_0,$$

where u_0 and v_0 are arbitrary real numbers. This follows **(5**, pp. 44–45**)** from the fact that the a_{ij} are integrable Lebesgue on $[s, t]$. By the null solution of (3.23) is meant the solution for which u and v vanish identically on an interval. It follows from the existence theorem cited that if u and v vanish at one point of (s, t) then (u, v) is the null solution on $[s, t]$. From (3.23) and (3.24) we have that for almost all x in [s, t].

$$v(x) = r(x)\, u'(x) + q(x)\, u(x) \qquad (3.25)$$

and

$$v'(x) = q(x)\, u'(x) + p(x)\, u(x). \qquad (3.26)$$

Furthermore, in a manner which runs along classical lines, one may show that the first conjugate point $c(a)$ of the point $x = a$ is the first zero beyond the point a of u of a nonnull solution (u, v) of (3.23) for which $u(a) = 0$. Hence if $c(a) \geqslant b$ the function

$$-\frac{v(x)}{u(x)} \qquad (3.27)$$

is absolutely continuous on every closed subinterval of (a, b) and satisfies the Riccati equation

$$z' - \frac{(z + q(x))^2}{r(x)} + p(x) = 0 \qquad (3.28)$$

for almost all x on (a, b). We have moreover that

$$J(y)\Big|_x^b = \int_x^b r(t)\left[y'(t) - \frac{u'(t)}{u(t)}\, y(t)\right]^2 dt - \frac{v(x)}{u(x)}\, y^2(x) \qquad (3.29)$$

for every y in $F_t[a, b]$. The proof of this formula is similar to that of the analogous formula in **(2**, p. 103, Th. 6.2**)**. A consequence of (3.29) is the following well-known result.

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THEOREM 3.8. *If $c(a) \geqslant b$ and if (u, v) is any nonnull solution of the system* (3.23) *such that* $u(b) = 0$ *then*

$$L(x, b) = -\frac{v(x)}{u(x)} \qquad (a < x < b). \tag{3.30}$$

We return to give a proof of Theorem 3.7. Let r_n denote the function such that

$$r_n(x) = r(x) + \frac{1}{n} \qquad (a < x < b), \tag{3.31}$$

$$J_n(y)\Big|_x^b = \int_x^b [r_n(t)\, y'^2 + 2q(t)\, yy' + p(t)\, y^2]\, dt. \tag{3.32}$$

Let $L_n(x, b)$ denote the Riccati functions of J_n. We then have that

$$J_n(y)\Big|_x^b \geqslant J_{n+1}(y)\Big|_x^b \geqslant J(y)\Big|_x^b \tag{3.33}$$

from which it follows that

$$L_n(x, b) \geqslant L_{n+1}(x, b) \geqslant L(x, b) \qquad (a < x < b). \tag{3.34}$$

Thus for each x on (a, b) there is a limit

$$\lim_{n=\infty} L_n(x, b) \geqslant L(x, b). \tag{3.35}$$

We assert that

$$\lim_{n=\infty} L_n(x, b) = L(x, b). \tag{3.36}$$

Since for each n, and every y in $F_t[x, b]$,

$$0 \leqslant r_n(t)\, y'^2(t) < r_1(t)\, y'^2(t) \qquad (x < t < b),$$

the Lebesgue limit theorem gives the result

$$\lim_{n=\infty} J_n(y)\Big|_x^b = J(y)\Big|_x^b \qquad (a < x < b). \tag{3.37}$$

Now let y be in $F_1[x, b]$ such that

$$J(y)\Big|_x^b < L(x, b) + \epsilon$$

and let m be an integer such that for $n > m$

$$J_n(y)\Big|_x^b < L(x, b) + \epsilon.$$

By (3.37) such an integer m exists. Thus

$$\lim_{n=\infty} L_n(x, b) \leqslant \lim_{n=\infty} J_n(y)\Big|_x^b \leqslant L(x, b) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\lim_{n=\infty} L_n(x, b) \leqslant L(x, b),$$

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and combining this result with (3.35) gives the assertion (3.36). Now since $1/r_n(x) \in L_1$ on every closed subinterval of (a, b) it follows from the remark (3.27) that $L_n(x, b)$ is a solution of the Riccati equation

$$z' - \frac{(z + q(x))^2}{r_n(x)} + p(x) = 0 \qquad (a < x < b). \tag{3.38}$$

Let T be the subset x on (a, b) for which $r(x) = 0$ and S the subset consisting of x on (a, b) for which $r(x) > 0$. Then

$$S \cup T = (a, b),$$

and since $r(x)$ is continuous on (a, b) S is open while T is closed in (a, b). If $m(T) < 0$ then by the inequality (3.18)

$$L(x, b) = -q(x)$$

for almost all of T, and hence

$$R[x, L(x, b)] = 0 \tag{3.39}$$

on almost all of T. If S is not empty then $m(S) > 0$. In the latter case let I_k be the components of S, let

$$I_k = (a_k, b_k) \qquad (k = 1, 2, \ldots).$$

It suffices to show that (3.39) holds on almost all of each interval I_k. Let k be a fixed but arbitrary integer and let $[s, t]$ be a closed subinterval of I_k. We have then by Theorem 3.8 that $L_n(x, b)$ is absolutely continuous on each closed subinterval of (a, b). Thus by (3.20) we have that

$$L_n(t, b) - L_n(s, b) = \int_s^t \frac{(L_n(x, b) + q(x))^2}{r(x)}\, dx - \int_s^t p(x)\, dx$$

