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RESEARCH ARTICLE

Multilayer network games: A cooperative approach

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Abstract
In this paper, we introduce the concept of a multilayer network game in a cooperative setup. We consider the notion of simultaneous contribution of individual players or links to two different networks (say, X and Z). Our model nests both classical network games and bi-cooperative network games. The calculation of the utility of players within a specific network in the presence of an additional/alternative network provides a broader spectrum of real-world decision dynamics. The subsequent challenge involves achieving an optimal distribution of payoffs among the players forming the networks. The link-based rule best fits to our model as it delves into the influence of the alternative links in the network. We have designed an extended Position value to address the complexities arising from scenarios where networks overlap. Further, it is shown that the Position value is uniquely characterized by the Efficiency and Balanced Link Contribution axioms.

Keywords: Position value; multilayer network game; network allocation rule

1. Introduction

In this paper, we study multilayer network games under a cooperative game theoretic setup. We investigate how the allocation rules can be designed and characterized based on players' involvement in multiple networks. In the recent times, we see that various agents (players) of the society are typically engaged in multiple networks simultaneously. Their interactions within each network have the potential to impact one another in their socioeconomic transactions (Lagesse et al., 2015; Neal, 2023). Consider, for example, the scenario where employers and potential employees are linked through Social and Business networks (Billand et al., 2023). In such instances, the hiring decisions made by employers are likely to be influenced by the social and personal behaviors exhibited by prospective employees on their social accounts. Such networks are modeled through "multigraphs," i.e., graphs having nodes with parallel edges, and are called multilayer networks (Billand et al., 2023; Joshi et al., 2023). In Billand et al. (2023) and Joshi et al. (2023), the interactions among the nodes (players) in multilayer networks are studied to understand how such networks form and evolve. They establish the necessary and sufficient conditions for the stability of a network (equilibria) using inter-network spillovers and network complementarities.

There are also instances where players interact within pre-defined network structures established through binding agreements. These agreements outline cooperation and value generation. In this framework, a network's stability depends on identifying a fair allocation mechanism for the value generated by the network. Consider the example of employer-employee networks in light of this framework. Here, the network's value quantifies the mutual trust among players, which is built over time through interactions in alternative networks. Using a fair allocation rule, the employer ranks the players in the network and decides whom to employ. This approach has its origin in the theory of cooperative games. Therefore, we call it "multilayer network games in a cooperative setup." In the existing theories of network games and cooperative games with transferable utilities, players' interactions are analyzed within a single network setting. The idea of how a player's involvement across multiple interconnected networks affects the overall distribution of value is not well explored. For example, in employer-employee networks, participants might be involved in multiple layers, such as formal professional relationships, informal social connections, or working together on specific projects. Single-layer network models are inadequate to describe such situations. Therefore, it is important to study the multilayer networks in a cooperative setup and investigate how the allocation rules can be designed and characterized for such scenarios.

Note that, our model is different from that of Billand et al. (2023) and Joshi et al. (2023) in the sense that (i) Unlike two fixed networks in their model, our value function is defined over the class of all ordered pairs of subnetworks that can form with the given player set. (ii) Secondly, they (Billand et al., 2023; Joshi et al., 2023) focus primarily on how networks evolve under equilibria when players are involved in multiple networks. On the other hand, in our model, we look at the collective behavior of the players in the networks, assuming that each network remains fixed for some time. It is beneficial for the players to be in the network and generate value rather than staying isolated and generating no value. The value function defined over the set of all such pairs of subnetworks quantifies the influence of one network over the other in that pair. In the next section, we provide an example to highlight these features more explicitly.

Our model is formulated as follows. We consider two networks, say g_1 and g_2 from the set of all possible networks with the players in N. The players in N can be partitioned into four sub-groups: players in g_1 only, players in g_2 only, players in both g_1 and g_2, and players neither in g_1 nor in g_2. Thus, in our model, the players are allowed to form multiple networks among themselves. This justifies the name of the corresponding network game as a "multilayer network game." The multilayer network games are defined over the set of all ordered pairs of networks, represented as (g_1, g_2), each assigned a value indicating the maximum gain or minimum loss when the players in N are partitioned into four groups as mentioned above. If either g_1 or g_2 is an empty network (a network without any links) in the pair (g_1, g_2), it represents a single classical network. On the other hand, if $g_1 \cap g_2 = \varnothing$, then the pair (g_1, g_2) represents a bi-cooperative network (Borkotokey and Gogoi, 2014; Gogoi et al., 2014). Therefore, the classical network games and bi-cooperative network games emerge as special cases of multilayer network games. Our work also resembles the multigraphs presented in Forlicz et al. (2018) where multiple links can connect nodes. However, our focus diverges in two key aspects. Firstly, in their model, each node is a tuple (x, b), where x represents a player and b denotes the number of direct links it possesses. Consequently, both the value function and allocation rule depend on the quantity b. In contrast, our model emphasizes the inter-network effects, analyzing how one network influences the overall structure and value of the other network, rather than focusing on individual player-level details within each network. In Section 6, we discuss the parallels and contrasts between these models in detail.

