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REMARKS ON IMMERSIONS IN THE METASTABLE DIMENSION RANGE

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Abstract In this work we present a generalization of an exact sequence of normal bordism groups given in a paper by H. A. Salomonsen (*Math. Scand.* **32** (1973), 87–111). This is applied to prove that if $h : M^n \to X^{n+k}$, $5 \leqslant n < 2k$, is a continuous map between two manifolds and $g : M^n \to BO$ is the classifying map of the stable normal bundle of h such that $(h, g)_* : H_i(M, \mathbb{Z}_2) \to H_i(X \times BO, \mathbb{Z}_2)$ is an isomorphism for $i < n - k$ and an epimorphism for $i = n - k$, then h bordant to an immersion implies that h is homotopic to an immersion. The second remark complements the result of C. Biasi, D. L. Gonçalves and A. K. M. Libardi (*Topology Applic.* **116** (2001), 293–303) and it concerns conditions for which there exist immersions in the metastable dimension range. Some applications and examples for the main results are also given.

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1. Introduction

Let $h : M^n \to X^{n+k}$ be a continuous map from a closed smooth connected n-manifold into a smooth connected $(n + k)$-manifold, $5 \leqslant n < 2k$. Let us assume that h is bordant to an immersion, in the sense of Conner and Floyd [**4**], and let $g : M \to BO$ be the classifying map of the stable normal bundle, $h^*(\tau_X) \oplus \nu_M$, of h, where τ_X denotes the tangent bundle of X and $\nu_M = -(\tau_M)$. One may ask on which conditions of (h, g) is h homotopic to an immersion?

Let $f : M \to N$ be a continuous map between two closed smooth connected n-dimensional manifolds and suppose that N immerses in $\mathbb{R}^{n+k}$, for some k, with $5 \leqslant n < 2k$. Under which conditions on f does M immerse in $\mathbb{R}^{n+k}$? The case when M immerses in $\mathbb{R}^{n+k}$ and in which one is looking for conditions on f such that N also immerses in $\mathbb{R}^{n+k}$ has been considered in [**2**] and [**5**–**7**].

For both problems, we use a normal bordism approach [**9**], and give an answer in terms of the induced maps of $\mathbb{Z}_2$-homology groups.

We prove the following main results.

Theorem A. *Let $h : M^n \to X^{n+k}$ be a continuous map from a closed smooth connected n-manifold into a smooth connected $(n+k)$-manifold, $5 \leqslant n < 2k$, and let $g : M \to BO$ be the classifying map of the stable normal bundle of h. Given*

$$(h, g) : M \to X \times BO,$$

suppose that the induced map

$$(h, g)_* : H_i(M, \mathbb{Z}_2) \to H_i(X \times BO, \mathbb{Z}_2)$$

is an isomorphism for $i < n - k$ and an epimorphism for $i = n - k$.

Then if h is bordant to an immersion, h is homotopic to an immersion.

Theorem B. *Let M and N be closed connected n-manifolds and let $f : M \to N$ be a continuous map such that*

$$f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$$

is an isomorphism for $i \geqslant 0$.

Then if N immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does M.

The paper is divided into four sections. In § 2 we present two exact sequences of bordism groups. One of them is a generalization of the exact sequence of normal bordism groups given by Salomonsen [**13**]; it will be applied to prove Theorem A.

In § 3 we prove Theorems A and B and in § 4 we present an application of Theorem B by using a non-standard obstruction theory, and we give some examples for Theorem A.

In this work, $\mathcal{C}$ will denote the class of all torsion groups where the torsion is odd.

2. Exact sequences of bordism groups

In this section we generalize an exact sequence given in [**13**], by using identifications of some normal bordism groups.

