BOOLEAN ALGEBRAS OF PROJECTIONS

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Bade, in (**1**), studied Boolean algebras of projections on Banach spaces and showed that a σ-complete Boolean algebra of projections on a Banach space enjoys properties formally similar to those of a Boolean algebra of projections on Hilbert space. (His exposition is reproduced in (**7**: XVII).) Edwards and Ionescu Tulcea showed that the weakly closed algebra generated by a σ-complete Boolean algebra of projections can be represented as a von Neumann algebra; and that the representation isomorphism can be chosen to be norm, weakly, and strongly bicontinuous on bounded sets (**8**): Bade's results were then seen to follow from their Hilbert space counterparts. I show here that it is natural to relax the condition of σ-completeness to weak relative compactness; indeed, a Boolean algebra of projections has a σ-completion if and only if it is weakly relatively compact (Theorem 1). Then, following the derivation of the theorem of Edwards and Ionescu Tulcea from the Vidav characterisation of abstract C^*-algebras (see (**9**)), I give a result (Theorem 2) which, with its corollary, includes (**1**: 2.7, 2.8, 2.9, 2.10, 3.2, 3.3, 4.5).

Let X be a complex Banach space with dual X'; let $L(X)$ be the Banach algebra of (bounded linear) operators on X. Two weak topologies are used in this paper; the weak topology $\sigma(X, X')$ on X, and the weak (operator) topology on $L(X)$. Note that a subset $\boldsymbol{E}$ of $L(X)$ is weakly relatively compact if and only if $\boldsymbol{E}x$ is weakly relatively compact (in X) for each x in X (**7**: VI.9.2). The strong (operator) topology on $L(X)$ will also be used.

A **projection** on X is an idempotent in $L(X)$. If E is a projection, so is its complement $I-E$. If E and F are commuting projections, they have a least upper bound $E \vee F(= E+F-EF)$ and a greatest lower bound $E \wedge F(= EF)$. A set of commuting projections is a Boolean algebra of projections if it contains 0 and I, and is closed under complementation and the operations $\vee$ and $\wedge$.

A Boolean algebra of projections $\boldsymbol{B}$ on X is **complete** (**σ-complete**) if for each subset (subsequence) $\{B_\lambda\}$ of $\boldsymbol{B}$ there is a direct sum decomposition $X = Y \oplus Z$, where Y is the closed subspace spanned by $\{B_\lambda X\}$, Z is the closed subspace $\bigcap (I-B_\lambda)X$, and the projection B onto Y along Z belongs to $\boldsymbol{B}$; then $B = \bigvee B_\lambda$. If $\boldsymbol{B}$ is complete (σ-complete) in this sense, then $\boldsymbol{B}$ is complete (σ-complete) as an abstract Boolean algebra.

If $\boldsymbol{B}$ is σ-complete (even only as an abstract Boolean algebra), then $\boldsymbol{B}$ is **bounded** (that is, there is a number M with $\| B \| \leqq M$ when $B \in \boldsymbol{B}$ (**1**: Theorem 2.2)).

If $\boldsymbol{B}$ is complete (σ-complete) and (B_λ) is an increasing net (sequence) in $\boldsymbol{B}$, then $\bigvee B_\lambda = \lim B_\lambda$ in the strong topology (**1**: Lemma 2.3).

Similarly, if $\boldsymbol{B}$ is weakly relatively compact, then $\boldsymbol{B}$ is bounded and $\boldsymbol{B}x$ is weakly relatively compact for each x in X. If (B_λ) is an increasing net in $\boldsymbol{B}$, then, by (**2**: Corollary 1), $\bigvee B_\lambda$ exists and $\bigvee B_\lambda = \lim B_\lambda$ in the strong topology.

Let Λ be the Stone representation space of $\boldsymbol{B}$. Write $\boldsymbol{K}$ for the set of open-and-closed subsets of Λ, $\boldsymbol{S}$ for the set of Baire subsets of Λ. Let us write the representation map $\boldsymbol{K}\to\boldsymbol{B}$ in the form $\tau\mapsto B(\tau)$. Because Λ is a totally disconnected compact Hausdorff space, $\boldsymbol{S}$ is the σ-algebra (alternatively, the monotone class) generated by $\boldsymbol{K}$; also, $\boldsymbol{L}$, the linear (which is also the algebra) span of the characteristic functions of sets in $\boldsymbol{K}$, is norm-dense in $C(\Lambda)$, the algebra of continuous functions on Λ.

