MONOGENIC ENDOMORPHISMS OF A FREE MONOID

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1. Introduction and summary. Free monoids play a central role in the theory of formal languages. Their endomorphisms appear naturally in the context of deterministic OL-schemes which trace their origin to biology. Closely related to such a scheme is a DOL-system which consists of a triple (X, ϕ, w) where X is a finite set, ϕ is an endomorphism of the free monoid X^* and $w \in X$. The associated language is defined as the set $\{w, \phi w, \phi^2 w, \ldots\}$ called a DOL-language. For a full discussion of this subject, we recommend the book **[2]** by Herman and Rozenberg.

The monoid of endomorphisms of the free monoid X^* on an arbitrary alphabet X has a certain interest in its own right. It ought to have a structure which bears some resemblance to the monoid of all transformations on a set, or the monoid of all linear transformations on a vector space. The underlying spaces in these two cases are: (1) a set without further structure, hence simpler than X^*, and (2) a vector space, hence a structure richer than a free monoid on a set. We may thus expect that the endomorphism monoid of X^* harbours interesting structural complexity.

Semigroups of transformations, partial transformations, partial one-to-one transformations, linear transformations on a vector space, binary relations on a set and numerous others have a densely embedded ideal which is a completely O-simple semigroup (except for the first one in which it is a left zero semigroup if the functions are written on the left), see **[3]**. An ideal I of a semigroup S is densely embedded if (1) the only congruence on S whose restriction to I is equality is the equality relation on S and (2) S is maximal with this property relative to I under set theoretical inclusion.

We call an endomorphism σ of X^* monogenic if its range is contained in a monogenic submonoid of X^*. The monogenic endomorphisms form a semigroup $\mathfrak{M}$ with many remarkable properties.

We study the structure of $\mathfrak{M}$ as well as its position in the monoid of all transformations $\mathcal{T}(X^*)$ on X^* (functions written on the left). Section 2 contains a construction of a Rees matrix semigroup S over the multiplicative semigroup of positive integers. It is then proved that this Rees matrix semigroup is isomorphic to $\mathfrak{M}$, thereby providing $\mathfrak{M}$ with a Rees matrix representation. Hence this case bears strong similarity with the instances mentioned above with the notable difference that we now have a Rees matrix semigroup over a semigroup which is not a group. The left and the right idealizers of $\mathfrak{M}$ in $\mathcal{T}(X^*)$ are identified in Section 4. The elements of the right idealizers are particularly interesting; they are called here generalized endomorphisms and are further investigated in Section 5. There is a curious phenomenon here of duality between $\mathcal{T}(X^*)$ and $\mathbb{N}^X$, the free commutative monoid on X. It is proved in Section 6 that $\mathfrak{M}$ is a densely embedded ideal of its idealizer in $\mathcal{T}(X^*)$ which means that the isomorphism of S onto $\mathfrak{M}$ is a dense embedding of the Rees matrix semigroup S into $\mathcal{T}(X^*)$.

2. The basic construction. *We fix a nonempty set X throughout the paper.* By X^* denote *the free monoid on* X, that is the set of all words over the alphabet X with concatenation as product. The empty word 1 is the identity of X^*. The semigroup X^+ consisting of nonempty words over X is *the free semigroup on* X. A word $w \in X^*$ is *primitive* if $w = u^n$ for any $u \in X^*$ implies $n = 1$, and thus $w = u$. Denote by $\mathcal{P}$ the set of all primitive words in X^* (or over X). For $w \in X^*$ and $x \in X$, let w_x denote the number of occurrences of x in w; let $\bar{w}$ be the X-tuple of nonnegative integers w_x. The mapping

$$\xi : w \rightarrow \bar{w} = (w_x) \qquad (w \in X^*)$$

is called the *Parikh mapping*. Clearly $\overline{uv} = \bar{u} + \bar{v}$ for all $u, v \in X^*$.

For any X-tuple $q = (q_x)$ of nonnegative integers and $w \in X^*$, define their *dot product* by

$$q \,.\, \bar{w} = \sum_{x \in X} q_x w_x.$$

Note that this sum is finite since $\bar{w}$ has only a finite number of nonzero entries. For X-tuples of nonnegative integers, p, q and $u, v \in X^*$, one easily verifies that the following relations hold:

$$(p+q) \,.\, \bar{u} = p \,.\, \bar{u} + q \,.\, \bar{u}, \tag{1}$$

$$q \,.\, \overline{uv} = q \,.\, (\bar{u} + \bar{v}) = q \,.\, \bar{u} + q \,.\, \bar{v},$$

$$p \,.\, \bar{u} = q \,.\, \bar{u} \text{ for all } u \in X^* \text{ implies } p = q, \tag{2}$$

$$q \,.\, \bar{u} = q \,.\, \bar{v} \text{ for all } q \text{ implies } \bar{u} = \bar{v}, \tag{3}$$

when the sum of X-tuples is by components.

Let $\mathcal{Q}$ denote the set of all X-tuples $q = (q_x)$ of nonnegative integers such that $\gcd\{q_x \mid x \in X\} = 1$. We will be interested only in the dot product $q \,.\, \bar{u}$ with $q \in \mathcal{Q}$ and $u \in \mathcal{P}$. All the relations above remain valid with these restrictions. We will use them freely without further reference. Denote by 0 the X-tuple all of whose entries are equal to zero. We also require that $0 \notin \mathcal{Q}$.

Denote by $\mathbb{N}$ the multiplicative semigroup of nonnegative integers and by $\mathbb{N}^+$ its subsemigroup of positive integers. We may now define a Rees matrix semigroup in the usual way

$$S = \mathcal{M}^0(\mathcal{P}, \mathbb{N}^+, Q; (q \,.\, \bar{p}))$$

with index sets $\mathcal{P}$ and $\mathcal{Q}$ over $\mathbb{N}^+$ with sandwich matrix denoted by $(q \,.\, \bar{p})$. Clearly, in every row and in every column $(q \,.\, \bar{p})$ has at least one nonzero entry.

For each nonzero element (p, n, q) of S define a mapping

$$\theta_{(p,n,q)} : w \rightarrow p^{n(q \,.\, \bar{w})} \qquad (w \in X^*)$$

and

$$\theta_0 : w \rightarrow 1 \qquad (w \in X^*).$$

We will generally write functions as left operators. Let $\mathcal{T}(X^*)$ denote the semigroup of all transformations on X^* written and composed as left operators. Denote by $\mathcal{E}$ its subsemigroup of endomorphisms of X^*. For each $w \in X^*$, let w^* stand for the submonoid of X^* generated by w. Finally let

$$\mathfrak{M} = \{\theta \in \mathscr{E} \mid \theta X^* \subseteq w^* \text{ for some } w \in X^*\}$$

and call its elements *monogenic endomorphisms* of X^*. Denote by ζ the trivial endomorphism $\zeta : w \to 1$ for all $w \in X^*$.

Obviously $\mathfrak{M}$ is a subsemigroup of $\mathscr{E}$. It represents the main subject of our study. In the succeeding sections, we will give a Rees matrix representation for it, its translational hull, its left and right idealizer in $\mathscr{T}(X^*)$ and finally, by means of it, construct a dense embedding of S into $\mathscr{T}(X^*)$.