Given a topological space X and a virtual bundle ϕ over X (i.e. an ordered pair of vector bundles ϕ^+ and ϕ^- over X, written $\phi^+ - \phi^-$), the nth normal bordism group of X with coefficient ϕ, denoted by $\Omega_n(X, \phi)$, is the bordism group of pairs $(h : M \to X, g)$, where g is the stable bundle isomorphism $\tau_M \oplus g^*(\phi^-) \simeq \varepsilon^n \oplus g^*(\phi^+)$ and ε^n denotes the trivial bundle of dimension n. We recall that $\Omega_n(X, \phi) = \Omega_n(X, \phi + \varepsilon^r)$, and if ϕ can be expressed in the form $\phi = \varepsilon^l - (\phi^-)^l$, there is an isomorphism $\Omega_n(X, \phi) \simeq \pi^S_{n+l}(T(\phi^-))$, where $T(\phi^-)$ is the disjoint union of the (total space) ϕ^- and a point ∞. For more details see [**13**] or [**9**]. We adopt the Salomonsen convention.

Let us now consider X, an $(n+k)$-manifold, and let $\nu^p_X = -(\tau_X)$ be the stable normal bundle of X, with p large enough. If $\phi^{p+k} = \varepsilon^{p+k} - \nu^p_X \times \gamma^k$, an element of $\Omega_n(X \times BO(k), \phi^{p+k})$ can be considered as $[(h, g) : M^n \to X \times BO(k), H]$, where

$$H : \tau_M \oplus h^*(\nu^p_X) \oplus g^*(\gamma^k) \to \varepsilon^{p+k} \oplus \varepsilon^n$$

is a stable bundle isomorphism and g is the classifying map of the stable normal bundle of h. This is equivalent to the isomorphism $\nu_M \simeq h^*(\nu_X^p) \oplus g^*(\gamma^k)$ and, since $\nu_X \oplus \tau_X$ is trivial, $h^*(\tau_X) \oplus \nu_M \simeq g^*(\gamma^k) \oplus \varepsilon^{p+n}$. In this case, the stable normal bundle of h has an $O(k)$-structure and then, by Hirsch [**8**], h is homotopic to an immersion. Let us denote $\Omega_n(X \times BO(k), \phi^{p+k})$ by $I_n(X)$ and let $\mathcal{F} : I_n(X) \to \eta_n(X)$ be the forgetful map. We remark that if $[M, f] \in \eta_n(X)$ is an element of $\mathcal{F}(I_n(X))$, then f is homotopic to an immersion.

Let $\psi = \psi^+ - \psi^-$ be a virtual bundle over X. We note that the geometric dimension $g\dim(\psi) \leqslant k$ if and only if there exists a k-dimensional vector bundle μ^k such that $\mu^k \oplus \psi^- = \varepsilon^k \oplus \psi^+$. We recall that if we consider $f : M^n \to X^{n+k}$ to be a continuous map between two closed smooth manifolds and $\psi = f^*\tau_X - \varepsilon^k \oplus \tau_M$, then $g\dim(\psi) \leqslant k$ if there exists a vector bundle μ^k such that $\mu^k \oplus \varepsilon^k \oplus \tau_M \simeq \varepsilon^k \oplus f^*\tau_X$. This isomorphism is equivalent to $\mu^k \oplus \tau_M \simeq f^*\tau_X$, and then, by [**8**], f is homotopic to an immersion.

In order to study whether $g\dim(\psi) \leqslant k$ we need to define a fibre bundle $\tilde{V}_k(\psi^q)$ over X. Consider the bundle $\mathrm{Iso}(\varepsilon^k \oplus \psi^-, \varepsilon^k \oplus \psi^+) \to X$, whose fibre consists of $\mathrm{Iso}(\mathbb{R}^k \oplus (\psi^-)_x, \mathbb{R}^k \oplus (\psi^+)_x)$. The linear group Gl_k acts freely on the right and then we define $V_k(\psi) = \mathrm{Iso}(\varepsilon^k \oplus \psi^-, \varepsilon^k \oplus \psi^+)/Gl_k$, which is a fibre bundle over X with fibre homotopy equivalent to a Stiefel manifold. For each t we can construct $V_k(\psi^+ \oplus \varepsilon^t - \psi^- \oplus \varepsilon^t)$ over X whose fibre is also $(k-1)$-connected. Then we define

$$\tilde{V}_k(\psi) = \bigcup_{t=0}^{\infty} V_k(\psi^+ \oplus \varepsilon^t - \psi^- \oplus \varepsilon^t)$$