Now since, for every n,

$$\frac{(L_n(x, b) + q(x))^2}{r_n(x)} \leqslant \frac{(|L_1(x, b)| + |q(x)|)^2}{r(x)},$$

the Lebesgue limit theorem may be invoked to infer that

$$L(t, b) - L(s, b) = \int_s^t \frac{(L(s, b) + q(x))^2}{r(x)}\, dx - \int_s^t p(x)\, dx \tag{3.40}$$

for every t and s such that

$$a_k < s < t < b_k.$$

On taking the derivative of (3.40) when $t < x$ we have almost everywhere on I_k that

$$L'(x, b) - \frac{(L(x, b) + q(x))^2}{r(x)} + p(x) = 0.$$

The proof is complete.

We remark that equation (3.40) contains the following result.

THEOREM 3.9. *If $c(a) \geqslant b$ then $L(x, b)$ is absolutely continuous on every closed interval in (a, b) where $r(x)$ is continuous and positive.*

As the results of the next section will show, even if $r(x)$ is continuous, the condition that $c(a) > b$ does not imply that $L(x, b)$ is continuous.

Let $r(x)$ be continuous on $(-\infty, \infty)$. Let $I_k = (a_k, b_k)$ be the components of the subset S of (a, b) on which $r(x) > 0$. Let $x = m_k$ be a point of (a_k, b_k) and T the complement of S in (a, b). We have the following theorem.

THEOREM 3.10. *Let $c(a) \geqslant b$. If $q(x)$ is continuous on T and if*

$$\int_{a_k}^{m_k} \frac{dx}{r(x)} = \infty, \quad \int_{m_k}^{b_k} \frac{dx}{r(x)} = \infty \qquad (k = 1, 2, \ldots) \tag{3.41}$$

then $L(x, b)$ is continuous on (a, b) and $L(x, b) = -q(x)$ for every x in T.

By Theorem 3.9 it follows at once that $L(x, b)$ is continuous at each point of S. To show that L is continuous at each point $x = c$ of T it is therefore sufficient to prove that

$$L(c, b) = -q(c) \tag{3.42}$$

and

$$\lim_{n=\infty} L(c_n, b) = -q(c) \tag{3.43}$$

for one sequence of numbers such that $c_n < c$ for all $n = 1, 2, \ldots$. This follows from the facts that $L(x, b)$ is right continuous on (a, b) and that it is of bounded variation on every closed subinterval of (a, b).

We assert first that

$$L(c, b) = -q(c)$$

for every point $x = c$ in T. By the hypothesis (3.41) and the formula

$$L(m_k, b) - L(a_k, b) = \int_{a_k}^{m_k} \frac{(L(x, b) + q(x))^2}{r(x)}\, dx - \int_{a_k}^{m_k} p(x)\, dx,$$

it follows that

$$L(a_k, b) = \lim_{x=a_k} L(x, b) = -q(a_k). \tag{3.44}$$

Let $x = c$ be any point of T. If $x = c$ is a right limit point of the sequence a_k the assertion (3.42) follows from (3.44). In the remaining case $x = c$ is isolated on the right from a_k. Then T contains an interval (c, a'), $(0 < a' < b)$. But since $r(x) = 0$ in $[c, a']$, Theorem 3.6 implies that

$$L(x, b) = -q(x)$$

almost everywhere in $[c, a']$, in particular on an everywhere dense subset of $[c, a']$. Then on taking c_n in $[c, a']$ such that

$$L(c_n, b) = -q(c_n) \qquad (n = 1, 2, \ldots),$$

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the assertion follows from the continuity of q and the right continuity of L. Now the desired result (3.43) is clear for every point of T which is a left limit point of T. If $x = c$ is isolated on the left from T, then $x = c$ is the right endpoint of an interval of S. Thus $c = b_k$ and by reasoning analogous to that used for the a_k we have

$$\lim_{x=b_k^-} L(x, b) = -q(b_k, b) = L(b_k, b).$$

4. The case $r = 0$. In this section it will be presumed throughout that $r(x)$ vanishes identically on $(-\infty, \infty)$. We will determine the Riccati functions completely and obtain a necessary and sufficient conjugate point condition.

For the functionals under consideration

$$R[x, z] = -(z + q(x))^2. \tag{4.1}$$

Further since $r = 0$ is everywhere continuous, $L(x, b)$, whenever it exists, satisfies the Riccati equation

$$R[x, z] = 0$$

almost everywhere on (a, b). Thus for these x,

$$L(x, b) = -q(x). \tag{4.2}$$

We have then as an immediate consequence of (4.2), Theorem 3.1, and Corollary 3.1 the following result.