For any $y \in X$, let $1_y = (q_x)$ where

$$q_x = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

For any $w \in X^+$, let $w = (\pi w)^{\varepsilon w}$ where w is the primitive word which raised to the exponent εw gives w. By definition, $\pi 1 = 1$, $\varepsilon 1 = 1$.

3. A Rees representation for $\mathfrak{M}$. The principal result here is the existence of an isomorphism of S onto $\mathfrak{M}$, giving a Rees representation for $\mathfrak{M}$. A few properties of S are then investigated, providing a further clue as to the structure of $\mathfrak{M}$.

THEOREM 3.1. *The mapping*

$$\theta : s \to \theta_s \qquad (s \in S)$$

is an isomorphism of S onto $\mathfrak{M}$.

Proof. First, for $s = (p, n, q) \in S$ and $u, v \in X^*$, we obtain

$$(\theta_s u)(\theta_s v) = p^{n(q\,.\,\bar{u})} p^{n(q\,.\,\bar{v})} = p^{n(q\,.\,\bar{u} + q\,.\,\bar{v})}$$
$$= p^{n(q\,.\,(\bar{u}+\bar{v}))} = p^{n(q\,.\,\overline{uv})} = \theta_s(uv)$$

and thus θ_s is an endomorphism of X^*. Trivially $\theta_0 = \zeta \in \mathfrak{M}$ so $\theta : S \to \mathfrak{M}$.

Next let $\mu \in \mathfrak{M}$ and suppose that $\mu \neq \zeta$. Then $\mu X^* \subseteq w^*$ for some $w \in X^+$. There is a unique primitive word p for which $w = p^t$ for some $t \geq 1$. For each $x \in X$, define t_x by $\mu x = p^{t_x}$. Let $n = \gcd\{t_x \mid x \in X\}$ and let $q_x = \dfrac{t_x}{n}$ for each $x \in X$. Then with $q = (q_x)$, we obtain for any $x \in X$,

$$\mu x = p^{t_x} = p^{nq_x} = p^{n(q\,.\,\bar{x})} = \theta_{(p,n,q)} x.$$

Therefore $\mu = \theta_{(p,n,q)}$ and thus θ maps S onto $\mathfrak{M}$.

Assume next that $\theta_{(p,n,q)} = \theta_{(r,m,t)}$. Applied to any $x \in X$, this gives

$$p^{nq_x} = p^{n(q\,.\,\bar{x})} = r^{m(t\,.\,\bar{x})} = r^{mt_x}.$$

Since both p and r are primitive words, it follows that $p = r$ and $nq_x = mt_x$ for all $x \in X$. Let $g = \gcd\{m, n\}$. Then $n = gd$ with $(d, m) = 1$. Also d divides mt_x, so that d divides t_x. This is true for all $x \in X$, and by hypothesis $\gcd\{t_x \mid x \in X\} = 1$, whence $d = 1$.

Consequently, $n = g$, so that n divides m. By symmetry, we conclude that $m = n$. But then $q_x = t_x$ for all $x \in X$, whence $g = t$. Therefore $(p, n, q) = (r, m, t)$ and θ is one-to-one.

Further, if $t \,.\, \bar{w} \neq 0$, then

$$\begin{aligned}\theta_{(p,n,q)}\theta_{(r,m,t)}w &= \theta_{(p,n,q)}r^{m(t\,.\,\bar{w})} = p^{n(q\,.\,\overline{r(t\,.\,\bar{w})})}\\ &= p^{n(q\,.\,\bar{r})(t\,.\,\bar{w})} = \theta_{(p,n(q\,.\,\bar{r}),t)}w\\ &= \theta_{(p,n,q)(r,m,t)}w;\end{aligned}$$

and if $t \,.\, \bar{w} = 0$, then $\theta_{(p,n,q)}\theta_{(r,m,t)}w = 1 = \theta_{(p,n,q)(r,n,t)}w$. Consequently $\theta_s\theta_{s'} = \theta_{ss'}$ for s, s' nonzero elements of S. This equation can be easily verified to be true if one or both of s and s' is equal to zero. Therefore θ is a homomorphism and thus an isomorphism of S onto $\mathfrak{M}$.

In view of the above theorem, the semigroup S plays a central role in our investigation. It is thus worth having a closer look at its structure. To this end, we characterise Green's relations, idempotents, inverses, regular elements and an embedding into a completely 0-simple semigroup as an order. It will be convenient to introduce the following concepts.

Definition 3.2. A word w in X^* is *monic* if there is a letter x in X which occurs in w only once. An element q of Q is *monic* if some component of q is equal to 1.

Proposition 3.3. *Let $s = (p, n, q)$ and $t = (z, m, r)$ be distinct (nonzero) elements of S.*

(i) *$s\mathcal{L}t \Leftrightarrow p$ and z are monic, $n = m$, $q = r$.*

(ii) *$s\mathcal{R}t \Leftrightarrow p = z$, $n = m$, q and r are monic.*

(iii) $s\mathcal{D}t \Leftrightarrow \begin{cases} \text{either } p \text{ and } z \text{ are monic or } p = z,\\ n = m,\\ \text{either } q \text{ and } r \text{ are monic or } q = r.\end{cases}$

(iv) *$s\mathcal{J}t \Leftrightarrow n = m$, p, z, q and r are monic.*

(v) *$\mathcal{H}$ is the equality relation.*

Proof. (i) Indeed,

$$\begin{aligned}&(p, n, q) = (u, k, x)(z, m, r),\ (z, m, r) = (v, l, y)(p, n, q)\\ &\quad\Leftrightarrow p = u,\ n = k(x\,.\,\bar{z})m,\ q = r,\ z = v,\ m = l(y\,.\,\bar{p})n,\ r = q\\ &\quad\Leftrightarrow p = u,\ n = m,\ k(x\,.\,\bar{z}) = l(y\,.\,\bar{p}) = 1,\ z = v,\ r = q,\end{aligned}$$

whence the assertion concerning $\mathcal{L}$.

(ii) The argument here is dual.

(iii) This follows directly from items (i) and (ii).

(iv) The assertion follows directly from the calculation

$$\begin{aligned}(p, n, q) &= (u, k, x)(z, m, r)(v, l, y)\\ &\Leftrightarrow p = u,\ n = k(x\,.\,\bar{z})m(r\,.\,\bar{v})l,\ q = y.\end{aligned}$$

(v) This is a direct consequence of items (i) and (ii).

Comparing parts (iii) and (iv), we conclude that $\mathcal{D} \neq \mathcal{J}$ in S.

PROPOSITION 3.4. *Let $s = (p, n, q)$ and $t = (z, m, r)$ be elements of S.*

(i) *s is idempotent if and only if $q_x p_x = 1$ for some $x \in X$ and $q_y p_y = 0$ if $y \neq x$ and $n = 1$.*

(ii) *s and t are mutually inverse if and only if $q \,.\, \bar{z} = r \,.\, \bar{p} = 1$ and $n = m = 1$.*

(iii) *s is regular if and only if both p and q are monic and $n = 1$.*

(iv) *$\{(x, 1, 1_y) \mid x, y \in X\} \cup \{0\}$ is a combinatorial Brandt semigroup.*

Proof. Straightforward verification.