over X with $(k-1)$-connected fibre. Since Gl_k acts freely on $\mathrm{Iso}(\varepsilon^k \oplus \psi^-, \varepsilon^k \oplus \psi^+)$ and effectively on $\mathbb{R}^k$, we have that $\mathrm{Iso}(\varepsilon^k \oplus \psi^-, \varepsilon^k \oplus \psi^+) \times_{Gl_k} \mathbb{R}^k$ is a k-dimensional vector bundle μ^k over $\tilde{V}_k(\psi)$ [**13**]. In this paper we will consider

$$\tilde{V}_k(\psi) \xrightarrow{\pi} X \times BO(q),$$

with $\psi = \gamma^q - \varepsilon^q$ a virtual bundle over $X \times BO(q)$ and where γ^q denotes the pull-back of the universal vector bundle over $BO(q)$, by the second projection $\pi_2 : X \times BO(q) \to BO(q)$.

Let us consider $\theta' : \tilde{V}_k(\psi) \to BO(k)$, the classifying map of the vector bundle μ^k, which is a high homotopy equivalence, for k large enough.

Let α^p be an arbitrary p-dimensional vector bundle over X, and, for each q, consider $\phi^{p+q} = \varepsilon^{p+q} - (\alpha^p \times \gamma^q)$, a virtual bundle over $X \times BO(q)$. We note that, for q large,

$$\Omega_n(X \times BO, \phi^{p+q}) \simeq \pi^S_{n+p+q}(T(\alpha) \wedge MO),$$

where $T(\alpha)$ is the Thom space [**9**] and, since $T(\alpha)$ is $(p-1)$-connected, we conclude that $\eta_n(X) \simeq \Omega_n(X \times BO, \phi^{p+q})$ and then this normal bordism group does not depend on α^p.

The following diagram is commutative:

$$\begin{array}{ccc} \Omega_n(\tilde{V}_k(\psi), \phi^{p+k}) & \longrightarrow & \Omega_n(\tilde{V}_k(\psi), \phi^{p+q}) \\ \downarrow \pi_* & \swarrow \theta_* & \downarrow \pi_* \\ \Omega_n(X \times BO(k), \phi^{p+k}) & \longrightarrow & \Omega_n(X \times BO(q), \phi^{p+q}) \end{array} \qquad \text{(I)}$$

where θ_*, induced by θ', is an isomorphism for q large, from remarks above.

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Let us suppose that $n \leqslant 2k+2$. These identifications and Diagram (I) fit in a sequence of Salomonsen [**13**] yielding the following exact sequence:

$$\text{(II)} \longrightarrow \Omega_{n-k}(X \times BO(q) \times P^{\infty}, \Gamma_k) \longrightarrow I_n(X) \xrightarrow{\mathcal{F}} \eta_n(X) \\ \xrightarrow{\tilde{\gamma}_{k-1}} \Omega_{n-k-1}(X \times BO(q) \times P^{\infty}, \Gamma_{k-1}) \longrightarrow \cdots,$$

where

$$\Gamma_k = \nu_X^p \times \gamma^q \oplus (\varepsilon^{q-n+k} - \gamma^q) \otimes \lambda - \varepsilon^{p+q-n+k}$$

and λ is the canonical bundle over the real projective space P^{∞}.

Next we take ψ a virtual vector bundle over M and suppose that $5 \leqslant n < 2k$. Then from the exact sequence of Salomonsen [**13**], we have the following exact sequence:

$$\text{(III)} \longrightarrow \Omega_n(\tilde{V}_k(\psi), \tau_M - \varepsilon^n) \xrightarrow{\pi_{M*}} \Omega_n(M, \tau_M - \varepsilon^n) \xrightarrow{\gamma_M} \Omega_{n-k-1}(M \times P^{\infty}, \Phi) \longrightarrow \cdots,$$

where $\Phi = -(n-k-1)\lambda - \lambda \otimes \psi + \tau_M - \varepsilon^n$ and γ_M is defined in the construction of the sequence (see Theorem 6.1 in [**13**]).