In the context of network games, a solution involves a rational allocation of utilities generated as a result of cooperation, which could manifest in terms of money, power, influence, etc., obtained within the specified network (Jackson, 2005). There are two distinct solution concepts, *viz.*, the player-based allocation rule and the link-based allocation rule (Bozzo et al., 2015; Jackson, 2005; Myerson, 1977; Slikker, 2005). However, unlike the classical network game, where it is the prerogative of the decision maker to choose between the player-based and link-based rules, in multilayer network games, such rules cannot be applied arbitrarily. A player-based allocation rule cannot capture situations involving the common links in networks g_1 and g_2 simultaneously. On the other hand, link-based allocation rules consider the relationships between players through their links and, thus, provide a more robust and realistic allocation procedure. As a result, collaboration, efficiency, and resilience to player substitutions, etc., can be captured, and a more effective resource distribution within the network can be obtained. Owing to all these challenges associated with the multilayer network game, this paper concentrates on

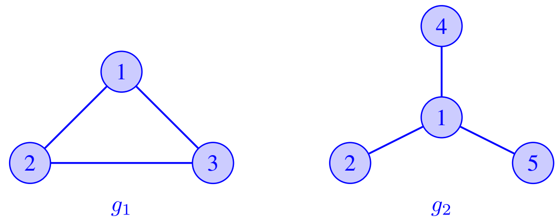

Figure 1. Facebook network: $g_1 = \{12, 13, 23\}$ and LinkedIn network: $g_2 = \{12, 14, 15\}$.

a link-based allocation rule: the Position value. Within the cooperative framework of networks, the Position value is recognized as one of the fairest link-based allocation rules (Borkotokey and Gogoi, 2014; Gogoi et al., 2014; Slikker, 2005, 2011; van den Nouweland and Slikker, 2012). We provide a characterization of the Position value using Efficiency and Balanced Link Contribution.

The rest of the paper is organized as follows: Section 2 details an example, that further highlights the motivation and application of our model. Section 3 contains the preliminary concepts and definitions required for the model's development. In Section 4, we present the model for multilayer network games, and in Section 5, we formulate the Position value specific to these games. Further, in Section 6, we provide a concluding discussion.

2. Example

Suppose a group of college friends, connected on Facebook, graduate and start their careers in different fields. Over time, they maintain their Facebook connections, sharing updates about their lives, jobs, and accomplishments. Take, for example, a member working as a graphic designer who posts on Facebook about a project they are currently working on. A member of the group with marketing expertise offers to help. They connect on LinkedIn to discuss potential collaborations. This connection generates a resource of their potential partnerships and professional collaborations which is dependent on both their Facebook and LinkedIn networks. Thus, by leveraging their Facebook connections, this group of individuals has created a valuable LinkedIn network that can generate various resources, through job opportunities, professional advice and collaborations. Suitable resource allocation within this network can enhance the benefits for all members.

Suppose players 1, 2, and 3 connect on Facebook and seek collaborations within their LinkedIn networks. Let 4 and 5 join them from 1's LinkedIn network, and 2 already have a connection with 1 on LinkedIn. We consider this situation as an ordered pair of networks $(\{12, 13, 23\}, \{12, 14, 15\})$ as shown in Figure 1. There can be 64 possible collaborations in this way,[1] viz., $(\{12, 13\}, \{12\})$, $(\{12, 13\}, \{14\})$, $(\{12, 13\}, \{14, 15\})$, $(\{12, 13, 23\}, \{15\})$ etc. All these possible collaborations generate resources, for example, in the form of market share through their networked efforts. Finally, based on all such possible interactions, the resource they generate needs to be shared among all the players. Existing theories on network games and cooperative games do not capture situations involving these multiple interconnected networks. Bi-cooperative network games are also limited in this regard, as they do not consider links like $\{12\}$, which belong to both the Facebook and LinkedIn networks in this example. On the other hand, multigraphs treat such links as parallel links. However, in reality, links like $\{12\}$ are single links that share different information on different platforms, like a person adopting different roles in different contexts. In classical network games, as already mentioned, the Position value is one of the fairest allocation rules, taking into account player interactions in the networks. It aggregates the contributions of the subnetworks of a given network. It does not consider the influence of multiple networks. On the other hand, although the Position value in bi-cooperative network games takes into account the value generated by the network pairs, they only consider disjoint network pairs, viz., $(\{12, 13\}, \{14\})$, $(\{12, 13\}, \{14, 15\})$ etc. Nevertheless, in case of multilayer network games, we also have to consider the values generated by pairs viz., $(\{12, 13\}, \{12, 15\})$ etc. Here, $\{12\}$ in the first component is a Facebook link and that in the second component is a LinkedIn link. Therefore, the Position value for multilayer network games needs to be defined in a more general setup such that the values of all possible ordered network pairs are taken into account. This further justifies that the study of link-based allocation rules in multilayer network games is more significant, as understanding the characteristics of various links across multiple networks helps designers identify the most valuable connections.

Government agencies can apply this concept for national accounting or tax and subsidy purposes when dealing with conglomerates. As conglomerates operate in multiple industries, the trade networks involving conglomerates can precisely be modeled by a multilayer network. The Position value naturally captures the value added by every trade link and thus gives good estimate for industrial policy perspective.