Simple verification also shows that S is an order in the completely 0-simple semigroup $\mathcal{M}^0(\mathcal{P}, \mathbb{Q}^+, \mathcal{Q}; (q \,.\, \bar{p}))$ where $\mathbb{Q}^+$ is the multiplicative group of positive rationals. For a full discussion of these concepts, we refer the reader to [**1**].

4. The left and the right idealizer of $\mathfrak{M}$ in $\mathcal{T}(X^*)$. The (*left, right*) *idealizer* of a subsemigroup T of a semigroup S is the greatest subsemigroup $i_S(T)(li_S(T), ri_S(T))$ of S in which T is a (left, right) ideal. For these, we have the following simple expressions:

$$i_S(T) = \{s \in S \mid sT, Ts \subseteq T\},$$
$$li_S(T) = \{s \in S \mid sT \subseteq T\},$$

and analogously for $ri_S(T)$. It follows that $i_S(T) = li_S(T) \cap ri_S(T)$.

We start with the left idealizer. Recall the notation πw and εw from Section 2.

PROPOSITION 4.1.

$$li_{\mathcal{T}(X^*)}(\mathfrak{M}) = \{\sigma \in \mathcal{T}(X^*) \mid \sigma(w^n) = (\sigma w)^n \text{ for all } w \in X^*, n \geq 1\}.$$

Proof. First let $\sigma \in li_{\mathcal{T}(X^*)}(\mathfrak{M})$. Let $w \in X^*$ and $n \geqslant 1$. By the above formula and Theorem 3.1, $\sigma\theta_s$ is an endomorphism of X^* for any $s \in S$. Hence, for any $x \in X$,

$$\begin{aligned} \sigma w^n &= \sigma\theta_{(\pi w, (\varepsilon w)n, 1_x)} x = \sigma\theta_{(\pi w, \varepsilon w, 1_x)} x^n \\ &= (\sigma\theta_{(\pi w, \varepsilon w, 1_x)} x)^n = (\sigma w)^n. \end{aligned}$$

Conversely, let σ be in the set on the right hand side in the statement of the proposition. Then for any $(p, n, q) \in S$ and $w \in X^*$, we obtain

$$\sigma\theta_{(p,n,q)} w = \sigma p^{n(q \,.\, \bar{w})} = (\sigma p)^{n(q \,.\, \bar{w})} = \theta_{(\pi\sigma p, (\varepsilon\sigma p)n, q)} w,$$

which proves that $\sigma\theta_{(p,n,q)} = \theta_{(\pi\sigma p, [\varepsilon(\sigma(p))]n, q)}$. Furthermore, for any $w \in X^*$ and $n \geqslant 1$, we have

$$\sigma\theta_0 w = \sigma\zeta w = \sigma 1 = \sigma 1^n = (\sigma 1)^n,$$

which evidently implies that $\sigma 1 = 1$. Hence $\sigma\theta_0 = \theta_0$. Consequently $\sigma \in li_{\mathcal{T}(X^*)}(\mathfrak{M})$, as required.

We now consider the right idealizer. Recall the definition of the Parikh mapping ξ and the notation 1_y from Section 2.

Denote by $\mathbb{N}^X$ the semigroup of all X-tuples of nonnegative integers with only a finite number of nonzero entries under componentwise addition. For any set Y, denote by $\mathcal{T}'(Y)$ the semigroup of all transformations on Y written and composed as operators on the right. We say that $\sigma \in \mathcal{T}(X^*)$ and $\tau \in \mathcal{T}'(\mathbb{N}^X)$ are *adjoints* of each other if

$$r \,.\, \overline{\sigma w} = r\tau \,.\, \bar{w} \qquad (r \in \mathbb{N}^X,\ w \in X^*).$$

THEOREM 4.2.

$$\begin{aligned} ri_{\mathcal{T}(X^*)}(\mathfrak{M}) &= \{\sigma \in \mathcal{T}(X) \mid \zeta\sigma : X^* \to \mathbb{N}^X \text{ is a homomorphism}\} \\ &= \{\sigma \in \mathcal{T}(X) \mid \sigma \text{ has an adjoint in } \mathbb{N}^X\}. \end{aligned}$$

Proof. Let $\sigma \in ri_{\mathcal{T}(X^*)}(\mathfrak{M})$. Then for $s = (p, n, q) \in S$, we have $\theta_s\sigma \in \mathfrak{M}$ and hence for any $u, v \in X^*$, we get

$$\theta_s\sigma(uv) = (\theta_s\sigma u)(\theta_s\sigma v),$$

so that

$$p^{n(q\,.\,\overline{\sigma(uv)})} = p^{n(q\,.\,\overline{\sigma u})}p^{n(q\,.\,\overline{\sigma v})} = p^{n(q\,.\,(\overline{\sigma u}+\overline{\sigma v}))} = p^{n(q\,.\,\overline{(\sigma u)(\sigma v)})},$$

whence $q \,.\, \overline{\sigma(uv)} = q \,.\, \overline{(\sigma u)(\sigma v)}$. Since this holds for all $q \in \mathcal{Q}$, by (3) we deduce that $\overline{\sigma(uv)} = \overline{(\sigma u)(\sigma v)}$. It follows that

$$\xi\sigma(uv) = \overline{\sigma(uv)} = \overline{(\sigma u)(\sigma v)} = \overline{\sigma u} + \overline{\sigma v} = \xi\sigma u + \xi\sigma v$$

and $\xi\sigma$ is a homomorphism.

Now assume that $\xi\sigma$ is a homomorphism. We define τ on $\mathbb{N}^X$ as follows. For any $x, y \in X$, let $(1_y\tau)_x = \overline{\sigma x}_y$, that is, the xth component of the value of τ at 1_y is equal to the yth component of $\overline{\sigma x}$. This defines τ on the set $\{1_y \mid y \in X\}$. Since $\mathbb{N}^X$ is the free commutative monoid on X, we may extend τ uniquely to an endomorphism of $\mathbb{N}^X$, again denoted by τ. We will now show that τ is an adjoint of σ in $\mathcal{T}'(\mathbb{N}^X)$.

Any $r \in \mathbb{N}^X$ can be written as $r = r_{z_1}1_{z_1} + r_{z_2}1_{z_2} + \ldots + r_{z_n}1_{z_n}$ so that

$$\begin{aligned} r\tau &= r_{z_1}(1_{z_1}\tau) + r_{z_2}(1_{z_2}\tau) + \ldots + r_{z_n}(1_{z_n}\tau) \\ &= r_{z_1}(\overline{\sigma x}_{z_1})_{x\in X} + r_{z_2}(\overline{\sigma x}_{z_2})_{x\in X} + \ldots + r_{z_n}(\overline{\sigma x}_{z_n})_{x\in X} \\ &= (r_{z_1}(\overline{\sigma x})_{z_1} + r_{z_2}(\overline{\sigma x})_{z_2} + \ldots + r_{z_n}(\overline{\sigma x})_{z_n})_{x\in X}, \end{aligned}$$

whence for $w = x_1^{w_{x_1}}x_2^{w_{x_2}}\ldots x_m^{w_{x_m}}$, we get

$$\begin{aligned} r\tau \,.\, \bar{w} &= \sum_{x\in X} (r_{z_1}(\overline{\sigma x})_{z_1} + r_{z_2}(\overline{\sigma x})_{z_2} + \ldots + r_{z_m}(\overline{\sigma x})_{z_n})w_x \\ &= \sum_{i=1}^{m} (r_{z_1}(\overline{\sigma x_i})_{z_1} + r_{z_2}(\overline{\sigma x_i})_{z_2} + \ldots + r_{z_n}(\overline{\sigma x_i})_{z_n})w_{x_i} \\ &= \sum_{j=1}^{n} r_{z_j}((\overline{\sigma x_1})_{z_j}w_{x_1} + (\overline{\sigma x_2})_{z_j}w_{x_2} + \ldots + (\overline{\sigma x_m})_{z_j}w_{x_m}). \end{aligned} \tag{4}$$