We recall that if $\psi = h^*\tau_X - \varepsilon^k \oplus \tau_M$, where $h : M \to X$ is a continuous map, $5 \leqslant n < 2k$, then $\gamma_M([M])$ is the invariant $\omega_k(\nu_h)$ defined by Koschorke [**10, 11**], which is an obstruction to the existence of a monomorphism from $M \times \mathbb{R}^{\ell}$ into ν_h. With this notation, h is homotopic to an immersion if and only if $\gamma_M([M]) = 0$.

Here, $[M] = [M, 1_M, t_M] \in \Omega_n(M, \tau_M - \varepsilon^n)$ is the fundamental class of M, $t_M : \tau_M \oplus \varepsilon^n \to \varepsilon^n \oplus \tau_M$ being the isomorphism which interchanges factors.

3. Proofs of Theorems A and B

Proof of Theorem A. Let $h : M \to X$ be a continuous map from a closed connected smooth n-dimensional manifold M into a smooth connected $(n+k)$-dimensional manifold X.

Let us now consider the following commutative diagram, where the left-hand vertical sequence is (III) with $\psi = h^*\tau_X - \varepsilon^k \oplus \tau_M$, the right-hand vertical sequence is (II) and $(h,g)_*$ and $((h,g) \times \mathrm{Id})_*$ are induced maps of (h,g) in convenient normal bordism groups:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\Omega_n(M, \tau_M - \varepsilon^n) & \xrightarrow{(h,g)_*} & \eta_n(X) \\
\downarrow{\scriptstyle \gamma_M} & & \downarrow{\scriptstyle \tilde{\gamma}_{k-1}} \\
\Omega_{n-k-1}(M \times P^{\infty}, \Phi) & \xrightarrow{((h,g)\times \mathrm{Id})_*} & \Omega_{n-k-1}(X \times BO(q) \times P^{\infty}, \Gamma_{k-1}) \\
\downarrow & & \downarrow
\end{array}$$

Suppose that h is bordant to an immersion. Then

$$0 = \tilde{\gamma}_{k-1}([M,h]) = ((h,g) \times \mathrm{Id})_*(\gamma_M([M])).$$

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Since, by assumption,

$$(h,g)_*: H_i(M,\mathbb{Z}_2) \to H_i(X \times BO, \mathbb{Z}_2)$$

is an isomorphism for $i < n-k$ and an epimorphism for $i = n-k$, we conclude that $((h,g)\times \mathrm{Id})_*$ is a $\mathcal{C}$-isomorphism for $i = n-k-1$ and then $\ker((h,g)\times \mathrm{Id})_* \in \mathcal{C}$.

We recall that the order of the elements of the image of γ_M is a power of 2 [**9, 13**]. Therefore, $\gamma_M([M,h]) = 0$ and h is homotopic to an immersion [**10**]. □

Proof of Theorem B. We recall that under the hypotheses of Theorem B,

$$f_*: \Omega_n(M, f^*\tau_N - \varepsilon^n) \to \Omega_n(N, \tau_N - \varepsilon^n)$$

is a $\mathcal{C}$-isomorphism and $f^*(\beta_2) = \alpha_2$, where $\alpha = \nu_M$, and $\beta = \nu_N$ are the stable normal bundles of M and N, and α_2 and β_2 are the respective 2-localization [**2**].

Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\Omega_n(\tilde{V}_k(\psi'_M), f^*\tau_N - \varepsilon^n) & \xrightarrow{G_*} & \Omega_n(\tilde{V}_k(\psi_N), \tau_N - \varepsilon^n) \\
\downarrow{\scriptstyle (\pi'_M)_*} & & \downarrow{\scriptstyle (\pi_N)_*} \\
\Omega_n(M, f^*\tau_N - \varepsilon^n) & \xrightarrow{f_*} & \Omega_n(N, \tau_N - \varepsilon^n) \\
\downarrow{\scriptstyle \gamma'_M} & & \downarrow{\scriptstyle \gamma_N} \\
\Omega_{n-k-1}(M \times P^\infty, f^*(\phi_N)) & \xrightarrow{F_*} & \Omega_{n-k-1}(N \times P^\infty, \phi_N)
\end{array}$$

where the right-hand sequence is obtained from (III), $\psi_N = \varepsilon^{n+k} - \tau_N \oplus \varepsilon^k$, $\psi'_M = \varepsilon^{n+k} - f^*\tau_N \oplus \varepsilon^k$. The left-hand sequence is induced from the right-hand sequence by f and by G and F, which are induced by f and are given in [**13**].