3. Preliminaries

In this section, we present the definitions and results from (Boruah et al., 2023; Grabisch and Labreuche, 2002; Jackson and Wolinsky, 1996; Jackson, 2005; Labreuche and Grabisch, 2008) necessary for the development of our model. Specifically, definitions 1, 2, 4, 5 and their consequences will be utilized in designing the value function for multilayer network games. Additionally, definitions 3, 6, 7 will be used in the formulation of the Position value for multilayer network games.

Let $N = \{1, 2, \ldots, n\}$ be a finite player set. Every subset S of N is termed as a coalition and N is referred to as the grand coalition. We will use lowercase letters, such as s and t, to represent the cardinality of sets S and T, respectively, i.e., '$\#S$' is equivalent to "s" throughout the paper. Sometimes, depending upon the context, we will use "$|S|$" to denote the cardinality of the set S.

3.1 Overlapping coalitional games

The concept of an overlapping coalitional game is introduced in Boruah et al. (2023). Below, we formally present the model. Let there be two attributes, call them X and Z, where players can contribute independently.

Given a player set N, let $2^N \times 2^N = \{(S, T) : S, T \subseteq N\}$ be the set of pairs of coalition. We call each $(S, T) \in 2^N \times 2^N$ an overlapping bi-coalition. For any $(S, T) \in 2^N \times 2^N$, we assume that players in S are X contributors, players in T are Z contributors, players in $S \cap T$ contribute to both X and Z, call them X–Z contributors, and players in $N \setminus (S \cup T)$ absentees.

Definition 1. *An overlapping coalitional game is a function* $o : 2^N \times 2^N \mapsto \mathbb{R}$ *with* $o(\emptyset, \emptyset) = 0$.

For each $(S, T) \in 2^N \times 2^N$, $o(S, T)$ represents the worth (gain or loss) of (S, T), when the players in S are X contributors, players in T are Z contributors and $S \cap T$ may or may not be empty and finally the remaining $N \setminus (S \cup T)$ are absentees. Let $\mathbb{O}^N$ be the real vector space of all overlapping coalitional games on N. Now, consider the order relation $\sqsubseteq$ on $2^N \times 2^N$ as follows: $(A, B) \sqsubseteq (C, D) \iff A \subseteq C$ and $B \subseteq D$. For $(S', T') \in 2^N \times 2^N$ with $(S', T') \neq (\emptyset, \emptyset)$, consider the following special games in $\mathbb{O}^N$.

Definition 2. *The superior unanimity game* $\bar{u}_{(S',T')} : 2^N \times 2^N \mapsto \mathbb{R}$ *is defined as follows*

$$\bar{u}_{(S',T')}(A, B) = \begin{cases} 1 & \text{if } (S', T') \sqsubseteq (A, B), (A, B) \neq (\emptyset, \emptyset), \\ 0 & \textit{otherwise}. \end{cases} \tag{1}$$

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The set of superior unanimity games forms a basis for $\mathbb{O}^N$ and so every $o \in \mathbb{O}^N$ can be expressed as a linear combination of the superior unanimity games as follows,

$$o = \sum_{(S',T')\in 2^N \times 2^N} c_{(S',T')} \bar{u}_{(S',T')} \tag{2}$$

where $c_{(S',T')}$ are the real constants. For every $o \in \mathbb{O}^N$, we can associate a cooperative game V defined on $S \cup T$ such that $V(P) = o(P \cap S, P \cap T)$, for all $P \subseteq S \cup T$. Consequently, V possesses a corresponding representation in terms of unanimity cooperative games $\{U_P | P \subseteq N, P \neq \emptyset\}$ as follows (Harsanyi, 1963; Shapley, 1953).

$$V = \sum_{\emptyset \neq P \subseteq S \cup T} c_P U_P \tag{3}$$

where $c_P = \sum_{M \subseteq P} (-1)^{p-m} V(M)$ and the unanimity cooperative games U_P are defined by

$$U_P(S) = \begin{cases} 1 & \text{if } P \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\sum_{\emptyset \neq P \subseteq S \cup T} c_P = V(S \cup T) = o((S \cup T) \cap S, (S \cup T) \cap T) = o(S, T). \tag{4}$$

Definition 3. *A one-point solution concept or a value for overlapping coalitional games is a function that assigns an n-dimensional real vector to each overlapping coalitional game. This vector serves as a representation of a payoff distribution among the players.*

Here, we introduce a value for overlapping coalitional games similar to the LG value (Labreuche and Grabisch, 2008). This value is termed to as the Overlapping LG value, abbreviated as OLG, and is defined as follows.

For any $o \in \mathbb{O}^N$, $(S, T) \in 2^N \times 2^N$ such that for all $i \in N$

$$\Phi_i^{OLG}(o)(S, T) = \sum_{P \subseteq (S \cup T) \setminus i} \frac{p!(s + t - st - p - 1)!}{(s + t - st)!} \big(V(P \cup i) - V(P)\big) \tag{5}$$

where for $P \subseteq S \cup T$, $V(P) = o(P \cap S, P \cap T)$ and $\#S \cap T := st$.

