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Index Formulas for Ramified Elliptic Units

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Abstract. We compute the index of certains groups of elliptic units. These groups are the analoguous of Sinnott's groups of circular units.

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1. Introduction

In 1978 and 1980, Sinnott published two important papers on the cyclotomic units of Abelian number fields ([Sin1] and [Sin2]). Its constructions inspired Kubert and Lang and Kersey who tried to develop an equivalent approach for elliptic units, cf. [K-L] chapters 12 and 13. However, their main results are obtained under some very restrictive hypotheses. Galovich and Rosen [Ga-R] were also influenced by Sinnott's work. They obtained analoguous results for finite Abelian extensions of a rational function field. The roots of unity are replaced by the torsion points of Carlitz Modules. But it was Yin ([Yin1] and [Yin2]) who gave a complete response to this question in the case of global function fields. In such a situation, the material used are the torsion points of Drinfel'd Modules of rank one. Let us come back to elliptic units. The aim of this paper is to clear away almost all the restrictions imposed in [K-L]. Our main results are Theorem A and Theorem B stated below. The former is proved in Sections 3 and 4. Propositions 8 and 9 are crucial steps in this proof. We showed them by using ideas from [Yin1], Proposition 5.1. To state these theorems, we need some notation. Let $K \subset \mathbb{C}$ be a imaginary quadratic field and let $K^{\mathrm{ab}} \subset \mathbb{C}$ be the maximal Abelian extension of K in $\mathbb{C}$. Let $F \subset K^{\mathrm{ab}}$ be a finite Abelian extension of K and let $\mathcal{O}_F$ (resp. $\mathcal{O}_F^{\times}$) be the ring of integers (resp. the group of units) of F. Let μ_F be the group of roots of unity in F and let $w_F := \#\mu_F$. Let $\mathfrak{m}$ be the conductor of F/K. For each ideal $\mathfrak{n}$ of $\mathcal{O}_K$ dividing $\mathfrak{m}$, we let $f_{\mathfrak{n}}$ be the positive generator of $\mathbb{Z} \cap \mathfrak{n}$ and we put $w_{\mathfrak{n}} := \#\{\zeta \in \mu_K,\ \zeta \equiv 1 \text{ modulo } \mathfrak{n}\}$. Moreover, if $\mathfrak{n} \neq (1)$, we define $\tilde{\varphi}_{F,\mathfrak{n}} := N_{K_{\mathfrak{n}}/F \cap K_{\mathfrak{n}}}(\varphi_{\mathfrak{n}})^{w_K f_{\mathfrak{m}}/w_{\mathfrak{n}} f_{\mathfrak{n}}}$, where $K_{\mathfrak{n}} \subset K^{\mathrm{ab}}$ is the ray class field modulo the ideal $\mathfrak{n}$ and $\varphi_{\mathfrak{n}}$ is the Siegel–Ramachandra–Robert invariant (cf. Definition 2). Let $\tilde{\varphi}_F$ be the Galois submodule of $F^{\times}$ generated by $\tilde{\varphi}_{F,\mathfrak{n}}$, $\mathfrak{n}|\mathfrak{m}$ and $\mathfrak{n} \neq (1)$. Let h_F (resp. h) be the ideal class number of F (resp. K). Let us also denote, for each maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, $K_{\mathfrak{p}^\infty}$, the union of the ray class fields $K_{\mathfrak{p}^n}$ modulo $\mathfrak{p}^n$, $n \geqslant 0$. Let H be the Hilbert class field of K. Then we have

THEOREM A. *Let Ω_F be the subgroup of $\mathcal{O}_F^\times$ generated by μ_F, $\tilde{\varphi}_F \cap \mathcal{O}_F^\times$ and by all the norms*

$$N_{H/F\cap H}\left(\frac{\Delta(\mathcal{O}_K)\Delta(\mathfrak{a}\mathfrak{b})}{\Delta(\mathfrak{a})\Delta(\mathfrak{b})}\right)^{f_\mathfrak{m}},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of K and $\Gamma \longmapsto \Delta(\Gamma)$ is the discriminant function of lattices Γ of $\mathbb{C}$. Let $F_{(1)} := F \cap H$ and suppose that either $F \subset H$ or $H \subset F$, then

$$[\mathcal{O}_F^\times : \Omega_F] = \frac{h_F}{[H : F_{(1)}]} \frac{(12 w_K f_\mathfrak{m})^{[F:K]-1}}{\frac{w_F}{w_K}} \frac{\prod_\mathfrak{p}[F \cap K_{\mathfrak{p}^\infty} : F_{(1)}]}{[F : F_{(1)}]} (\mathbb{Z}[G_F] : U), \tag{1}$$

where $G_F := \mathrm{Gal}(F/K)$, U is a certain G_F-submodule of $\mathbb{Q}[G_F]$, cf. Definition 5, and $(\mathbb{Z}[G_F] : U)$ is Sinnott's index.

The G_F-module U naturally appears when computing the image of the elliptic units by the logarithm map. It is also related to Iwasawa ordinary distribution attached to K ([Yin3] or [B-O]). Some of the properties of the index $(\mathbb{Z}[G_F] : U)$ are given in Section 6 (cf. Proposition 16). Let us recall that the formula (1) is already known when $F \subset H$, ([Rob1], Section 3). When $\mathfrak{m} = \mathfrak{p}^e$ for some prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and $e \in \mathbb{N} - \{0\}$, this formula can be easily derived from Theorem 2.1 in Chapter 13 of [K-L].