THEOREM 4.1. *If $c(a) \geqslant b$ then $q(x)$ must agree almost everywhere on (a, b) with a function which is right continuous on (a, b) and which is of bounded variation on every closed subinterval of (a, b).*

With no loss of generality we may always assume that whenever $c(a) \geqslant b$ that $q(x)$ is right continuous on (a, b) and is of bounded variation on every closed subinterval of (a, b).

THEOREM 4.2. *In order that $c(a) \geqslant b$ it is necessary that the function*

$$q(x) + \int_x^b p(t)\, dt \tag{4.3}$$

be nonincreasing on (a, b).

LEMMA 4.1. *If $x = c$ is a point where $q(x)$ is right continuous then*

$$\lim_{t=c^+} J\left(\frac{x-t}{c-t}\right)\Bigg|_c^t = -q(c) \tag{4.4}$$

and

$$\lim_{t=c^+} J\left(\frac{x-c}{t-c}\right)\Bigg|_c^t = q(c). \tag{4.5}$$

We shall provide a demonstration of equation (4.4).

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It is sufficient to show that

$$\lim_{t=c^+} \int_c^t 2q(x) \frac{(x-t)}{(c-t)^2}\, dx = -q(c) \tag{4.6}$$

since

$$\left| \int_c^t p(t) \frac{(x-t)^2}{(c-t)^2}\, dx \right| \leqslant \int_c^t |p(x)|\, dx = O(1)$$

as t tends to c^+. But on an integration by parts we find that

$$\frac{2}{(c-t)^2} \int_c^t q(x)(x-t)\, dx = \frac{-2}{(c-t)^2} \int_c^t \int_c^s q(x)\, dx\, ds.$$

By l'Hospital's theorem one has

$$\lim_{t=c^+} \frac{2}{(c-t)^2} \int_c^t q(x)(x-t)\, dx = \lim_{t=c^+} \frac{-1}{t-c} \int_c^t q(x)\, dx = -q(c),$$

since $q(x)$ is right continuous at $x = c$.

The lemma is proved. We continue with a proof of the theorem.

Let $F(x)$ denote the function (4.3). As previously remarked we may assume that $F(x)$ is right continuous and of bounded variation on every closed subinterval of (a, b). Let $x = s$ and $x = u$ be two points of (a, b) such that $s < u$, and let numbers t and v be chosen such that

$$s < t < u < v. \tag{4.7}$$

We define a function y in $F_0[a, b]$ as follows:

$$y(x) = \begin{cases} 0, & a \leqslant x \leqslant s, \\ \dfrac{x-s}{t-s}, & s \leqslant x \leqslant t, \\ 1, & t \leqslant x \leqslant u, \\ \dfrac{x-v}{u-v}, & u \leqslant x \leqslant v, \\ 0, & v \leqslant x \leqslant b. \end{cases}$$

Since $c(a) \geqslant b$, we have

$$J(y)\Big|_a^b \geqslant 0,$$

or what is the same thing:

$$J\left(\frac{x-s}{t-s}\right)\Bigg|_s^t + \int_t^u p(x)\, dx + J\left(\frac{x-v}{u-v}\right)\Bigg|_u^v \geqslant 0 \tag{4.8}$$

for every s, t, u, and v which satisfy the inequalities (4.7). Now let t tend to s^+ and v then to u^+. By Lemma 4.1 the inequality (4.8) becomes

$$q(s) + \int_s^u p(x)\, dx - q(u) \geqslant 0,$$

that is,

$$F(s) - F(u) \geqslant 0 \qquad (a < s < u < b),$$

and the theorem follows.

We now establish the converse of the preceding Theorem.

THEOREM 4.3. *If the function*

$$q(x) + \int_x^b p(t)\, dt \tag{4.9}$$

is nonincreasing on (a, b) *then* $c(a) \geqslant b$.

Again we denote (4.9) by $F(x)$. The theorem will be proved if we show that

$$J(y)\Big|_a^b \geqslant 0$$

for every y in $F_0[a, b]$. On an integration by parts one finds that for every y in $F_0[a, b]$

$$J(y)\Big|_a^b = 2\int_a^b F(x)\, y(x)\, y'(x)\, dx. \tag{4.10}$$

Since $F(x)$ is nonincreasing we may infer from the Second Mean-Value Theorem that there exists c on (a, b) such that

$$\begin{aligned} J(y)\Big|_a^b &= 2\,F(a)\int_a^c y(x)\, y'(x)\, dx + 2F(b)\int_c^b y(x)\, y'(x)\, dx \\ &= (F(a) - F(b))\, y^2(c) \geqslant 0. \end{aligned}$$

The proof is complete.

We remark in closing that if

$$L(x, b) = -q(x)$$

on a set of positive measure T then there exists a subset T' such that

$$m(T') = m(T)$$

such that

$$L(x, c) = -q(x)$$

for all x in T' and all c such that

$$x < c < b.$$

Hence under these circumstances $L(x, b)$ is independent of b. Further, if $r(x) \equiv 0$, $L(x, b)$ is independent of the function p subject only to the condition that the function

$$q(x) + \int_x^b p(t)\, dt$$

be nonincreasing on (a, b).

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