Applying Eq.s (2), (3), (4) on (5), we find the expression of the OLG value for $o \in \mathbb{O}^N$ in terms of the Harsanyi's dividend (Harsanyi, 1963) for Φ^{Sh}, the Shapley value (Shapley, 1953) of the associated cooperative game $(S \cup T, V)$ as follows.

$$\Phi_i^{OLG}(o)(S, T) = \Phi_i^{Sh}(S \cup T, V) = \sum_{P \subseteq S \cup T\,:\, i \in P} \frac{c_P}{p}. \tag{6}$$

3.2 Network games

Given the player set N and distinct players $i, j \in N$, a link ij is the pair $\{i, j\}$ which represents an undirected relationship between i and j. Clearly, ij is equivalent to ji. The set of all possible links with the player set N denoted by $g^N = \{ij \mid i, j \in N \text{ and } i \neq j\}$ is called the complete network. Let $G^N = \{g | g \subseteq g^N\}$ be the set of all possible networks on N. The network $\varnothing$ is the network without any links, which we refer to as the empty network. Let the number of links in a network g be denoted by $\ell(g)$. We can consider the links in a network g as hypothetical players. Thus g can be thought of as a coalition of the players representing these links in g. However, to distinguish the two representations, we use $[g]$ to denote the coalition. Obviously, $\ell(g^N) = \binom{n}{2} = \frac{1}{2}n(n-1)$

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and $\ell(\varnothing)=0$. Let $L_i(g)=\{ij|\{i,j\}\in g\}$ be the set of links in which player i is involved. Denote $\ell_i(g)$, the number of links in player i's link set, and $\#L_i(g)=\ell_i(g)$. Therefore, it follows that $\ell(g)=\frac{1}{2}\sum_{i\in N}\ell_i(g)$. Let $N(g)$ be the set of all players in g. A value function v for networks is such that $v: G^N \mapsto \mathbb{R}$ with $v(\varnothing)=0$. For a network g, $v(g)$ specifies the total worth generated by the network g. Denote $\mathbb{V}^N=\{v|v: G^N \mapsto \mathbb{R}, v(\varnothing)=0\}$, the set of all value functions.

Definition 4. *A network game is defined as a pair (N, v), where N is the player set and v is the value function.*

When the player set N is fixed and there is no ambiguity, we represent a network game simply as v.

Definition 5. *For $g\in G^N$, the unanimity value function u_g is given by*

$$u_g(g')=\begin{cases}1 & \text{if } g\subseteq g'\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

The unanimity value functions form a basis for $\mathbb{V}^N$. Thus $v\in\mathbb{V}^N$ can be written as a linear combination of unanimity value functions u_g, i.e., $v=\sum_{g\subseteq g^N} c_g u_g$, where $c_g\in\mathbb{R}$ are the unanimity coefficients of v.

Definition 6. *A network allocation rule is a function $Y: \mathbb{G}^N\times\mathbb{V}^N\to\mathbb{R}^n$ such that for every $g\in\mathbb{G}^N$ and every $v\in\mathbb{V}^N$, $Y_i(g,v)$ represents the payoff to player i with the condition that $Y_i(\varnothing, v)=0$.*

Definition 7. *For a given network g, let v be a value function with unanimity coefficients $(c_{g'})_{g'\subseteq g}$. Then the Position value (Slikker, 2005), $Y^{PV}(g,v)$ is defined by*

$$Y_i^{PV}(g,v)=\sum_{g'\subseteq g}\frac{c_{g'}\ell_i(g')}{2\ell(g')},\ \forall i\in N. \tag{8}$$

Let v^g be the associated cooperative game with respect to the network game v considering the links in g^N as players. It follows that for every link l of g, the Shapley value Φ^{Sh} (Shapley, 1953) of l with respect to v^g is given by

$$\Phi_l^{Sh}(g,v^g)=\sum_{g'\subseteq g\,:\,l\subseteq g'}\frac{c_{g'}}{\ell(g')}. \tag{9}$$

Now, from Eq.s (8) and (9), we obtain

$$Y_i^{PV}(g,v)=\sum_{l\in L_i(g)}\frac{1}{2}\Phi_l^{Sh}(g,v^g),\ \forall i\in N. \tag{10}$$

4. Multilayer network games

In this section, we introduce the notion of our proposed multilayer network games. Let $g_1, g_2\in G^N$ be two networks with possibly non-empty $g_1\cap g_2$. Then (g_1,g_2) is called a multilayer network. We assume that players in g_1 and g_2 as X and Z contributors, respectively. Players belonging to $g_1\cap g_2$ are X–Z contributors, and players in the set $N\setminus N(g_1\cup g_2)$ are absentees. Following ideas similar to those of overlapping coalitions, we call a link in g_1 and g_2 an X-link and a Z link, respectively, and refer to them as single-layer links. On the other hand, a link in $g_1\cap g_2$ exhibits both X and Z attributes is called an X–Z link, which we refer as multilayer link. In Figure 2, we present a multilayer network (g_1,g_2) on the player set $N=\{1,2,3,4,5\}$ where $g_1=\{12,23\}$ and $g_2=\{23,24,34\}$. Both g_1 and g_2 contains two common players 2 and 3, and therefore, (g_1,g_2)