In Sections 5 and 7, we focus on ray class fields $K_\mathfrak{m}$ modulo a ideal $\mathfrak{m}$ prime to 6. We prove the following

THEOREM B. *Let $\mathfrak{m}$ be a ideal of $\mathcal{O}_K$ prime to 6 and put $L := K_{(12 f_\mathfrak{m}^2)}$. Let $V_\mathfrak{m}$ be the largest subgroup of $\mathcal{O}_L^\times$ such that $\mu_L V_\mathfrak{m}^{12 w_K f_\mathfrak{m}} = \mu_L \Omega_{K_\mathfrak{m}}$. Then the group $\mathcal{E}_\mathfrak{m} := V_\mathfrak{m} \cap K_\mathfrak{m}$ satisfies*

$$[\Omega_{K_\mathfrak{m}} : \mu_{K_\mathfrak{m}} \mathcal{E}_\mathfrak{m}^{12 w_K f_\mathfrak{m}}] = \begin{cases} \frac{w_{K_\mathfrak{m}}}{w_K} & \text{if } s = 0 \text{ or } s = 1, \\ w_{K_\mathfrak{m}} & \text{if } s \geqslant 2. \end{cases} \tag{2}$$

Moreover, we have

$$[\mathcal{O}_{K_\mathfrak{m}}^\times : \mathcal{E}_\mathfrak{m}] = \begin{cases} h_{K_\mathfrak{m}}, & \text{if } s \leqslant 2, \\ h_{K_\mathfrak{m}} w_K^{e(2^{s-2}-1)+2-s}, & \text{if } s \geqslant 3 \text{ and } h \text{ odd}, \end{cases} \tag{3}$$

where s is the number of prime ideals of $\mathcal{O}_K$ that divide $\mathfrak{m}$ ($s = 0$ if $\mathfrak{m} = (1)$) and e is the index in $\mathrm{Gal}(H/K)$ of the group generated by the Frobenius elements at these ideals.

To get formula (2), we used the results from [Rob2], [Rob3], [Sch] and [H-V], which enabled us to construct explicit generators for $\mathcal{E}_\mathfrak{m}$. Perhaps these generators may be useful for a better understanding of the group of elliptic units considered by Rubin in [Rub].

The following supplementary notations are used throughout this paper. We will put $r_{\mathfrak{m}} := w_{\mathfrak{m}} f_{\mathfrak{m}}$. Let $\mathfrak{a}$ be a fractional ideal of K. Then $\bar{\mathfrak{a}}$ will denote the image of $\mathfrak{a}$ by the complex conjugation. If $\mathfrak{a}$ is prime to $\mathfrak{m}$, then by $(\mathfrak{a}, F/K)$ we mean the automorphism of F/K associated to $\mathfrak{a}$ by the Artin map. If $\mathfrak{a} \subset \mathcal{O}_K$, then $N(\mathfrak{a}) := [\mathcal{O}_K : \mathfrak{a}]$ is the norm of $\mathfrak{a}$. In case $\mathfrak{m} \neq (1)$ we will denote $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the prime ideals that divide $\mathfrak{m}$, thus $\mathfrak{m} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$, for some $e_i \in \mathbb{N} - \{0\}$. If $n \geqslant 1$, we denote by μ_n the group of nth roots of unity in $\mathbb{C}$.

2. Preliminaries

2.1. Let Γ be a lattice of $\mathbb{C}$. It is well known that the field of elliptic functions with respect to Γ is generated over $\mathbb{C}$ by the Weierstrass function $\wp_\Gamma$ and it's derivative $\wp'_\Gamma$. Moreover, the points $(\wp_\Gamma(z), \wp'_\Gamma(z)), z \in \mathbb{C}/\Gamma - \{0\}$, parametrize the complex solutions of the equation $y^2 = 4x^3 - g_2 x - g_3$ that defines the elliptic curve associated with Γ, where the coefficients g_2 and g_3 are defined as follows:

$$g_2 = 60 \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = 140 \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \frac{1}{\omega^6}.$$

The discriminant $g_2^3 - 27g_3^2$ of the Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ is usually denoted $\Delta(\Gamma)$ and called the discriminant of Γ. In particular, we have $\Delta(\lambda\Gamma) = \lambda^{-12}\Delta(\Gamma)$ for all $\lambda \in \mathbb{C}^\times$. Let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$. Let $[\tau, 1]$ be the lattice of $\mathbb{C}$ generated over $\mathbb{Z}$ by the basis $(\tau, 1)$. Then the function $\tau \longmapsto \Delta(\tau) := \Delta([\tau, 1])$ is a cusp form of weight 12, and satisfies the Jacobi's product expansion

$$\Delta(\tau) = (2\pi)^{12} \mathrm{e}^{2i\pi\tau} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2i\pi n\tau})^{24}.$$

The function $\tau \longmapsto \eta(\tau) := \mathrm{e}^{\frac{2i\pi\tau}{24}} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2i\pi n\tau})$ is the so-called Dedekind's eta function.

PROPOSITION 1. *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of K. Then the quotient $\Delta(\mathfrak{a})/\Delta(\mathfrak{b}) \in H$ and generates the ideal $(\mathfrak{b}\mathfrak{a}^{-1}\mathcal{O}_H)^{12}$. Moreover, we have*

$$\left(\frac{\Delta(\mathfrak{a})}{\Delta(\mathfrak{b})} \right)^{(\mathfrak{c}, H/K)} = \frac{\Delta(\mathfrak{c}^{-1}\mathfrak{a})}{\Delta(\mathfrak{c}^{-1}\mathfrak{b})}.$$

Proof. See [Lan], chapter 12, Theorems 1 and 5. □

DEFINITION 1. Let $\tau \in \mathrm{Gal}(H/K)$ and $\mathfrak{b}$ be a ideal of K such that $(\mathfrak{b}, H/K) = \tau^{-1}$. Let $x \in K$ be a generator of $\mathfrak{b}^h$. Then we put

$$\partial(\tau) := x^{12} \Delta(\mathfrak{b})^h.$$

Let us remark that $\partial(\tau)$ is well defined since $\mathcal{O}_K^\times$ is of an order dividing 12.

COROLLARY 1. *Let* $\tau_1, \tau_2 \in \mathrm{Gal}(H/K)$. *Then* $\partial(\tau_1)/\partial(\tau_2) \in \mathcal{O}_H^\times$ *and we have*

$$\left(\frac{\partial(\tau_1)}{\partial(\tau_2)}\right)^\tau = \frac{\partial(\tau_1\tau)}{\partial(\tau_2\tau)}$$

for all $\tau \in \mathrm{Gal}(H/K)$.