On the other hand,

$$r \,.\, \overline{\sigma w} = r_{z_1}(\overline{\sigma w})_{z_1} + r_{z_2}(\overline{\sigma w})_{z_2} + \ldots + r_{z_n}(\overline{\sigma w})_{z_n}. \tag{5}$$

For equality of (4) and (5), it thus suffices to prove that

$$(\overline{\sigma x_1})_{z_j} w_{x_1} + (\overline{\sigma x_2})_{z_j} w_{x_2} + \ldots + (\overline{\sigma x_m})_{z_j} w_{x_m} = (\overline{\sigma w})_{z_j}, \tag{6}$$

for $j = 1, 2, \ldots, n$. Indeed, using the hypothesis that $\xi\sigma$ is a homomorphism, we obtain

$$\begin{aligned}(\overline{\sigma w})_{z_j} &= (\xi\sigma(x_1^{w_{x_1}} x_2^{w_{x_2}} \ldots x_m^{w_{x_m}}))z_j \\ &= (w_{x_1}(\overline{\sigma x_1}) + w_{x_2}(\overline{\sigma x_2}) + \ldots + w_{x_m}(\overline{\sigma x_m}))_{z_j} \\ &= w_{x_1}(\overline{\sigma x_1})_{z_j} + w_{x_2}(\overline{\sigma x_2})_{z_j} + \ldots + w_{x_m}(\overline{\sigma x_m})_{z_j},\end{aligned}$$

$j = 1, 2, \ldots, n$. This proves (6) and hence equality of (4) and (5) follows.

We have shown the equality $r \,.\, \overline{\sigma w} = r\tau \,.\, \bar{w}$ in the special case when $w = x_1^{w_{x_1}} x_2^{w_{x_2}} \ldots x_m^{w_{x_m}}$. However, for any $w \in X^*$, we have $\bar{w} = \bar{u}$ where $u = x_1^{w_{x_1}} x_2^{w_{x_2}} \ldots x_m^{w_{x_m}}$ with $x_1, x_2, \ldots, x_m$ distinct elements of X occurring in w. In fact, $u = x_1^{u_{x_1}} x_2^{u_{x_2}} \ldots x_m^{u_{x_m}}$. Moreover

$$\begin{aligned}\overline{\sigma w} &= \xi\sigma w = w_{x_1}\overline{\sigma x_1} + w_{x_2}\overline{\sigma x_2} + \ldots + w_{x_m}\overline{\sigma x_m} \\ &= \xi\sigma u = \overline{\sigma u}.\end{aligned}$$

We may thus conclude that

$$r \cdot \overline{\sigma w} = r \,.\, \overline{\sigma u} = r\tau \,.\, \bar{u} = r\tau \,.\, \bar{w},$$

that is σ and τ are adjoints.

Finally assume that σ has an adjoint τ in $\mathbb{N}^X$. For any $s = (p, n, q) \in S$ and $u, v \in X^*$, we obtain

$$\begin{aligned}\theta_s\sigma(uv) &= p^{n(q \,.\, \bar{\sigma}(\overline{uv}))} = p^{n(q\tau \,.\, \overline{uv})} = p^{n(q\tau \,.\, (\bar{u}+\bar{v}))} \\ &= p^{n(q\tau \,.\, \bar{u} + q\tau \,.\, \bar{v})} = p^{n(q\tau \,.\, \bar{u})} p^{n(q\tau \,.\, \bar{v})} \\ &= p^{n(q \,.\, \overline{\sigma u})} p^{n(q \,.\, \overline{\sigma v}))} = (\theta_s\sigma u)(\theta_s\sigma v),\end{aligned}$$

which proves that $\theta_s\sigma$ is an endomorphism of X^*. Clearly $(\theta_s\sigma)X^* \subseteq p^*$ and thus $\theta_s\sigma \in \mathfrak{M}$. Trivially $\theta_0\sigma = \theta_0$. Therefore $\sigma \in ri_{\mathcal{T}(X^*)}(\mathfrak{M})$, as required.

5. Generalized endomorphisms. We will now elaborate upon the transformations on X^* which appear in Theorem 4.2. As a motivation, we first prove the following simple result.

PROPOSITION 5.1. *A transformation $\sigma \in \mathcal{T}(X^*)$ has an adjoint in $\mathbb{N}^X$ if and only if $\xi\sigma$ is a homomorphism. If τ is such an adjoint, then τ is an endomorphism of $\mathbb{N}^X$ and is unique. For any $s = (p, n, q) \in S$, the adjoint of θ_s, again denoted by θ_s, is given by*

$$r\theta_s = (r \,.\, \bar{p})nq \qquad (r \in \mathbb{N}^X),$$

and $\theta_0 = 0$.

Proof. The first assertion is part of Theorem 4.2. Let τ be an adjoint of σ, let $r, r' \in \mathbb{N}^X$ and $w \in X^*$. Then, using (1),

$$\begin{aligned}(r + r')\tau \cdot \bar{w} &= (r + r') \,.\, \overline{\sigma w} = r \cdot \overline{\sigma w} + r' \cdot \overline{\sigma w} \\ &= r\tau \,.\, \bar{w} + r'\tau \,.\, \bar{w} = (r\tau + r'\tau) \,.\, \bar{w};\end{aligned}$$

since w is arbitrary, (2) implies that $(r+r')\tau = r\tau + r'\tau$. Therefore τ is an endomorphism of $\mathbb{N}^X$.

Let τ' be also an adjoint of σ. Then for any $r \in \mathbb{N}^X$ and $w \in X^*$, we get

$$r\tau \,.\, \bar{w} = r \,.\, \overline{\sigma w} = r\tau' \,.\, \bar{w}.$$

Again, w being arbitrary, (2) yields $r\tau = r\tau'$. Consequently $\tau = \tau'$, establishing uniqueness.

With the notation in the statement of the proposition and $r \in \mathbb{N}^X$, $w \in X^*$, we have

$$\begin{aligned} r\theta_s \,.\, \bar{w} &= (r \,.\, \bar{p})nq \,.\, \bar{w} = (r \,.\, \bar{p})n(q \,.\, \bar{w}) = r \,.\, \bar{p}n(q \,.\, \bar{w}) \\ &= r \,.\, \overline{p^{n(q \,.\, \bar{w})}} = r \,.\, \overline{\theta_s w}; \end{aligned}$$

this holds trivially for $s = 0$.