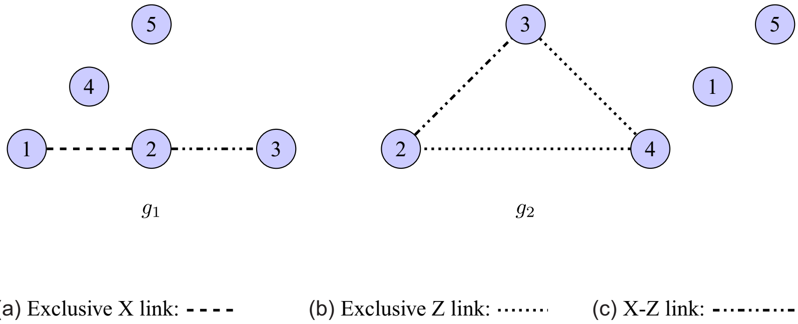

Figure 2. An example of a multilayer network (g_1, g_2) on $N = \{1, 2, 3, 4, 5\}$.

having an X–Z link {23} forms a multilayer network. Let $2^{g^N} \times 2^{g^N} = \{(g_1, g_2)|g_1, g_2 \subseteq g^N\}$ be the set of all multilayer networks on N.

For $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$, let $L(g_1, g_2)$ denote the set of links in g_1 and g_2, i.e., $L(g_1, g_2) = L(g_1 \cup g_2)$ and $\ell(g_1, g_2)$ represents the total number of links in $L(g_1, g_2)$. Let $L_i(g_1, g_2) = L_i(g_1 \cup g_2)$ be the set of links that involve player i in g_1 and g_2. Let $\ell_i(g_1, g_2) = \#L_i(g_1, g_2)$. For any link l, by $(g_1, g_2) \setminus l$, we mean $(g_1 \setminus l, g_2 \setminus l)$. We denote by $N(g_1, g_2) = N(g_1 \cup g_2)$, the set of players in g_1 and g_2.

A value function is a function $v^m : 2^{g^N} \times 2^{g^N} \mapsto \mathbb{R}$ such that $v^m(\varnothing, \varnothing) = 0$. Thus, a value function assigns each $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$ a real number, its generated worth, when players in g_1 are X contributors, players in g_2 are Z contributors, players in $g_1 \cap g_2$ act as X–Z contributors, and the remaining players are considered absentees. Let $\mathcal{MG}$ denotes the set of all value functions on $2^{g^N} \times 2^{g^N}$.

Definition 8. *A multilayer network game is a pair* (N, v^m), *of a player set* N *and a value function* $v^m : 2^{g^N} \times 2^{g^N} \mapsto \mathbb{R}$.

If the player set N is fixed and there is no ambiguity, we simply write v^m to denote a multilayer network game.

Definition 9. *An allocation rule for multilayer network game is a function* $Y : (2^{g^N} \times 2^{g^N}) \times \mathcal{MG} \mapsto \mathbb{R}^n$ *such that* $Y_i((\varnothing, \varnothing), v^m) = 0$ *for all* $v^m \in \mathcal{MG}$.

For each i, $Y_i((g_1, g_2), v^m)$ represents the payoff to the player i with respect to $v^m \in \mathcal{MG}$ and $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$.

5. The Position value for multilayer network games

Here, we initially present the link game associated with a multilayer network game, where the set of links serves as the set of players.

Let $[g^N]$ represent the hypothetical player set corresponding to the links in g^N. For $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$, let $[g_1]$ and $[g_2]$ refer to the set of all hypothetical players representing the links in g_1 and g_2 respectively. For a given multilayer network game (N, v^m), the associated link game $([g^N], v^{m*})$ (where $v^{m*} : \mathbb{O}^{[g^N]} \mapsto \mathbb{R}$ is a function) of (N, v^m) is defined as follows.

For each $(S, T) \in 2^{[g^N]} \times 2^{[g^N]} = \{(S, T)|S, T \subseteq [g^N]\}$ there is a $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$ with $S = [g_1]$, $T = [g_2]$ such that $v^{m*}(S, T) = v^m(g_1, g_2)$. Conversely, for each $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$, there is a pair $(S, T) \in 2^{[g^N]} \times 2^{[g^N]}$ such that $S = [g_1]$ and $T = [g_2]$.

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Definition 10. *An allocation rule Y is called a link-based allocation rule if there is some $Y^L : \left(2^{g^N} \times 2^{g^N}\right) \times \mathcal{MG} \mapsto \mathbb{R}^{\frac{n(n-1)}{2}}$ such that for all $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$, $v^m \in \mathcal{MG}$, $i \in N$,*

$$\sum_{l \in ([g_1] \cup [g_2])} Y_l^L((g_1, g_2), v^m) = v^m(g_1, g_2),$$

and

$$Y_i((g_1, g_2), v^m) = \sum_{l \in L_i(g_1, g_2)} \frac{Y_l^L((g_1, g_2), v^m)}{2}.$$

Definition 11. *The Position value in a multilayer network game, denoted as Y^{MNPV}, is the allocation rule where each player $i \in N$ in a multilayer network game (N, v^m) receives half of the OLG value for each of their links considered as hypothetical players in the associated link game $([g^N], v^{m*})$, i.e.,*

$$Y_i^{MNPV}((g_1, g_2), v^m) = \sum_{l \in L_i(g_1, g_2)} \frac{1}{2} \Phi_l^{OLG}(v^{m*})([g_1], [g_2]).$$

5.1 Characterization of the Position value

Here, we provide an axiomatic characterization of the Position value for the class of multilayer network games using axioms of Efficiency and Balanced Link Contribution.