2.2. Let us now recall the definition of Siegel–Ramachandra–Robert invariants and some of their properties. They are the essential material when constructing elliptic units in Abelian extensions of imaginary quadratic fields. One obtains them as special values of the classical φ-functions whose definition we now recall. If (ω_1, ω_2) is a 'positive' $\mathbb{Z}$-basis of the lattice Γ (i.e. such that $\mathrm{Im}(\omega_1/\omega_2) > 0$) then following Schertz ([Sch] formula (1.1)), we define

$$\varphi(t; \omega_1, \omega_2) = \kappa(t, \Gamma)\eta\left(\frac{\omega_1}{\omega_2}\right)^2 \omega_2^{-1},$$

where $t \longmapsto \kappa(t, \Gamma)$ is the Klein form ([K-L], Chapter 2, Section 1) and η is Dedekind's eta function introduced above. Robert in [Rob1], Section 1, proved many interesting properties of these φ-functions. (His notation is different from ours. More precisely his $\varphi(t; \omega_2, \omega_1)$ is our $-i\varphi(t; \omega_1, \omega_2)$.) Stark also used these functions in [Sta]. Indeed, let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$. If $t = u\tau + v$, where u and v are real numbers, then $i\varphi(t; \tau, 1)$ is denoted $\varphi(u, v, \tau)$ in [Sta] Equation (10). Formula (17) of [Sta] may be written as:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $\omega_1' = a\omega_1 + b\omega_2$, $\omega_2' = c\omega_1 + \mathrm{d}\omega_2$. Then we have

$$\varphi(t; \omega_1', \omega_2') = \varepsilon(A)\varphi(t; \omega_1, \omega_2), \tag{2.1}$$

where $\varepsilon(A)$ is a 12th root of unity depending only on A and such that ε: $SL_2(\mathbb{Z}) \longrightarrow \mu_{12}$ is a group homomorphism. See [Sch] formula (2.6) for an explicit description of $\varepsilon(A)$. On the other hand, if $\gamma = b_1\omega_1 + b_2\omega_2 \in \Gamma$ and $t = a_1\omega_1 + a_2\omega_2$ with $a_1, a_2 \in \mathbb{Q}$, then

$$\varphi(t + \gamma; \omega_1, \omega_2) = (-1)^{b_1b_2+b_1+b_2} \mathrm{e}^{-\pi i(b_1a_2-b_2a_1)}\varphi(t; \omega_1, \omega_2), \tag{2.2}$$

cf. [Sch] formula (2.3) or [K-L], formula **K 2**, page 28. Finally, we have

$$\varphi(at; a\omega_1, a\omega_2) = \varphi(t; \omega_1, \omega_2), \quad \text{for all } a \in \mathbb{C} - \{0\}.$$

See [Rob3], Section 2, where $z \longmapsto \varphi(z; \omega_1, \omega_2)$ is defined as a theta function with some special properties.

PROPOSITION 2. *Suppose we have* $\Gamma = \mathfrak{m}$, *where* $\mathfrak{m}$ *is a proper ideal of* $\mathcal{O}_K$ *and let* (ω_1, ω_2) *be a positive* $\mathbb{Z}$*-basis of* $\mathfrak{m}$. *Then*

(i) $\varphi(1;\omega_1,\omega_2)$ *is a algebraic integer in* $K_{(12f_{\mathfrak{m}}^2)}$.
(ii) $\varphi(1;\omega_1,\omega_2)^{12f_{\mathfrak{m}}}\in K_{\mathfrak{m}}$.

Proof. We have $f_{\mathfrak{m}}=r_1\omega_1+r_2\omega_2$ for some $r_1,r_2\in\mathbb{Z}$. Thus

$$\varphi(1;\omega_1,\omega_2)=\varphi\left(\frac{r_1}{f_{\mathfrak{m}}}\tau+\frac{r_2}{f_{\mathfrak{m}}};\tau,1\right)$$

with $\tau:=\omega_1/\omega_2\in K$. Since r_1, r_2 and $f_{\mathfrak{m}}$ are coprime the function $\tau\longmapsto\varphi(r_1/f_{\mathfrak{m}}\tau+r_2/f_{\mathfrak{m}}\tau,1)$ is a modular function of level $12f_{\mathfrak{m}}^2$. It is analytic inside $\mathfrak{h}:=\{z\in\mathbb{C},\ \mathrm{Im}(z)>0\}$ and its q-expansions at every cusp have coefficients in the ring of integers of $\mathbb{Q}(\mu_{12f_{\mathfrak{m}}^2})$, cf. [Sta], Section 4. Therefore (i) is a consequence of Lemma 1 and Theorem 3 of [Sta]. As for the part (ii) of the proposition we refer to the proof of Lemma 7 of [Sta]. Let us remark that our $\varphi(1;\omega_1,\omega_2)^{12f_{\mathfrak{m}}}$ is denoted by $E(\mathfrak{c}_0)$ in [Sta].

DEFINITION 2. We put $\varphi_{\mathfrak{m}}:=\varphi(1,\omega_1,\omega_2)^{12f_{\mathfrak{m}}}$, where (ω_1,ω_2) is any positive $\mathbb{Z}$-basis of $\mathfrak{m}$.

PROPOSITION 3. *Let* $\mathfrak{q}$ *be a maximal ideal of* $\mathcal{O}_K$. *Then we have*

$$N_{K_{\mathfrak{mq}}/K_{\mathfrak{m}}}(\varphi_{\mathfrak{mq}})^{w_{\mathfrak{m}}/w_{\mathfrak{mq}}}=\begin{cases}\varphi_{\mathfrak{m}}^{f_{\mathfrak{mq}}/f_{\mathfrak{m}}}, & \text{if } \mathfrak{q}|\mathfrak{m},\\ [\varphi_{\mathfrak{m}}^{f_{\mathfrak{mq}}/f_{\mathfrak{m}}}]^{(1-\sigma_{\mathfrak{q}}^{-1})}, & \text{if } \mathfrak{q}\nmid\mathfrak{m} \text{ and } \mathfrak{m}\neq(1)\\ \left(\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{q})}\right)^{f_{\mathfrak{q}}}, & \text{if } \mathfrak{m}=(1),\end{cases}$$

where $\sigma_{\mathfrak{q}}:=(\mathfrak{q},K_{\mathfrak{m}}/K)$.

Proof. See [Rob1], Théorème 2, p. 17.