We now consider the dual situation: which transformations on $\mathbb{N}^X$ have an adjoint in X^*? This is answered in the theorem below. It will be convenient to first prove an auxiliary statement of some independent interest.

LEMMA 5.2. *Let δ be a homomorphism of X^* into $\mathbb{N}^X$. For each $x \in X$, let $\sigma x = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ if $\delta x = r_1 1_{x_1} + r_2 1_{x_2} + \ldots + r_n 1_{x_n}$ (in some ordering of x_i's). Extend σ to an endomorphism of X^*, again denoted by σ. Then $\xi\sigma = \delta$.*

Proof. With the notation introduced, we have

$$\xi\sigma x = \xi(x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}) = r_1 1_{x_1} + r_2 1_{x_2} + \ldots + r_n 1_{x_n} = \delta x,$$

so that for $w = y_1 y_2 \ldots y_m$ with $y_1, y_2, \ldots, y_m \in X$, we obtain

$$\begin{aligned} \xi\sigma w &= \xi((\sigma y_1)(\sigma y_2) \ldots (\sigma y_m)) \\ &= (\xi\sigma)y_1 + (\xi\sigma)y_2 + \ldots + (\xi\sigma)y_m \\ &= \delta y_1 + \delta y_2 + \ldots + \delta y_m = \delta w. \end{aligned}$$

Therefore $\xi\sigma = \delta$, as asserted.

The claim of the lemma can be expressed by saying that every homomorphism of X^* into $\mathbb{N}^X$ can be "lifted" by ξ to an endomorphism of X^*.

THEOREM 5.3. *A transformation $\tau \in \mathcal{T}'(\mathbb{N}^X)$ has an adjoint in X^* if and only if τ is an endomorphism of $\mathbb{N}^X$. If σ is such an adjoint, then $\xi\sigma$ is a homomorphism and is unique; in addition, τ has an adjoint which is an endomorphism of X^*.*

Proof. If a transformation $\tau \in \mathcal{T}'(\mathbb{N}^X)$ has an adjoint σ in X^*, then by Proposition 5.1, τ is an endomorphism of $\mathbb{N}^X$. Conversely, assume that τ is an endomorphism of $\mathbb{N}^X$. We first define a function δ mapping X^* into $\mathbb{N}^X$ as follows. For any $x, y \in X$, let $(\delta y)_x = (1_x\tau)_y$, that is, the xth component of the value of δ at y is equal to the yth component of the value of τ at 1_x. This defines δ on the set X. Now extend δ to a homomorphism from X^* into $\mathbb{N}^X$, again denoted by δ. We show next that

$$r \,.\, \delta w = r\tau \,.\, \bar{w} \qquad (r \in \mathbb{N}^X,\ w \in X^*). \tag{7}$$

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For $w = y_1 y_2 \ldots y_n$, we get for any $x \in X$,

$$(\delta w)_x = (\delta y_1 + \delta y_2 + \ldots + \delta y_n) = (\delta y_1)_x + (\delta y_2)_x + \ldots + (\delta y_n)_x$$
$$= (1_x\tau)_{y_1} + (1_x\tau)_{y_2} + \ldots + (1_x\tau)_{y_n}.$$

For $r \in \mathbb{N}^X$, say $r_{x_1}1_{x_1} + r_{x_2}1_{x_2} + \ldots + r_{x_m}1_{x_m}$, we then obtain on the one hand,

$$\begin{aligned} r \,.\, \delta w &= (r_{x_1}1_{x_1} + r_{x_2}1_{x_2} + \ldots + r_{x_m}1_{x_m})((\delta w)_{x_1}1_{x_1} + (\delta w)_{x_2}1_{x_2} + \ldots + (\delta w)_{x_m}1_{x_m}) \\ &= r_{x_1}[(1_{x_1}\tau)_{y_1} + (1_{x_1}\tau)_{y_2} + \ldots + (1_{x_1}\tau)y_n] \\ &\quad + r_{x_2}[(1_{x_2}\tau)_{y_1} + (1_{x_2}\tau)_{y_2} + \ldots + (1_{x_2}\tau)y_n] \\ &\quad + \ldots \\ &\quad + r_{x_m}[(1_{x_m}\tau)_{y_1} + (1_{x_m}\tau)_{y_2} + \ldots + (1_{x_m}\tau)_{y_n}] \\ &= r_{x_1}(1_{x_1}\tau)_{y_1} + r_{x_2}(1_{x_2}\tau)_{y_1} + \ldots + r_{x_m}(1_{x_m}\tau)_{y_1} \\ &\quad + r_{x_1}(1_{x_1}\tau)_{y_2} + r_{x_2}(1_{x_2}\tau)_{y_2} + \ldots + r_{x_m}(1_{x_m}\tau)_{y_2} \\ &\quad + \ldots \\ &\quad + r_{x_1}(1_{x_1}\tau)_{y_n} + r_{x_2}(1_{x_2}\tau)_{y_n} + \ldots + r_{x_m}(1_{x_m}\tau)_{y_n}, \end{aligned} \tag{8}$$

and on the other hand,

$$\begin{aligned} r\tau \,.\, \bar{w} &= (r_{x_1}1_{x_1} + r_{x_2}1_{x_2} + \ldots + r_{x_m}1_{x_m})\tau \,.\, \overline{y_1 y_2 \ldots y_n} \\ &= [r_{x_1}(1_{x_1}\tau) + r_{x_2}(1_{x_2}\tau) + \ldots + r_{x_m}(1_{x_m}\tau)] \,.\, (1_{y_1} + 1_{y_2} + \ldots + 1_{y_n}) \\ &= (r_{x_1}(1_{x_1}\tau) + r_{x_2}(1_{x_2}\tau) + \ldots + r_{x_m}(1_{x_m}\tau))_{y_1} \\ &\quad + ((r_{x_1}(1_{x_1}\tau) + r_{x_2}(1_{x_2}\tau) + \ldots + r_{x_m}(1_{x_m}\tau))_{y_2} \\ &\quad + \ldots \\ &\quad + (r_{x_1}(1_{x_1}\tau) + r_{x_2}(1_{x_2}\tau) + \ldots + r_{x_m}(1_{x_m}\tau))_{y_n}. \end{aligned} \tag{9}$$

We now see that expressions (8) and (9) are equal, which then proves relation (7).

Since δ is a homomorphism of X^* into $\mathbb{N}^X$, Lemma 5.2 yields that for some endomorphism σ of X^* we have $\xi\sigma = \delta$. Substitution in (7) gives $r \,.\, \overline{\sigma w} = r\tau \,.\, \bar{w}$ for all $r \in \mathbb{N}^X$ and $w \in X^*$, which shows that σ is an adjoint of τ. This establishes the converse part of the first assertion and the last assertion of the theorem.

Now let σ be any adjoint of τ. Then for any $u, v \in X^*$ and $r \in \mathbb{N}^X$, we obtain

$$\begin{aligned} r \,.\, \overline{\sigma(uv)} &= r\tau \,.\, \overline{uv} = r\tau \,.\, (\bar{u} + \bar{v}) = r\tau \,.\, \bar{u} + r\tau \,.\, \bar{v} \\ &= r \,.\, \overline{\sigma u} + r \,.\, \overline{\sigma v} = r \,.\, (\overline{\sigma u} + \overline{\sigma v}). \end{aligned}$$

Since r is arbitrary, (3) implies that $\overline{\sigma(uv)} = \overline{\sigma u} + \overline{\sigma v}$ and $\xi\sigma$ is indeed a homomorphism.