We observe that $(\pi'_M)_*$ is the induced map of π_M in normal bordism groups with virtual bundle $f^*\tau_N - \varepsilon^n$.

If N immerses in $\mathbb{R}^{n+k}$, then $(\pi_N)_*$ is surjective [**13**] and, since $f_*: H_i(M,\mathbb{Z}_2) \to H_i(N,\mathbb{Z}_2)$ is an isomorphism for $i \geqslant 0$, F_* is a $\mathcal{C}$-monomorphism. Therefore, $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism and since the order of every element of the image of γ'_M is a power of 2 [**13**], we conclude that $(\pi'_M)_*$ is an epimorphism.

Now, we only to need to show that $(\pi_M)_*: \Omega_n(\tilde{V}_k(\psi_M), \tau_M - \varepsilon^n) \to \Omega_n(M, \tau_M - \varepsilon^n)$ is a $\mathcal{C}$-epimorphism, where $\psi_M = \varepsilon^{n+k} - \tau_M \oplus \varepsilon^k$. For this, we consider the commutative diagram

$$\begin{array}{ccc}
\pi^s_{n+p}(T\hat{\alpha}) & \longrightarrow & \pi^s_{n+p}(Tf^*(\hat{\beta})) \\
\downarrow{\scriptstyle (\pi_M)_*} & & \downarrow{\scriptstyle (\pi'_M)_*} \\
\pi^s_{n+p}(T\alpha) & \longrightarrow & \pi^s_{n+p}(Tf^*\beta)
\end{array}$$

where $\hat{\beta}$ and $\hat{\alpha}$ denote the pull-back of β and α by π_N and π_M, respectively. The two horizontal maps are $\mathcal{C}$-isomorphisms [**2**] and $(\pi_M)_*$ is a $\mathcal{C}$-epimorphism. □

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4. Applications

Let M and N be closed smooth manifolds of dimension n and $(n+k)$, respectively, and let $f : M \to N$ be a continuous map. Define $U_f \in H^k(N, \mathbb{Z}_2)$ to be the image of the fundamental class $[M] \in H_n(M, \mathbb{Z}_2)$ by the composite map

$$H_n(M, \mathbb{Z}_2) \xrightarrow{f_*} H_n(N, \mathbb{Z}_2) \xrightarrow{D_N^{-1}} H^k(N, \mathbb{Z}_2),$$

where D_N denotes the Poincaré duality isomorphism.

We also consider the following commutative diagram:

$$\begin{array}{ccc} H^p(N, \mathbb{Z}_2) & \xrightarrow{\cup U_f} & H^{p+k}(N, \mathbb{Z}_2) \\ \downarrow{\scriptstyle D_M \circ f^*} & & \downarrow{\scriptstyle D_N} \\ H_{n-p}(M, \mathbb{Z}_2) & \xrightarrow{f_*} & H_{n-p}(N, \mathbb{Z}_2) \end{array}$$

where '$\cup$' denotes the cup product.

Theorem 4.1. *Let M and N be closed smooth manifolds of dimension n. Suppose that*

$$H_i(M, \mathbb{Z}_2) \simeq H_i(N, \mathbb{Z}_2), \quad \text{for all } i \geqslant 0,$$

and there exists $f : M \to N$ with $\deg_2 f = 1$. Then $f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is an isomorphism, for $i \geqslant 0$.

Proof. Since the dimension of M and of N is n, we have that $U_f \in H^0(N, \mathbb{Z}_2)$ and $U_f = \deg_2 f$.