The above results may be used to determine the ideal generated in $K_{\mathfrak{m}}$ by the invariant $\varphi_{\mathfrak{m}}$. The following corollary makes this ideal explicit:

COROLLARY 2. *Suppose that* $\mathfrak{m}=\mathfrak{q}^e$, *where* $e\geqslant 1$ *and* $\mathfrak{q}$ *is a maximal ideal of* $\mathcal{O}_K$, *and let* $\mathfrak{q}_{K_{\mathfrak{m}}}$ *be the product of the maximal ideals of* $K_{\mathfrak{m}}$ *which contain* $\mathfrak{q}$. *Then* $\varphi_{\mathfrak{m}}$ *generates in* $\mathcal{O}_{K_{\mathfrak{m}}}$ *the* $(12/w_K)r_{\mathfrak{m}}$*-st power of the ideal* $\mathfrak{q}_{K_{\mathfrak{m}}}$. *Otherwise* $\varphi_{\mathfrak{m}}$ *is a unit of* $\mathcal{O}_{K_{\mathfrak{m}}}$.

Proof. By Proposition 3, above we have

$$N_{K_{\mathfrak{m}}/H}(\varphi_{\mathfrak{m}})^{w_K/w_{\mathfrak{m}}}=\left(\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{q})}\right)^{f_{\mathfrak{m}}}.$$

This implies the first statement of the corollary since $\Delta(\mathcal{O}_K)/\Delta(\mathfrak{q})$ generates the ideal $(\mathfrak{q}\mathcal{O}_H)^{12}$, thanks to Proposition 1, and $K_{\mathfrak{m}}/H$ is totally ramified at $\mathfrak{q}$. Now suppose that $\mathfrak{m}$ is divisible by at least two ideals. Then $N_{K_{\mathfrak{m}}/H}(\varphi_{\mathfrak{m}})$ must be a unit as follows from the norm formulas of Proposition 3. But recall that $\varphi_{\mathfrak{m}}$ is a algebraic integer, cf. Proposition 2. Hence, $\varphi_{\mathfrak{m}}$ is a unit too.

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2.3. Let χ be a character of $G_F := \mathrm{Gal}(F/K)$, where F is a finite Abelian extension of K. Let $F_\chi \subseteq F$ be the fixed field of $\ker\chi$. The character χ factors through $\mathrm{Gal}(F/F_\chi) = \ker\chi$ and yields a character χ' of $\mathrm{Gal}(F_\chi/K)$. Let $\mathfrak{m}_\chi$ be the conductor of the Abelian extension F_χ/K. Let $\mathfrak{a}$ be an ideal of K. If $\mathfrak{a}$ is prime to $\mathfrak{m}_\chi$ then we put $\chi(\mathfrak{a}) := \chi'((\mathfrak{a}, F_\chi/K))$. Otherwise we set $\chi(\mathfrak{a}) = 0$.

If $\chi \neq 1$, then one can associate to χ the L-function $L(\cdot, \chi)$: $s \longmapsto L(s, \chi)$, defined in the half-plane $\mathrm{Re}(s) > 1$ by the Euler product

$$L(s, \chi) := \prod_{\mathfrak{p} \nmid \mathfrak{m}_\chi} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}.$$

It is well known that $L(\cdot, \chi)$ has a analytic continuation to the whole complex plane. Moreover, $L(0, \chi) = 0$ and $L'(0, \chi) \neq 0$, cf. [Tat], Proposition 3.4, p. 24. Let ζ_F (resp. ζ_K) be the zeta function of F (resp. K), then we have the following decomposition $\zeta_F(s) = \zeta_K(s)\prod_{\chi\neq 1} L(s, \chi)$, cf. loc. cit. page 12, from which we deduce the analytic class number formula

$$\frac{h_F \mathrm{Reg}(F)}{w_F} = \frac{h}{w_K}\prod_{\chi\neq 1} L'(0, \chi), \tag{2.3}$$

where $\mathrm{Reg}(F)$ is the regulator of F. If $F = K_\mathfrak{m}$, then we have the Kronecker limit formulas

$$\prod_{\mathfrak{p}|\mathfrak{m}} (1 - \chi(\mathfrak{p}))L'(0, \chi) = \begin{cases} \dfrac{-1}{12 r_\mathfrak{m}} \displaystyle\sum_{\sigma\in G_\mathfrak{m}} \log(|\varphi_\mathfrak{m}(\sigma)|^2)\chi(\sigma), & \text{if } \mathfrak{m} \neq (1), \\ \dfrac{-1}{12 w_K h} \displaystyle\sum_{\sigma\in G_\mathfrak{m}} \chi(\sigma)\log(|\partial(\sigma)|^2), & \text{if } \mathfrak{m} = (1), \end{cases} \tag{2.4}$$

where $G_\mathfrak{m} := \mathrm{Gal}(K_\mathfrak{m}/K)$, ([Gr-R], Propositions 7.15 and 7.19).

3. The Groups of Elliptic Units C_F and C_F^0

Let F be a finite Abelian extension of K of conductor $\mathfrak{m}$. For each ideal $\mathfrak{n}$ of $\mathcal{O}_K$ we put $F_\mathfrak{n} := K_\mathfrak{n} \cap F$. Moreover, if $\mathfrak{n}|\mathfrak{m}$ and is such that $\mathfrak{n} \neq (1)$, then we define

$$\varphi_{F,\mathfrak{n}} := N_{K_\mathfrak{n}/F_\mathfrak{n}}(\varphi_\mathfrak{n})^{d(\mathfrak{m},\mathfrak{n})} = \tilde{\varphi}_{F,\mathfrak{n}}^h,$$

where $d(\mathfrak{m}, \mathfrak{n}) := w_K f_\mathfrak{m} h/r_\mathfrak{n}$. The invariants $\varphi_{F,\mathfrak{n}}$ were introduced for the first time in [K-L], p. 307. They are called the *Kersey invariants*. An easy calculation based on Proposition 3 above shows that for all ideals $\mathfrak{n}$ and $\mathfrak{q}$ such that $\mathfrak{q}$ is prime and $\mathfrak{n}\mathfrak{q}|\mathfrak{m}$, we have

$$N_{F_{\mathfrak{n}\mathfrak{q}}/F_\mathfrak{n}}(\varphi_{F,\mathfrak{n}\mathfrak{q}}) = \begin{cases} \varphi_{F,\mathfrak{n}}, & \text{if } \mathfrak{q}|\mathfrak{n}, \\ [\varphi_{F,\mathfrak{n}}]^{1-(\mathfrak{q},F_\mathfrak{n}/K)^{-1}}, & \text{if } \mathfrak{q}\nmid\mathfrak{n} \text{ and } \mathfrak{n} \neq (1), \\ N_{H/F\cap H}\left(\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{q})}\right)^{hf_\mathfrak{m}}, & \text{if } \mathfrak{n} = (1). \end{cases}$$

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DEFINITION 3. Let Δ be the subgroup of $\mathcal{O}_H^\times$ generated by the units $\partial(\tau_1)/\partial(\tau_2)$, $\tau_1, \tau_2 \in \mathrm{Gal}(H/K)$. We define P_F to be the G_F-submodule of $F^\times$ generated by μ_F, $N_{H/F\cap H}(\Delta)^{f_\mathfrak{m}}$ and by all $\varphi_{F,\mathfrak{n}}$, $\mathfrak{n}|\mathfrak{m}$ and $\mathfrak{n} \neq (1)$. Also we put $C_F := P_F \cap \mathcal{O}_F^\times$.