Assume, in addition, that σ' is also an adjoint of τ. Then for any $r \in \mathbb{N}^X$ and $w \in X^*$, we have

$$r \,.\, \overline{\sigma w} = r\tau \,.\, \bar{w} = r \,.\, \overline{\sigma' w},$$

so again by (3), we conclude that $\overline{\sigma w} = \overline{\sigma' w}$. It follows that $\xi\sigma$ is unique.

Motivated by the above theorem, we introduce the following concept.

DEFINITION 5.4. A transformation σ of X^* is a *generalized endomorphism* if $\xi\sigma|$ is a homomorphism.

In the notation employed above, this means that $\overline{\sigma(uv)} = \overline{(\sigma u)(\sigma v)}$ for all $u, v \in X^*$. According to Proposition 5.1, $\sigma \in \mathcal{T}(X^*)$ is a generalized endomorphism if and only if it has an adjoint in $\mathbb{N}^X$. Generalized endomorphisms of X^* may be constructed in a similar fashion, as the following simple result shows.

PROPOSITION 5.5. *Let $\sigma: X \to X^*$ be any mapping. For $w = x_1x_2 \ldots x_n \in X^*$ let σw be any word in X^* satisfying*

$$\overline{\sigma w} = \overline{(\sigma x_1)(\sigma x_2) \ldots (\sigma x_n)}.$$

Then σ is a generalized endomorphism of X^*. Also, every generalized endomorphism of X^* can be so constructed.

Proof. Let $u = x_1x_2 \ldots x_m$ and $v = y_1y_2 \ldots y_n$ be words in X^*. Then

$$\begin{aligned}\overline{\sigma u} + \overline{\sigma v} &= \overline{(\sigma x_1)(\sigma x_2) \ldots (\sigma x_m)} + \overline{(\sigma y_1)(\sigma y_2) \ldots (\sigma y_n)} \\ &= \overline{\sigma x_1} + \overline{\sigma x_2} + \ldots + \overline{\sigma x_m} + \overline{\sigma y_1} + \overline{\sigma y_2} + \ldots + \overline{\sigma y_n} \\ &= \overline{(\sigma x_1)(\sigma x_2) \ldots (\sigma x_m)(\sigma y_1)(\sigma y_2) \ldots (\sigma y_n)} \\ &= \overline{(\sigma(uv)},\end{aligned}$$

as required.

Conversely, let σ be a generalized endomorphism of X^*. Then for all $w = x_1x_2 \ldots x_m$ in X^*, we have

$$\overline{\sigma w} = \overline{\sigma(x_1x_2 \ldots x_n)} = \overline{(\sigma x_1)(\sigma x_2) \ldots (\sigma x_m)},$$

as required.

It should now be clear how much generalized endomorphisms are indeed more general than endomorphisms. Nevertheless, we have the following statement.

PROPOSITION 5.6. *Let σ be a generalized endomorphism of X^* such that $\sigma X^* \subseteq w^*$ for some $w \in X^*$. Then σ is an endomorphism.*

Proof. If $w = 1$, the assertion is trivial. Assume that $w \neq 1$. For every $x \in X$, we have $\sigma x = w^{r_x}$ for a positive integer r_x. Hence

$$\begin{aligned}\overline{\sigma(x_1x_2 \ldots x_n)} &= \overline{\sigma x_1} + \overline{\sigma x_2} + \ldots + \overline{\sigma x_n} \\ &= r_{x_1}\bar{w} + r_{x_2}\bar{w} + \ldots + r_{x_n}\bar{w} \\ &= (r_{x_1} + r_{x_2} + \ldots + r_{x_n})\bar{w},\end{aligned}$$

so that $\sigma(x_1x_2 \ldots x_n) = w^{r_{x_1}+r_{x_2}+\cdots+r_{x_n}}$. But this implies that

$$\begin{aligned}\sigma(x_1 \ldots x_n y_1 \ldots y_m) &= w^{r_{x_1}+\cdots+r_{x_n}+r_{y_1}+\cdots+r_{y_m}} \\ &= w^{r_{x_1}+\cdots+r_{x_m}} w^{r_{y_1}+\cdots+r_{y_m}} \\ &= \sigma(x_1 \ldots x_n)\sigma(y_1 \ldots y_m),\end{aligned}$$

which gives $\sigma(uv) = (\sigma u)(\sigma v)$ for all $u, v \in X^*$. Therefore σ is an endomorphism.

The following examples illustrate the nature of some of the transformations studied.

EXAMPLE 5.7. Let $|X|>1$ and fix $a \in X^+$. For every $w \in X^*$, let $\sigma w = a^{\varepsilon w}$ where εw was defined in Section 2. For any $w \in X^*$ and $n \geqslant 1$, we obtain

$$\sigma w^n = a^{\varepsilon w^n} = a^{n(\varepsilon w)} = (a^{\varepsilon w})^n = (\sigma w)^n.$$

For any $x, y \in X$, $x \neq y$, we further get

$$\sigma(xy) = a \neq a^2 = (\sigma x)(\sigma y)$$

and σ is not an endomorphism. Hence Proposition 5.7 is not valid under the hypothesis that $\sigma w^n = (\sigma w)^n$ for all $w \in X^*$ and all $n \geqslant 1$ instead of σ being a generalized endomorphism.

EXAMPLE 5.8. Let $X = \{a, b\}$ and define σ by:

$$\sigma w = w \quad \text{if} \quad w \notin (ab)^* \qquad (w \in X^*),$$
$$\sigma(ab)^n = (ba)^n \qquad (n \geqslant 1).$$

For $m, n \geqslant 1$, we get $\sigma w^n = (\sigma w)^n$ trivially if $w \notin (ab)^*$ and

$$\sigma((ab)^m)^n = \sigma(ab)^{mn} = (ba)^{mn} = ((ba)^m)^n = (\sigma(ab)^m)^n.$$

Therefore $\sigma w^n = (\sigma w)^n$ for all $w \in X^*$, $n \geqslant 1$.

Next let $u, v \in X^*$. Then for $x \in X$,

$$(\sigma(uv))_x = (uv)_x = u_x + v_x = (\sigma u)_x + (\sigma v)_x,$$

which implies that $\overline{\sigma(uv)} = \overline{\sigma u} + \overline{\sigma v}$ and σ is a generalized endomorphism. However,

$$\sigma(ab) = ba = (\sigma b)(\sigma a)$$

and σ is not an endomorphism. This shows that $\mathscr{E}$ is properly contained in $i_{\mathscr{T}(X^*)}(\mathfrak{M})$.

6. A dense embedding. Let I be an ideal of a semigroup S. We say that S is an *ideal extension* of I. If the equality relation on S is the only congruence on S whose restriction to I is the equality on I, then S is a *dense* (*ideal*) *extension* of I. If, in addition, S is, under inclusion, a maximal dense extension of I, then I is a *densely embedded ideal* of I.

