

EXPLICIT ESTIMATES FOR THE DISTRIBUTION OF PRIMES

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(Received 19 May 2023; first published online 22 June 2023)

2020 *Mathematics subject classification*: primary 11N05; secondary 11M06, 11M26, 11N37.

Keywords and phrases: distribution of primes, primes in intervals, prime number theorem, Riemann–von Mangoldt explicit formula, Riemann zeta function.

This thesis presents new explicit results on the distribution of prime numbers. The results largely fall into the categories of error estimates for the prime number theorem (PNT) and interval estimates for primes. The error in the PNT can be estimated with the truncated Riemann–von Mangoldt explicit formula

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + E(x, T),$$

where $\rho = \beta + i\gamma$ represents the nontrivial zeros of the Riemann zeta-function. A new explicit version of Goldston’s estimate for $E(x, T)$ is proved, of order

$$E(x, T) = O\left(\frac{x \log x \log \log x}{T}\right).$$

This estimate is used to update two short-interval results: we prove that there are primes between cubes, that is, in intervals $(n^3, (n+1)^3)$ for all $n \geq \exp(\exp(32.537))$, and primes between n^{155} and $(n+1)^{155}$ for all $n \geq 1$. These results are published in [1]. The proof follows the original method of Ingham and builds on work of Dudek [6]. We also use, and prove, updated versions of Bertrand’s postulate of primes in $(n, 2n-2)$ for integers $n > 3$. This work is published in [4], with corrections in [5]. The methods of Ramaré and Saouter [8] and Kadiri and Lumley [7] are used to give new pairs (Δ, x_0) for which there exist at least one prime in $((1-\Delta^{-1})x, x]$ for all $x \geq x_0$. For instance, we can take $(x_0, \Delta) = (e^{150}, 2.07 \times 10^{11})$. Lastly, new conditional results are proved for the error in the PNT. Under the Riemann hypothesis (RH), we prove an explicit error estimate and an explicit mean-value estimate for the PNT in short intervals. The former

Thesis submitted to the University of New South Wales in March 2023; degree approved on 12 May 2023; primary supervisor Timothy Trudgian, secondary supervisor Harvinder Sidhu.

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is published in [2] and the latter is in the preprint [3]. The mean-value estimate is based on Selberg's work [9], and is of particular interest for its applications, of which two are given. We first prove that under the RH, there is a prime in $(y, y + 37 \log^2 y]$ for at least half the $y \in [x, 2x]$ and all $x \geq 2$. The second application is to Goldbach numbers: we prove that under the RH, there is a Goldbach number in $(x, x + 864 \log^2 x]$ for all $x \geq 2$.

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