Now we give a technical lemma which is helpful in the proof of Lemma 2.

LEMMA 1. *Suppose that* $\mathfrak{m} \neq (1)$ *and let* $x \in P_F$*. Then there exist* $\alpha \in K$*, a finite Abelian extension* M *of* K *and* $y \in M$ *such that*

(i) $x^{w_M} = \alpha^{12 f_\mathfrak{m} w_M} y^d$ *with* $d := 12 w_K w_M f_\mathfrak{m} h$.
(ii) *The valuation of* α *at every prime ideal of* $\mathcal{O}_K$ *is divisible by* h.

Proof. It suffices to show the claim for the generators of $P_{K_\mathfrak{m}}$. Let $\mathfrak{n}$ be a proper ideal of $\mathcal{O}_K$ such that $\mathfrak{n}|\mathfrak{m}$. Let $\mathfrak{n}'$ be a integral ideal of $\mathcal{O}_K$ such that $\mathfrak{n}|\mathfrak{n}'$, $\mathfrak{n}$ and $\mathfrak{n}'$ are divisible by the same prime ideals of $\mathcal{O}_K$ and $w_{\mathfrak{n}'} = 1$. Then Proposition 2 implies that $N_{K_{\mathfrak{n}'}/K_\mathfrak{n}}(\varphi_{\mathfrak{n}'})^{w_\mathfrak{n}} = \varphi_\mathfrak{n}^{f_{\mathfrak{n}'}/f_\mathfrak{n}}$.

By construction, we have $\varphi_{\mathfrak{n}'} \in [K_{(12 f_{\mathfrak{n}'}^2)}]^{12 f_{\mathfrak{n}'}}$. Thus the Lemma is true for $x = \varphi_\mathfrak{n}^{d(\mathfrak{m},\mathfrak{n})}$, with $\alpha = 1$ and $M = K_{(12 f_{\mathfrak{n}'}^2)}$. Now let us prove the lemma for the generators of $\Delta^{f_\mathfrak{m}}$. If $w_K \neq 2$, then $\Delta \subset \mathcal{O}_K^\times = \mu_K$. Hence, we may suppose $w_K = 2$. Let $\tau \in \mathrm{Gal}(H/K)$. Let $\mathfrak{a}$ be a integral primitive ideal of $\mathcal{O}_K$, prime to 6 and such that $\tau^{-1} = (\mathfrak{a}, H/K)$. Here primitive means that $\mathfrak{a}$ is not of the form $t\mathfrak{a}'$ for some integer $t > 1$ and some integral ideal $\mathfrak{a}'$ of $\mathcal{O}_K$. Let $z \in \mathcal{O}_K$ be a generator of $\mathfrak{a}^h$. Then we have

$$\frac{\partial(\tau)}{\partial(1)} = z^{12}\left(\frac{\Delta(\mathfrak{a})}{\Delta(\mathcal{O}_K)}\right)^h = (za^{-h})^{12}\left(\frac{\eta(\bar{\mathfrak{a}})}{\eta(\mathcal{O}_K)}\right)^{24h},$$

where $a = N(\mathfrak{a})$ and $\mathfrak{v} \longmapsto \eta(\mathfrak{v})$ is the η-function on primitive ideals of $\mathcal{O}_K$ that are prime to 6. ([H-V], Definition 8). Now the assertion (ii) of Proposition 10 of loc. cit. implies that our lemma is true for $x = (\partial(\tau)/\partial(1))^{f_\mathfrak{m}}$, with $\alpha = za^{-h}$. The lemma is now proved. □

DEFINITION 4. Let Δ^0 be the subgroup of $\mathcal{O}_H^\times$ formed of all the quotients

$$\frac{\partial(1)\partial(\tau_1\tau_2)}{\partial(\tau_1)\partial(\tau_2)}, \quad \tau_1, \tau_2 \in \mathrm{Gal}(H/K).$$

We define P_F^0 to be the G_F-submodule of $F^\times$ generated by μ_F, $N_{H/F\cap H}(\Delta^0)^{f_\mathfrak{m}}$, and by all $\varphi_{F,\mathfrak{n}}$, $\mathfrak{n}|\mathfrak{m}$. The group $P_F^0 \cap \mathcal{O}_F^\times$ will be denoted C_F^0.

PROPOSITION 4. *The group* Ω_F *of Theorem A is the largest subgroup of* $\mathcal{O}_F^\times$ *such that* $\mu_F \Omega_F^h = C_F^0$.

Proof. It is clear because the group $N_{H/F\cap H}(\Delta^0)^{f_\mathfrak{m}}$ is generated by the following units of $F \cap H$:

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$$N_{H/F\cap H}\left(\frac{\Delta(\mathcal{O}_K)\Delta(\mathfrak{ab})}{\Delta(\mathfrak{a})\Delta(\mathfrak{b})}\right)^{hf_{\mathfrak{m}}},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of K.

To go further we need to describe the image of P_F by the logarithm map l_F: $F^\times \longmapsto \mathbb{R}[G_F]$, where $\mathbb{R}[G_F]$ is the group ring of G_F over the field of the real numbers, defined for $x \in F^\times$ by $l_F(x) := -\sum_{\sigma\in G_F}\log(|x^\sigma|^2)\sigma^{-1}$. The map l_F is a G_F-homomorphism with the property $\ker l_F \cap \mathcal{O}_F^\times = \mu_F$.