Claim 1 (Efficiency). *An allocation rule Y satisfies Efficiency if for any $v^m \in \mathcal{MG}$ and $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$ we have,*

$$\sum_{i \in N(g_1, g_2)} Y_i((g_1, g_2), v^m) = v^m(g_1, g_2).$$

The Efficiency axiom implies that the payoffs received by players under the allocation rule should sum to the total worth generated by the multilayer network (g_1, g_2). This principle is consistent with the notion that an efficient allocation ensures the accurate distribution of the total worth within a multilayer network among its individual players in accordance with the specified allocation rules.

Lemma 1. *The Position value in a multilayer network game satisfies Efficiency.*

Proof. It directly follows from the definition 11. □

Claim 2 (Balanced Link Contribution). *An allocation rule Y satisfies Balanced Link Contribution if for any $v^m \in \mathcal{MG}$ and $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$ and $i, j \in N$, we have*

$$\sum_{l \in L_j(g_1, g_2)} \left(Y_i((g_1, g_2), v^m) - Y_i((g_1, g_2) \setminus l, v^m)\right) = \sum_{l \in L_i(g_1, g_2)} \left(Y_j((g_1, g_2), v^m) - Y_j((g_1, g_2) \setminus l, v^m)\right).$$

The Balanced Link Contribution axiom states that, within a multilayer network game and under an allocation rule, the impact of losing a link, whether multilayer or single-layer is identical for the involved players. Moreover, the net effects of removal of a X–Z link, remain balanced between players, ensuring fairness in the allocation process. The unique feature highlighted here is that the removal of a multilayer link results in its removal from both the networks g_1 and g_2.

Lemma 2. *The Position value in a multilayer network game satisfies Balanced Link Contribution.*

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Proof. For a given $v^m \in \mathcal{MG}$, $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$ and $i, j \in N$, let Y_i^{MNPV} and Y_j^{MNPV} be the i^{th} and j^{th} components of the Position value. Now,

$$
\begin{aligned}
&\sum_{l \in L_j(g_1,g_2)} \left(Y_i^{MNPV}((g_1, g_2), v^m) - Y_i^{MNPV}((g_1, g_2) \setminus l, v^m)\right) \\
&= \sum_{l \in L_j(g_1,g_2)} \left(\sum_{l_1 \in L_i(g_1,g_2)} \frac{1}{2} \Phi_{l_1}^{OLG}(v^{m*})([g_1], [g_2]) - \sum_{l_1' \in L_i(g_1 \setminus l, g_2 \setminus l)} \frac{1}{2} \Phi_{l_1'}^{OLG}(v^{m*})([g_1 \setminus l], [g_2 \setminus l]) \right) \\
&= \sum_{l \in L_j(g_1,g_2)} \left(\sum_{l_1 \in L_i(g_1,g_2)} \frac{1}{2} \Phi_{l_1}^{Sh}([g_1] \cup [g_2], V) - \sum_{l_1' \in L_i(g_1 \setminus l, g_2 \setminus l)} \frac{1}{2} \Phi_{l_1'}^{Sh}([g_1 \setminus l] \cup [g_2 \setminus l], V) \right) \\
&= \frac{1}{2} \sum_{l \in L_j(g_1,g_2)} \left(\sum_{l_1 \in L_i(g_1,g_2)} \sum_{P \subseteq ([g_1] \cup [g_2]) : l_1 \in P} \frac{c_P}{|P|} - \sum_{l_1' \in L_i(g_1 \setminus l, g_2 \setminus l)} \sum_{P' \subseteq ([g_1 \setminus l] \cup [g_2 \setminus l]) : l_1' \in P'} \frac{c_{P'}}{|P'|} \right) \\
&= \frac{1}{2} \left(\sum_{P \subseteq [g_1] \cup [g_2]} |P_j| \frac{c_P}{|P|} |P_i| - \sum_{P' \subseteq [g_1 \setminus l] \cup [g_2 \setminus l]} |P_j'| \frac{c_{P'}}{|P'|} |P_i'| \right) \\
&= \sum_{l \in L_i(g_1,g_2)} \left(Y_j^{MNPV}((g_1, g_2), v^m) - Y_j^{MNPV}((g_1, g_2) \setminus l, v^m)\right),
\end{aligned}
$$

where the first equality follows by definition, the second and third equalities are derived from Eq.(6), and the fourth equality is obtained through the rearrangement of terms. Here, for each $S \subseteq N$, $|S_i|$ denotes the number of hypothetical players in S corresponding to the links involving player $i \in N$. Note that the expression after the fourth equality sign is symmetric in both i and j. □

Next, we will establish that the Position value for the class of multilayer network games is the unique allocation rule that satisfies these two axioms: Efficiency and Balanced Link Contribution.