An isomorphism ϕ of a semigroup S into a semigroup T is a *dense embedding* if $S\phi$ is a densely embedded ideal of its idealizer in T.

Our aim here is to prove that the mapping θ in Theorem 3.1, considered as an isomorphism of S into $\mathscr{T}(X^*)$, is a dense embedding. In order to achieve this goal, we will make use of the following useful tool.

Let S be a semigroup. A transformation λ (respectively ρ) written on the left (respectively right) is a *left* (respectively *right*) *translation* of S if $\lambda(xy) = (\lambda x)y$ (respectively $(xy)\rho = x(y\rho)$) for all $x, y \in S$; the two translations are *linked* if $(x\rho)y = x(\lambda y)$ for all $x, y \in S$ in which case (λ, p) is a *bitranslation*. The set $\Omega(S)$ of all bitranslations of S under the componentwise multiplication is the *translational hull* of S.

For any $a \in S$, let $\lambda_a x = ax$ and $x\rho_a = xa$ for all $x \in S$. Then $\pi_a = (\lambda_a, \rho_a)$ is an *inner bitranslation* of S. The mapping $\pi: a \to \pi_a$ $(a \in S)$ is the *canonical homomorphism* of S into $\Omega(S)$ with image $\Pi(S)$. Finally, S is *weakly reductive* if π is one-to-one, that is, for any $a, b \in S$, $ax = bx$ and $xa = xb$ for all $x \in S$ implies that $a = b$. The proof of the above stated goal is based on the following well-known

RESULT 6.1. *If a semigroup S is weakly reductive, then $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$.*

Since in this case π is an isomorphism of S onto $\Pi(S)$, the proof will be effected by constructing an isomorphism of the idealizer of $S\phi$ onto $\Omega(S)$ which maps $S\phi$ onto $\Pi(S)$. For a complete discussion concerning these concepts, consult [**4**, Chapter II].

As the first part of our program, we prove

LEMMA 6.2. *The semigroup S is weakly reductive.*

Proof. Let (p, n, q), (z, m, r) be nonzero elements of S and assume that

$$(u, k, x)(p, n, q) = (u, k, x)(z, m, r),$$
$$(p, n, q)(u, k, x) = (z, m, r)(u, k, x)$$

for all $(u, k, x) \in S$. We can choose $x \in \mathcal{Q}$ such that $x \,.\, \bar{p} \neq 0$ which gives $k(x \,.\, \bar{p})n = k(x \,.\, \bar{z})m \neq 0$ and $q = r$. Also, we can find $u \in \mathcal{P}$ such that $q \,.\, \bar{u} \neq 0$ which implies that $p = z$ and $n(q \,.\, \bar{u})k = m(r \,.\, \bar{u})k \neq 0$. But $q = r$ then gives $n = m$. Therefore $(p, n, q) = (z, m, r)$, as required.

For any set Y, denote by $\mathcal{F}(Y)$ (respectively $\mathcal{F}'(Y)$) the semigroup of all partial transformations on the set Y written and composed as left (respectively right) operators. For $\phi \in \mathcal{F}(Y) \cup \mathcal{F}'(Y)$, denote by $\mathbf{d}\phi$ the domain of ϕ.

The following lemma gives a description of left and right translations of the semigroup S.

LEMMA 6.3. (i) *For $\alpha \in \mathcal{F}(\mathcal{P})$ and $\phi: \mathbf{d}\alpha \to \mathbb{N}^+$, the function λ defined by*

$$\lambda(p, n, q) = \begin{cases} (\alpha p, (\phi p)n, q) & \text{if } p \in \mathbf{d}\alpha \\ 0 & \text{otherwise} \end{cases}, \qquad \lambda 0 = 0,$$

is a left translation of S. Conversely, every left translation of S has this form.

(ii) *For $\beta \in \mathcal{F}'(\mathcal{Q})$ and $\psi: \mathbf{d}\beta \to \mathbb{N}^+$, the function ρ defined by*

$$(p, n, q)\rho = \begin{cases} (p, n(q\psi), q\beta) & \text{if } q \in \mathbf{d}\beta \\ 0 & \text{otherwise} \end{cases}, \qquad 0\rho = 0$$

is a right translation of S. Conversely, every right translation of S has this form.

(iii) *With the above notation, λ and ρ are linked if and only if for any $p \in \mathcal{P}$ and $q \in \mathcal{Q}$,*

$$p \in \mathbf{d}\alpha,\ q \,.\, \overline{\alpha p} \neq 0 \Leftrightarrow q \in \mathbf{d}\beta,\ q\beta \,.\, \bar{p} \neq 0$$

and if one side holds, then

$$(q\,.\,\overline{\alpha p})(\phi p)=(q\psi)(q\beta\,.\,\bar{p}).$$

Proof. The proofs in [**4**, V.3] remain valid in any Rees matrix semigroup over a semigroup with a zero adjoined. The above is an application of this remark to the semigroup $S=\mathcal{M}^0(\mathcal{P},\mathbb{N}^+,\mathcal{Q};(q\,.\,\bar{p}))$.

THEOREM 6.4. *Let $\mathcal{I}=i_{\mathcal{T}(X^*)}(\mathfrak{M})$ and define a mapping χ by*

$$\chi:\sigma\to(\lambda,\rho)\qquad(\sigma\in\mathcal{I})$$

where

$$\sigma\theta_s=\theta_{\lambda s},\ \theta_s\sigma=\theta_{s\rho}\qquad(s\in S).$$

Then χ is an isomorphism of $\mathcal{I}$ onto $\Omega(S)$ and the following diagram commutes:

$$\begin{array}{ccccc} \Omega(S) & \supseteq & & & \Pi(S) \\ \uparrow{\scriptstyle\chi} & & & & \uparrow{\scriptstyle\pi} \\ \mathcal{I} & \supseteq & \mathfrak{M} & \xleftarrow{\ \theta\ } & S. \end{array}$$

Proof. First note that χ is well-defined since $\sigma\in\mathcal{I}$ so that $\sigma\mathfrak{M}$, $\mathfrak{M}\sigma\subseteq\mathfrak{M}$ and by Lemma 6.2, $\theta_{\lambda s}$ and $\theta_{s\rho}$ uniquely determine λs and $s\rho$, respectively. Further, for any $s,t\in S$, we have

$$\theta_{(\lambda s)t}=\theta_{\lambda s}\theta_t=(\sigma\theta_s)\theta_t=\sigma(\theta_s\theta_t)=\sigma\theta_{st}=\theta_{\lambda(st)},$$

so that $(\lambda s)t=\lambda(st)$ and analogously $s(t\rho)=(st)\rho$ and $(s\rho)t=s(\lambda t)$. Consequently $(\lambda,\rho)\in\Omega(S)$ and hence χ maps $\mathcal{I}$ into $\Omega(S)$.

Let $\sigma,\tau\in\mathcal{I}$ and $\sigma\chi=(\lambda,\rho),\tau\chi=(\alpha,\beta)$. Then for any $s\in S$, we have

$$\sigma\tau\theta_s=\sigma\theta_{\alpha s}=\theta_{\lambda\alpha s},\qquad\theta_s\sigma\tau=\theta_{s\rho}\tau=\theta_{s\rho\beta},$$

whence

$$(\sigma\tau)\chi=(\lambda\alpha,\beta\rho)=(\lambda,\rho)(\alpha,\beta)=(\sigma\chi)(\tau\chi)$$

and χ is a homomorphism.