Now we introduce some notations useful in the sequel. If X is a subset of G_F we put $s(X) := \sum_{\sigma\in X}\sigma \in R := \mathbb{Z}[G_F]$. Moreover, to every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$ we associate the element $(\mathfrak{p}, F) := \mathcal{F}_{\mathfrak{p}}^{-1}s(T_{\mathfrak{p}})/|T_{\mathfrak{p}}|$ of $\mathbb{Q}[G_F]$, where $T_{\mathfrak{p}}$ denotes the inertia group of $\mathfrak{p}$ and $\mathcal{F}_{\mathfrak{p}} \in G_F/T_{\mathfrak{p}}$ the Frobenius automorphism. For any R-module A, we denote by A_0 the kernel in A of multiplication by $s(G_F)$. □

DEFINITION 5. We denote by U the R-submodule of $\mathbb{Q}[G_F]$ generated by the element $\alpha_{(1)} := s(G_1)$, where $G_1 := \mathrm{Gal}(F/F\cap H)$ and by $\alpha_{\mathfrak{n}} := s(\mathrm{Gal}(F/F_{\mathfrak{n}}))\prod_{\mathfrak{p}|\mathfrak{n}}(1-(\mathfrak{p},F))$, where $\mathfrak{n}$ is any proper ideal of $\mathcal{O}_K$.

PROPOSITION 5. *If $F \subset H$, then we have $U = R$. Otherwise U is generated as an R-module by $\alpha_{\mathfrak{n}}, \mathfrak{n}|\mathfrak{m}$. Moreover, U is a free $\mathbb{Z}$-module of rank $[F:K]$.*

Proof. The first two assertions are obvious. On the other hand, since U is torsion free and finitely generated as a $\mathbb{Z}$-module it is $\mathbb{Z}$-free. Now recall that U is a R-submodule of $\mathbb{Q}[G_F]$. Thus, we can use character theory to compute its $\mathbb{Z}$-rank. Let χ be a complex character of G_F and let ρ_χ be the ring homomorphism $\mathbb{C}[G_F] \longrightarrow \mathbb{C}$ induced by χ. If $\mathfrak{n}_\chi$ is the conductor of χ, then we have $\rho_\chi(\alpha_{\mathfrak{n}_\chi}) = \#\mathrm{Gal}(F/F_{\mathfrak{n}_\chi})$. In particular, $\rho_\chi(U) \neq 0$. This implies that the $\mathbb{Z}$-rank of U must be equal to $[F:K]$. □

For each character χ of G_F we let $\mathcal{I}_\chi := 1/|G_F|\sum_{\sigma\in G_F}\chi(\sigma)\sigma^{-1}$ be the idempotent associated to χ in $\mathbb{C}[G_F]$. The element $\omega := 12w_K f_{\mathfrak{m}} h\sum_{\chi\neq 1}L'(0,\bar{\chi})\mathcal{I}_\chi$ of $\mathbb{C}[G_F]$ is uniquely determined by the conditions $\rho_\chi(\omega) = 12w_K f_{\mathfrak{m}} h_K\ L'(0,\bar{\chi})$ for all $\chi \in \hat{G}_F - \{1\}$ and $\rho_1(\omega) = 0$. Since the complex conjugate of $L'(0,\chi)$ is $L'(0,\bar{\chi})$ we see that $\omega \in \mathbb{R}[G_F]$. Let $l_F^* := (1-\mathcal{I}_1)l_F$, where $\mathcal{I}_1$ is the idempotent associated to the trivial character, then we have

PROPOSITION 6. *Let $\mathfrak{n}$ be a proper ideal of $\mathcal{O}_K$ such that $\mathfrak{n}|\mathfrak{m}$ and let $\tau \in \mathrm{Gal}(H/K)$. Then we have*

$$l_F^*(\varphi_{F,\mathfrak{n}}) = \omega\alpha_{\mathfrak{n}} \quad \textit{and} \quad f_{\mathfrak{m}} l_F^*\left(N_{H/F\cap H}\left(\frac{\partial(1)}{\partial(\tau)}\right)\right) = \omega\alpha_{(1)}(1-\tilde{\tau}),$$

where $\tilde{\tau}$ is any automorphism of F/K which coincide with τ on $F\cap H$. In particular we have $l_F^*(P_F) = \omega U_0$.

Proof. It clearly suffices to show that $\rho_\chi(l_F^*(\varphi_{F,\mathfrak{n}})) = \rho_\chi(\omega\alpha_\mathfrak{n})$ and

$$f_\mathfrak{m}\rho_\chi\left(l_F^*\left(N_{H/F\cap H}\left(\frac{\partial(1)}{\partial(\tau)}\right)\right)\right) = \rho_\chi(\omega s(G_1)(1-\tilde{\tau})).$$

for all $\chi \in \hat{G}_F$. But this is an easy consequence of (2.4). □

4. The Indices $[\mathcal{O}_F^\times : C_F]$ and $[C_F : C_F^0]$

Let V be a vector space of finite dimension over $L = \mathbb{Q}$ or $\mathbb{R}$. By a lattice in V we mean a finitely generated subgroup X of V such that $\mathrm{rank}_\mathbb{Z}(X) = \dim_L(V)$ and $LX = V$. Moreover, if A and B are lattices of V, then the index $(A : B)$ is by definition $|\det\gamma|$, where γ is any linear transformation of V mapping A onto B. In other words, we must have $\gamma(A) = B$. This implies in particular that γ is nonsingular since we have $\gamma(V) = \gamma(LA) = L\gamma(A) = V$. If $B \subset A$, then $(A : B)$ is the usual group index.

PROPOSITION 7. *U_0 is a lattice of $\mathbb{R}[G_F]_0$. Moreover, we have*

$$(U_0 : l_F^*(P_F)) = (12w_K f_\mathfrak{m} h)^{[F:K]-1}\,\frac{w_K}{w_F}\frac{\mathrm{Reg}(F)h_F}{h}. \tag{4.1}$$

Proof. We only have to prove that U_0 is a lattice of $\mathbb{Q}[G_F]_0$ or, equivalently, the $\mathbb{Z}$-rank of U_0 is $[F:K]-1$. But since we have $\mathrm{rank}_\mathbb{Z}(U_0) = \mathrm{rank}_\mathbb{Z}(U) - 1$, Proposition 5 above implies the conclusion. Now recall that U_0 is also an R-submodule of $\mathbb{R}[G_F]$. Therefore, since $\rho_\chi(\omega) = 12w_K f_\mathfrak{m} hL'(0,\bar{\chi}) \neq 0$ for the characters $\chi \neq 1$, we have by [Sin2] Lemma 1.2 (b)

$$(U_0 : l_F^*(P_F)) = (U_0 : \omega U_0) = |\det\omega| = \prod_{\chi\neq 1}\rho_\chi(\omega).$$

The claim now follows from the analytic class number formula (2.3).