Therefore, $\cup U_f$ is a multiple of $\deg_2 f = 1$, so that

$$\cup U_f : H^p(N, \mathbb{Z}_2) \to H^p(N, \mathbb{Z}_2) \text{ is the identity map}$$

for $p \geqslant 0$ and

$$f_* : H_{n-p}(M, \mathbb{Z}_2) \to H_{n-p}(N, \mathbb{Z}_2) \text{ is onto}$$

for all $p \geqslant 0$. But $H_i(M, \mathbb{Z}_2) \simeq H_i(N, \mathbb{Z}_2)$, $i \geqslant 0$, and the result follows. □

Corollary 4.2. *Let M and N be closed smooth n-manifolds with isomorphic homology groups. Suppose that there exists $f : M \to N$ with $\deg_2 f = 1$. Then M immerses in $\mathbb{R}^{n+k}$, $5 \leqslant n < 2k$, if and only if N does.*

Let M and N be closed smooth n-manifolds. Given $x_0 \in M^n$ and $y_0 \in N^n$, let us take D_1^n and D_2^n discs containing x_0 and y_0, respectively, for which there exists a homeomorphism $h : D_1^n \to D_2^n$ with $h(x_0) = y_0$.

Put $A = \partial D_1$, $M_{n-1} = M^{(n-1)} \cup A$, where $M^{(n-1)}$ is the $(n-1)$-skeleton of M, $Y = N - h(\mathring{D}_1)$, $f_0 = h|_A$, and let

$$\chi_n^{n-1} : H^n(M, A, \pi_{n-1}(Y)) \to H^n(M, A, H_{n-1}(Y))$$

be the homomorphism induced in cohomology by the Hurewicz homomorphism.

Let us suppose that f_0 extends to M_{n-1}, Y is $(n-1)$-simple and $H_{n-1}(A,\mathbb{Z})$ is a free group.

Theorem 4.3. *Suppose that M^n and N^n are such that $H_*(M,\mathbb{Z}_2) \simeq H_*(N,\mathbb{Z}_2)$. If χ_n^{n-1} is a monomorphism and there exists a homomorphism $\psi : H_n(M,\mathbb{Z}) \to H_n(N,\mathbb{Z})$ such that $(f_0)_* = \psi \circ i_*$, with $i_* : H_n(A,\mathbb{Z}) \to H_n(M,\mathbb{Z})$ induced by the inclusion, then there exists $f : M \to N$ with $\deg_2 f = 1$.*

Proof. Under these conditions, f_0 extends to $f : M \to N$ (see [**1**]) with $f(M - \mathring{D}_1) = N - f(\mathring{D}_1)$. By excision, $H_n(M,\mathbb{Z}_2)$ (respectively, $H_n(N,\mathbb{Z}_2)$) is isomorphic to $H_n(M, M - x_0, \mathbb{Z}_2)$ (respectively, $H_n(N, N - y_0, \mathbb{Z}_2)$), which is isomorphic to $H_n(D_1, D_1 - x_0, \mathbb{Z}_2)$ (respectively, $H_n(f(D_1), f(D_1) - y_0, \mathbb{Z}_2)$) and the result follows. □

We finish with some examples which illustrate Theorem A. In these examples, we are supposing that $h : M^n \to X^{n+k}$ is bordant to an immersion.

Example 4.4. Let us consider $n \geqslant 5$ and $k = n - 2$. In order for

$$(h,g)^* : H^1(X,\mathbb{Z}_2) \oplus H^1(BO,\mathbb{Z}_2) \to H^1(M,\mathbb{Z}_2)$$

to be an isomorphism, one needs to take M such that $w_1(M) \neq 0$, because otherwise $(h,g)^*(w_1(X) + w_1(\gamma)) = 0$. For example, $M^n = P^n$, n even, and $H^1(X,\mathbb{Z}_2) = 0$.

Example 4.5. If $n \geqslant 7$ and $k = n - 3$, we take M^n as the real Grassmannian manifold $G_{l+2,2}$ with $l > 3$ and X sufficiently highly connected that $H^i(X \times BO, \mathbb{Z}_2) = H^i(BO,\mathbb{Z}_2)$. Then, by [**12**], $H^i(BO,\mathbb{Z}_2) \to H^i(G_{l+2,2},\mathbb{Z}_2)$ is an isomorphism for $i \leqslant 3$.

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