Theorem 1. *Let $\boldsymbol{B}$ be a Boolean algebra of projections on a Banach space. Then $\boldsymbol{B}$ is weakly relatively compact if and only if $\boldsymbol{B}$ has a σ-completion.*

Proof. Let $\boldsymbol{B}$ be weakly relatively compact. Consider a sequence (τ_n) in $\boldsymbol{K}$. By a remark above, $\left(B\left(\bigcup_1^n \tau_k\right)\right)$ increases, and converges strongly, to a projection in $\boldsymbol{B}^s$, the strong closure of $\boldsymbol{B}$. Countable iteration of this process will give a projection $\tilde{B}(\tau)$ for each τ in S. It is easy to see that $\tilde{B}$: $\boldsymbol{S}\to\boldsymbol{B}^s$ extends B, that $\tilde{B}$ is a spectral measure and that $\tilde{B}(\boldsymbol{S})$ is the σ-completion of $\boldsymbol{B}$.

Conversely, assume that $\boldsymbol{B}$ has a σ-completion $\tilde{\boldsymbol{B}}$. Then, as in the preceding paragraph, B: $\boldsymbol{K}\to\boldsymbol{B}$ has an extension $\tilde{B}$: $\boldsymbol{S}\to\tilde{\boldsymbol{B}}$, showing that $\tilde{\boldsymbol{B}}$ is the range of a spectral measure. (This observation was of prime importance in (**1**).) Now the range of a vector measure is weakly relatively compact (**3**: Theorem 2.9), so $\tilde{\boldsymbol{B}}x$ is weakly relatively compact for each x in X. Thus $\tilde{\boldsymbol{B}}$ is weakly relatively compact; whence $\boldsymbol{B}$ is.

Theorem 2. *Let $\boldsymbol{B}$ be a Boolean algebra of projections on a Banach space X. Suppose that $\boldsymbol{B}$ has a σ-completion (or equivalently, that $\boldsymbol{B}$ is weakly relatively compact). Then $\boldsymbol{B}$ has a completion contained in $\boldsymbol{B}^s$ (which is a complete Boolean algebra of projections), and the weak and strong topologies agree on $\boldsymbol{B}^s$. Let A be the norm-closed algebra generated by $\boldsymbol{B}$; let A^w be the weak closure of A. Then $\boldsymbol{B}$ is complete if and only if $\boldsymbol{B} = \boldsymbol{B}^s$, if and only if $A = A^w$.*

Proof. X has an equivalent norm for which the members of $\boldsymbol{B}$ are **hermitian** (in that they have real numerical range (see (**6**)) (**4**: Lemmas 2.2, 2.3). We may assume that X has this norm. By (**5**: Theorem 2.1), the map B: $\boldsymbol{K}\to\boldsymbol{B}$ has an isometric linear extension to a map $\boldsymbol{L}\to A$; this extension is an algebra homomorphism. Therefore B: $\boldsymbol{K}\to\boldsymbol{B}$ extends to an isometric algebra isomorphism $\tilde{\tilde{B}}$: $C(\Lambda)\to A$; and $\tilde{\tilde{B}}(f) = \int_\Lambda f d\tilde{\tilde{B}}$ $(f\in C(\Lambda))$. So, in the terminology of (**9**), A is representable by a spectral measure. By (**9**: Theorem 2), A^w is a W^*-algebra; moreover, there are a Hilbert space H, a von Neumann algebra $\tilde{A}$ on

H, and a C^*-isomorphism $A^w \to \tilde{A}$ which is weakly and strongly bicontinuous on bounded sets. The theorem now follows from the corresponding Hilbert space results.

Corollary. *Let $\boldsymbol{B}$ be a bounded Boolean algebra of projections on a weakly complete Banach space X. Then $\boldsymbol{B}$ satisfies the hypotheses of the theorem.*

Proof. The map $\tilde{\tilde{B}}$: $C(\Lambda) \to A$ may be defined as in the proof of the theorem. Then $C(\Lambda) \to X$: $f \mapsto B(f)x$ is weakly compact (3: Theorem 3.2), whence $\boldsymbol{B}x$ is weakly relatively compact (for each x in X); so $\boldsymbol{B}$ is weakly relatively compact.

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