Theorem 1. *The Position value Y^{MNPV} is the unique allocation rule determined by the axioms of Efficiency and Balanced Link Contribution.*

Proof. The proof proceeds through induction on $\ell(g_1, g_2)$, where $(g_1, g_2) \in 2^{g^N} \times 2^{g^N}$. Let $N = \{i, j\}$ such that $\ell(g_1, g_2) = 1$. Then, one of the following scenarios arises:

Case (i) $g_1 = \{ij\}$, $g_2 = \varnothing$,

Case (ii) $g_1 = \varnothing$, $g_2 = \{ij\}$ and

Case (iii) $g_1 = \{ij\}$, $g_2 = \{ij\}$.

Case (i): If possible let, Y, Y' be two allocation rules that satisfy Efficiency and Balanced Link Contribution. It follows directly from Balanced Link Contribution that,

$$
\begin{aligned}
\sum_{l \in L_j(g_1,g_2)} \left(Y_i((g_1, g_2), v^m) - Y_i((g_1, g_2) \setminus l, v^m)\right) &= \sum_{l \in L_i(g_1,g_2)} \left(Y_j((g_1, g_2), v^m) - Y_j((g_1, g_2) \setminus l, v^m)\right) \\
\Rightarrow Y_i((g_1, g_2), v^m) - Y_i((g_1, g_2) \setminus ij, v^m) &= Y_j((g_1, g_2), v^m) - Y_j((g_1, g_2) \setminus ij, v^m) \\
\Rightarrow Y_i((g_1, g_2), v^m) - Y_i((\varnothing, \varnothing), v^m) &= Y_j((g_1, g_2), v^m) - Y_j((\varnothing, \varnothing), v^m) \\
\Rightarrow Y_i((g_1, g_2), v^m) &= Y_j((g_1, g_2), v^m).
\end{aligned}
$$

Similarly, $Y_i'((g_1, g_2), v^m) = Y_j'((g_1, g_2), v^m)$. Therefore, by Efficiency, we have $Y = Y'$.

Cases (ii) and (iii) are similar to Case (i), and so the proofs are omitted here. Thus, the result holds for a single link.

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Let $\ell(g_1,g_2)>1$. For $\ell(g_1,g_2)\le k-1$, suppose the allocation rule for v^m which satisfies Efficiency and Balanced Link Contribution is unique. From Balanced Link Contribution, we have

$$\begin{aligned}
\sum_{l\in L_2(g_1,g_2)} \big(Y_1((g_1,g_2),v^m)-Y_1((g_1,g_2)\setminus l,v^m)\big) &= \sum_{l\in L_1(g_1,g_2)} \big(Y_2((g_1,g_2),v^m)-Y_2((g_1,g_2)\setminus l,v^m)\big),\\
\sum_{l\in L_3(g_1,g_2)} \big(Y_1((g_1,g_2),v^m)-Y_1((g_1,g_2)\setminus l,v^m)\big) &= \sum_{l\in L_1(g_1,g_2)} \big(Y_3((g_1,g_2),v^m)-Y_3((g_1,g_2)\setminus l,v^m)\big),\\
&\vdots\\
\sum_{l\in L_n(g_1,g_2)} \big(Y_1((g_1,g_2),v^m)-Y_1((g_1,g_2)\setminus l,v^m)\big) &= \sum_{l\in L_1(g_1,g_2)} \big(Y_n((g_1,g_2),v^m)-Y_n((g_1,g_2)\setminus l,v^m)\big).
\end{aligned}$$

This would further imply,

$$\begin{aligned}
\ell_2(g_1,g_2)Y_1((g_1,g_2),v^m)-\ell_1(g_1,g_2)Y_2((g_1,g_2),v^m) &= \sum_{l\in L_2(g_1,g_2)} Y_1^{MNPV}((g_1,g_2)\setminus l,v^m)\\
&\quad - \sum_{l\in L_1(g_1,g_2)} Y_2^{MNPV}((g_1,g_2)\setminus l,v^m),\\
\ell_3(g_1,g_2)Y_1((g_1,g_2),v^m)-\ell_1(g_1,g_2)Y_3((g_1,g_2),v^m) &= \sum_{l\in L_3(g_1,g_2)} Y_1^{MNPV}((g_1,g_2)\setminus l,v^m)\\
&\quad - \sum_{l\in L_1(g_1,g_2)} Y_3^{MNPV}((g_1,g_2)\setminus l,v^m),\\
&\vdots\\
\ell_n(g_1,g_2)Y_1((g_1,g_2),v^m)-\ell_1(g_1,g_2)Y_n((g_1,g_2),v^m) &= \sum_{l\in L_n(g_1,g_2)} Y_1^{MNPV}((g_1,g_2)\setminus l,v^m)\\
&\quad - \sum_{l\in L_1(g_1,g_2)} Y_n^{MNPV}((g_1,g_2)\setminus l,v^m),
\end{aligned}$$

and by Efficiency,

$$\sum_{i\in N(g_1,g_2)} Y_i((g_1,g_2),v^m)=v^m(g_1,g_2).$$

Here, we have a system of n linearly independent equations with n variables: Y_1, Y_2, . . ., Y_n. The coefficient matrix is of full rank, n. Hence, the system possesses a unique solution. Therefore, for $\ell(g_1,g_2)=k$, the allocation rule that upholds Efficiency and Balanced Link Contribution is unique. In conclusion, the Position value stands as the unique allocation rule meeting the axioms of Efficiency and Balanced Link Contribution. □

6. Concluding discussions

In this section, we explore the distinctions and parallels of multilayer network games, bi-cooperative network games, and multigraphs. By understanding how links and nodes operate within these different frameworks, we can better address various network challenges.