Remark 1. In the case $\mathfrak{m} \neq (1)$ we choose for each $i \in \{1,\ldots,s\}$ a generator $x_{\mathfrak{p}_i}$ of the ideal $\mathfrak{p}_i^h$. Let $\mathfrak{n} := \mathfrak{p}_i^{e_i}$ then

$$N_{F_\mathfrak{n}/F\cap H}(\varphi_{F,\mathfrak{n}}) = x_{\mathfrak{p}_i}^{12 f_\mathfrak{m}[H:F\cap H]} N_{H/F\cap H}\left(\frac{\partial(1)}{\partial(\tau_i^{-1})}\right)^{f_\mathfrak{m}},$$

where $\tau_i := (\mathfrak{p}_i, H/K)$. In particular the group Q_F generated by $N_{H/F\cap H}(\Delta)^{f_\mathfrak{m}}$ and by all the $x_{\mathfrak{p}_i}^{12 f_\mathfrak{m}[H:F\cap H]}$ is a subgroup of P_F. If $\mathfrak{m} = (1)$ we put $Q_F = P_F$.

PROPOSITION 8. *We have*

$$[P_F^{w_F} \cap K : Q_F^{w_F} \cap K][l_F^*(P_F) : l_F(C_F)] = \prod_\mathfrak{p}[F \cap K_{\mathfrak{p}^\infty} : F \cap H], \tag{4.2}$$

where $\mathfrak{p}$ describes all the maximal ideals of $\mathcal{O}_K$.

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Proof. Here we take our inspiration from the proof of Proposition 5.1. of [Yin1]. If $F \subset H$, then we have $P_F = C_F$. Thus we can assume that $\mathfrak{m} \neq (1)$. Moreover, we can replace P_F by $P' := P_F^{w_F}$ and C_F by $C' := P' \cap \mathcal{O}_F^\times = C_F^{w_F}$ since we obviously have

$$[l_F^*(P_F) : l_F^*(C_F)] = [l_F^*(P') : l_F^*(C')].$$

Let us also put $Q' := Q_F^{w_F}$ and $\Delta' := Q' \cap \mathcal{O}_F^\times$. We have $Q' \cap \ker l_F^* = Q' \cap K$ and $P' \cap \ker l_F^* = P' \cap K$. Therefore we obtain the following commutative diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Q' \cap K & \longrightarrow & Q'/\Delta' & \xrightarrow{l_F^*} & l_F^*(Q')/l_F^*(\Delta') & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P' \cap K & \longrightarrow & P'/C' & \xrightarrow{l_F^*} & l_F^*(P')/l_F^*(C') & \longrightarrow & 1 \end{array}$$

which has exact rows and columns. The arrows are just the inclusion maps. The snake lemma applied to the above diagram gives us

$$\frac{[l_F^*(P') : l_F^*(C')]}{[l_F^*(Q') : l_F^*(\Delta')]} = \frac{[P'/C' : Q'/\Delta']}{[P' \cap K : Q' \cap K]}.$$

But since $K^\times \subset \ker l_F^*$ we have $l_F^*(Q') = l_F^*(\Delta')$. The next step now is to compute the index $[P'/C' : Q'/\Delta']$. Let $\mathfrak{p}_1', \ldots, \mathfrak{p}_s'$ be maximal ideals of $\mathcal{O}_F$ choosen so that $\mathfrak{p}_i \subset \mathfrak{p}_i'$. Then we have a well defined homomorphism $v_F \colon F^\times \to \mathbb{Z}^s$ which associates to $x \in F^\times$ the element $v_F(x) = (v_1(x), \ldots, v_s(x))$, where v_i is the valuation associated to $\mathfrak{p}_i'$. Moreover, since $C' = P' \cap \ker v_F$, we have $[P'/C' : Q'/\Delta'] = [v_F(P') : v_F(Q')]$. But we know, thanks to Corollary 2, that

$$v_F(P') = \prod_{i=1}^{s} (12 f_{\mathfrak{m}} h w_F e(F/F_{\mathfrak{p}_i^{e_i}})[H : F \cap H]\mathbb{Z}),$$

where $e(F/F_{\mathfrak{p}_i^{e_i}})$ is the ramification index at $\mathfrak{p}_i$ in $F/F_{\mathfrak{p}_i^{e_i}}$, and

$$v_F(Q') = \prod_{i=1}^{s} (12 f_{\mathfrak{m}} h w_F |T_{\mathfrak{p}_i}|[H : F \cap H]\mathbb{Z}).$$

Therefore we obtain the equality

$$[P'/C' : Q'/\Delta'] = \prod_{i=1}^{s} [F_{\mathfrak{p}_i^{e_i}} : F \cap H].$$

This concludes the proof of the proposition. □

LEMMA 2. *Suppose* $\mathfrak{m} \neq (1)$ *and let* $\mathcal{R}$ *be the subgroup of* $K^\times$ *generated by* $x_{\mathfrak{p}_i}^{12 f_{\mathfrak{m}} w_F}$, $i = 1, \ldots, s$. *Then* $\mathcal{R}$ *is free of rank* s. *Moreover, we have*

$$Q_F^{w_F} \cap K = \mathcal{R}^{[H:F\cap H]} \subset P_F^{w_F} \cap K \subset \mathcal{R}.$$

Proof. The claim that $\mathcal{R}$ is free of rank s is obvious. Now since $Q_F^{w_F}$ is generated by $\mathcal{R}^{[H:F\cap H]}$ and by $N_{H/F\cap H}(\Delta)^{f_{\mathfrak{m}}w_F}$, the group $Q_F^{w_F}\cap K$ is generated by $\mathcal{R}^{[H:F\cap H]}$ and by

$$N_{H/F\cap H}(\Delta)^{f_{\mathfrak{m}}w_F}\cap K = N_{H/F\cap H}(\Delta)^{f_{\mathfrak{m}}w_F}\cap \mathcal{O}_K^\times = N_{H/F\cap H}(\Delta)^{f_{\mathfrak{m}}w_F}\cap \mu_K = 1.$$