Before proving that χ is one-to-one, we make the following observation. For any $s=(p,n,q)\in S$ and $w\in X^*$, we get

$$\begin{aligned}\sigma\theta_s w=\sigma p^{n(q\,.\,\bar{w})}=(\sigma p)^{n(q\,.\,\bar{w})}&=[\pi(\sigma p)]^{[\varepsilon(\sigma p)]n(q\,.\,\bar{w})}\\&=\theta_{(\pi(\sigma p),[\varepsilon(\sigma p)]n,q)}w,\end{aligned}$$

which implies

$$\sigma\theta_{(p,n,q)}=\begin{cases}\theta_{(\pi(\sigma p),[\varepsilon(\sigma p)]n,q)} & \text{if } \sigma p\neq 1,\\ \zeta & \text{otherwise.}\end{cases}\tag{10}$$

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Now assume that for $\sigma, \tau \in \mathcal{I}$, we have $\sigma\theta_s = \tau\theta_s$ for all $s \in S$. In view of (10), we conclude that

$$\sigma p = 1 \Leftrightarrow \tau p = 1 \qquad (p \in \mathcal{P}),$$

and otherwise $\theta_{(\pi(\sigma p),[\varepsilon(\sigma p)]n,q)} = \theta_{(\pi(\tau p),[\varepsilon(\tau p)]n,q)}$, which yields

$$\sigma p = [\pi(\sigma p)]^{\varepsilon(\sigma p)} = [\pi(\tau p)]^{\varepsilon(\tau p)} = \tau p.$$

This evidently implies that $\sigma w = \tau w$ for all $w \in X^*$, whence $\sigma = \tau$. Therefore χ is one-to-one.

In order to show that χ is onto, we let $(\lambda, \rho) \in \Omega(S)$. By Lemma 6.3, to (λ, ρ) is associated a quadruple $(\alpha, \phi, \psi, \beta)$. We thus may define the following two functions. For $w \in X^+$, say $w = p^m$ where $p \in \mathcal{P}$, let

$$\sigma w = \begin{Bmatrix} (\alpha p)^{(\phi p)m} & \text{if } p \in \mathbf{d}\alpha \\ 1 & \text{otherwise} \end{Bmatrix}, \qquad \sigma 1 = 1.$$

For $r \in \mathbb{N}^X$, $r \neq 0$, say $r = nq$ where $q \in \mathcal{Q}$, let

$$r\tau = \begin{Bmatrix} (q\psi)n(q\beta) & \text{if } q \in \mathbf{d}\beta \\ 1 & \text{otherwise} \end{Bmatrix}, \qquad 0\tau = 0.$$

With this notation, we have for $k \geq 1$,

$$\sigma w^k = \sigma p^{mk} = \begin{Bmatrix} (\alpha p)^{(\phi p)mk} & \text{if } p \in \mathbf{d}\alpha \\ 1 & \text{otherwise} \end{Bmatrix} = (\sigma p^m)^k = (\sigma w)^k,$$

which proves that $\sigma \in li_{\mathcal{T}(X^*)}(\mathfrak{M})$ by Proposition 4.1. Furthermore,

$$r \cdot \overline{\sigma w} = \begin{cases} nq \cdot (\phi p)m\overline{\alpha p} & \text{if } p \in \mathbf{d}\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$r\tau \cdot \bar{w} = \begin{cases} (q\psi)m(q\beta) \cdot m\bar{p} & \text{if } q \in \mathbf{d}\beta, \\ 0 & \text{otherwise.} \end{cases}$$

The conditions on the quadruple $(\alpha, \phi, \psi, \beta)$ in Lemma 6.3(iii) imply that $r \cdot \overline{\sigma w} = r\tau \cdot \bar{w}$. Since this holds trivially for $r = 0$ or $w = 1$, we conclude that σ and τ are adjoints. By Theorem 4.2, we have that $\sigma \in ri_{\mathcal{T}(X^*)}(\mathfrak{M})$. Consequently $\sigma \in \mathcal{I}$.

It remains to prove that $\sigma\chi = (\lambda, \rho)$. Let $s = (p, n, q) \in S$ and $w \in X^*$. On the left we get

$$\begin{aligned} \sigma\theta w &= \sigma(p^{n(q \cdot \bar{w})}) = (\sigma p)^{n(q \cdot \bar{w})} = \theta_{(\pi(\sigma p),[\varepsilon(\sigma p)]n,q)}w \\ &= \theta_{(\alpha p,(\phi p)n,q)}w = \theta_{\lambda(p,n,q)}w = \theta_{\lambda s}w \end{aligned}$$

and thus $\sigma\theta_s = \theta_{\lambda s}$; this holds trivially if $s = 0$. On the right, we obtain

$$\theta_s \sigma w = \theta_s \sigma(\pi w)^{\varepsilon w} = \theta_s \begin{cases} [\alpha(\pi w)]^{[(\phi(\pi w)](\varepsilon w)} & \text{if } \pi w \in \mathbf{d}\alpha, \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} p^{n(q \cdot \overline{\alpha(\pi w)})[\phi(\pi w)](\varepsilon w)} & \text{if } \pi w \in \mathbf{d}\alpha, \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} p^{n(q\beta \cdot \overline{\pi w})(q\psi)(\varepsilon w)} & \text{if } q \in \mathbf{d}\beta, \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \theta_{(p, n(q\psi), q\beta)}(\pi w)^{\varepsilon w} & \text{if } q \in \mathbf{d}\beta, \\ 1 & \text{otherwise} \end{cases}$$

$$= \theta_{(p,n,q)\rho} w = \theta_{s\rho} w$$

and thus $\theta_s \sigma = \theta_{s\rho}$; this holds trivially for $s = 0$. Consequently $\sigma\chi = (\lambda, \rho)$.

Therefore χ is an isomorphism of $\mathscr{I}$ onto $\Omega(S)$. It remains to show that θ followed by χ equals π, that is $\theta_s \chi = \pi_s$ for every $s \in S$. Indeed, let $s, t \in S$. Then

$$\theta_s \theta_t = \theta_{st} = \theta_{\lambda_s t}, \qquad \theta_t \theta_s = \theta_{ts} = \theta_{t\rho_s}$$

and hence $\theta_s \chi = (\lambda_s, \rho_s) = \pi_s$, as required.

We can finally deduce the desired result.

COROLLARY 6.5. *The mapping*

$$\theta : s \to \theta_s \qquad (s \in S)$$

is a dense embedding of S into $\mathscr{T}(X^*)$.

Proof. By Lemma 6.2, S is weakly reductive. Hence by Result 6.1, $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$. According to Theorem 6.4, χ is an isomorphism of $\mathscr{I} = i_{\mathscr{T}(X^*)}(\mathfrak{M})$ onto $\Omega(S)$ which maps $\mathfrak{M}$ onto $\Pi(S)$. Therefore $\mathfrak{M}$ is a densely embedded ideal of $\mathscr{I}$ and hence θ is a dense embedding of S into $\mathscr{T}(X^*)$.

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