In a multilayer network, a link acts as an X–Z link, displaying both X and Z characteristics simultaneously. It is possible that the set of links for both multilayer network games and bi-cooperative network games are the same. However, in a bi-cooperative network, a link can either

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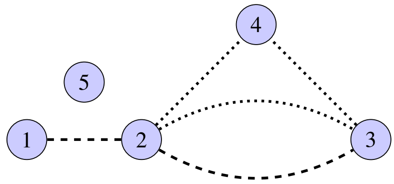

Figure 3. Multigraph: $g_1 + g_2 = \{12, 23, 23, 24, 34\}$.

be exclusively X or Z but not both at the same time, see Figure 2. Hence, even though the set of links are same, their roles may differ in each case. For instance, consider $(h_1, h_2) = (\{12, 23\}, \{23\})$ and $(g_1, g_2) = (\{12\}, \{23\})$. In this scenario, $L(h_1, h_2) = L(g_1, g_2) = \{12, 23\}$. However, $\{23\}$ acts as a X–Z link in (h_1, h_2) and as a Z link in (g_1, g_2). Consequently, (h_1, h_2) belongs to a multilayer network, while (g_1, g_2) belongs to a bi-cooperative network.

Multigraph is a graph in which multiple links (arcs) between two given nodes and loops (links ending at the same nodes) can occur (Forlicz et al., 2018). Consider, $g_1 = \{12, 23\}$ and $g_2 = \{23, 24, 34\}$ on $N = \{1, 2, 3, 4, 5\}$. Here, the pair (g_1, g_2) forms a multilayer network (see Figure 2). However, if we combine the two networks, viz., $g_1 + g_2 = \{12, 23, 23, 24, 34\}$, then $g_1 + g_2$ represents a multigraph as shown in Figure 3.

Further, an allocation rule (or a point-value solution) in a multigraph (Forlicz et al., 2018) or multilayer network allows for determining the role of each player within it. For example, players (2, 3, and 4 in Figure 3) connected by more links are considered more important in multigraphs than players with fewer links (such as player 1 in Figure 3). Similarly, in multilayer networks, players connected by X–Z links are more significant than those connected only by X or Z links. Therefore, an allocation rule in both multigraphs and multilayer networks should prioritize these players. However, the distinction between a multigraph and a multilayer network lies in how we deal with the links. In a multigraph, all links between nodes are considered to be distinct, and there are no overlaps among them (Forlicz et al., 2018). On the contrary, in the multilayer network games, we have "multilayer links" that belong to two different networks simultaneously. Although a multilayer link belongs to two different networks, it acts as a single link with two distinct attributes (say, X and Z). For example, in Figure 2 multilayer link {23} acts as a single link. Therefore, when determining the Position value, we count multilayer links only once. Forlicz et al. (2018) define the Shapley value for multigraphs that considers every individual link between two given nodes, including both single and multiple links, as well as loops. To illustrate, in the multigraph $g_1 + g_2$ depicted in Figure 3, nodes 2 and 3 are connected through two links. In the model proposed by Forlicz et al. (2018), each link between nodes 2 and 3 is evaluated separately when computing the Shapley value for each player in $g_1 + g_2$.

The multilayer network games, bi-cooperative network games, and multigraphs offer valuable insights into managing, optimizing, and decision-making in complex network systems. Multilayer network games highlight the interconnectedness across different network layers, acknowledging nodes' multiple roles in managing internal data and external communications simultaneously. This understanding supports integrated network management and optimizes resource allocation based on critical multilayer connections. Bi-cooperative network games simplify interactions within distinct network structures (X or Z), clarifying responsibilities in simpler contexts. Multilayer network games address decision-making complexities to comprehensively assess risks and balance security, efficiency, and operational continuity in dynamic environments. Multigraphs provide detailed insights into network structure and the significance of nodes, contributing to enhanced network security and efficiency and ensuring resilience across interconnected layers of network infrastructures.

In conclusion, we have studied the allocation rule of players in a network in presence of an another network having common or shared players. Our multilayer network game in the cooperative framework has proven to be more effective in addressing real life scenarios arising in the government and private sectors. The proposed framework is free from the limitations of bi-cooperative network game and classical network game. Our model offers more flexibility in terms of network connections while calculating utilities of players, enabling us to solve real life complexities.

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Note

1 g_1 has 3 links, so the total number of subnetworks of g_1 is given by $\sum_{i=0}^{3} \binom{3}{i} = 8$, including the empty network. Similarly, g_2 also has 3 links, so the total number of subnetworks of g_2 is also 8. Therefore, the total number of possible collaborations between g_1 and g_2 is $8 \times 8 = 64$.

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