Hence, it remains to prove that $P_F^{w_F}\cap K\subset\mathcal{R}$. So let $x\in P_F$ be such that $x^{w_F}\in P_F^{w_F}\cap K$. By Lemma 1, we can find $\alpha\in K$, a finite Abelian extension M of K and $y\in M$ such that $x^{w_M}=\alpha^{12f_{\mathfrak{m}}w_M}y^d$ with $d:=12w_Kw_Mf_{\mathfrak{m}}h$. Moreover, the valuation of α at every prime ideal of $\mathcal{O}_K$ is divisible by h. The Lemma 6 of [Sta] tells us that we necessarily have $y^d=\zeta z^{d_1}$ where $z\in K$, $\zeta\in\mu_K$ and $d_1=d/w_K$. (recall $K(y)/K$ is Abelian and $y^d\in K$). Actually we have $\zeta\in F^{w_F}\cap\mu_K=\{1\}$. But x is a unit outside $\mathfrak{p}_i, i=1,\dots,s$. This is also true for the element αz^h of K because we have $x^{w_M}=(\alpha z^h)^{12f_{\mathfrak{m}}w_M}$. Now recall that the valuation of α at every prime ideal of $\mathcal{O}_K$ is divisible by h. This means that

$$\alpha z^h\mathcal{O}_K=\mathfrak{p}_1^{hr_1}\cdots\mathfrak{p}_s^{hr_s}=\left(\prod_{i=1}^{s}x_{\mathfrak{p}_i}^{r_i}\right)\mathcal{O}_K,$$

for some $r_i\in\mathbb{N}$. In other words we have $x^{w_F}=\left(\prod_{i=1}^s x_{\mathfrak{p}_i}^{r_i}\right)^{12f_{\mathfrak{m}}w_F}$, and this proves that $x^{w_F}\in\langle x_{\mathfrak{p}_i}^{12f_{\mathfrak{m}}w_F}, i=1,\dots,s\rangle$. This concludes the proof of Lemma 2. □

THEOREM 1. *Let us put* $d(F):=[P_F^{w_F}\cap K:Q_F^{w_F}\cap K]$. *Then we have*

$$[\mathcal{O}_F^\times:C_F]=\frac{(12w_Kf_{\mathfrak{m}}h)^{[F:K]-1}}{\frac{w_F}{w_K}}\frac{h_F}{h}\frac{\prod_{\mathfrak{p}}[F\cap K_{\mathfrak{p}^\infty}:F\cap H]}{[F:F\cap H]}\frac{(R:U)}{d(F)}.$$

Proof. Since $\ker l_F\cap\mathcal{O}_F^\times=\mu_F$ we have

$$\begin{aligned}[\mathcal{O}_F^\times:C_F]&=[l_F(\mathcal{O}_F^\times):l_F(C_F)]\\&=\frac{(R_0:U_0)}{(R_0:l_F^*(\mathcal{O}_F^\times))}(U_0:l_F^*(P_F))(l_F^*(P_F):l_F(C_F)).\end{aligned}$$

It is not hard to check the identity $(R_0:l_F(\mathcal{O}_F^\times))=\mathrm{Reg}(F)$. On the other hand, the indices $(U_0:l_F^*(P_F))$ and $d(F)(l_F^*(P_F):l_F(C_F))$ have already been computed. Moreover, the identity

$$(R:U)=(s(G_F)R:s(G_F)U)(R_0:U_0),$$

together with the fact that $s(G_F)R=s(G_F)\mathbb{Z}$ and $s(G_F)U=|G_1|s(G_F)\mathbb{Z}$ shows that $(R:U)=[F:F\cap H](R_0:U_0)$. The theorem is now proved. □

Remark 2. If $F\subset H$, by definition we have $d(F)=1$. Actually we also have $d(F)=1$ in the case $H\subset F$, thanks to Lemma 2. In general there is no explicit formula for $d(F)$. However, if one of the following four conditions holds, then $d(F) = 1$.

(i) $s \in \{0, 1, 2\}$.
(ii) $\mathrm{Gal}(F/F \cap H)$ is the direct product of the inertia groups.
(iii) $\mathrm{Gal}(F/F \cap H)$ is cyclic.
(iv) $[F : F \cap H]$ is prime to $[H : F \cap H]$.

The proof of this claim is closely related to the theory of ordinary distributions ([Yin3] and [B-O]).

Let us now compute the index $[C_F : C_F^0]$. Let $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_t$ $(t \geqslant s)$ be prime ideals of $\mathcal{O}_K$ such that $\mathrm{Gal}(H/K) = \{(\mathfrak{p}_i, H/K), i = 1, \ldots, t\}$. Let Q^0 be the G_F-submodule of $F^\times$ generated by the elements

$$N_{H/F\cap H}\left(\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{p}_i)}\right)^{hf_{\mathfrak{m}}}, \quad i = 1, \ldots, t.$$

The G_F-submodule $\tilde{P}^0$ of $F^\times$ generated by P_F^0 and Q^0 is such that $l_F^*(\tilde{P}^0) = l_F^*(P_F)$ and $C_F^0 = \tilde{P}^0 \cap \mathcal{O}_F^\times$. In particular

$$[l_F^*(\tilde{P}^0) : l_F(C_F^0)] = [l^*(P_F) : l_F(C_F)][C_F : C_F^0].$$

Let us put $d^0(F) := [(\tilde{P}^0)^{w_F} \cap K : (Q^0)^{w_F} \cap K]$. Then, by slightly modifying the proof of Proposition 8, one may prove the identity

$$d^0(F)[l_F^*(\tilde{P}^0) : l_F(C_F^0)] = \prod_{i=1}^{s}[F_{\mathfrak{p}_i^{e_i}} : F \cap H][l_F^*(Q^0) : l_F(Q^0 \cap \mathcal{O}_F^\times)].$$

But $Q^0 \cap \mathcal{O}_F^\times = N_{H/F\cap H}(\Delta^0)^{f_{\mathfrak{m}}}$. Therefore we have

$$[l_F^*(Q^0) : l_F(Q^0 \cap \mathcal{O}_F^\times)] = [\omega s(G_1)R_0 : \omega s(G_1)R_0^2] = [F \cap H : K].$$

Thus we have proved

PROPOSITION 9. *We have* $d^0(F)[C_F : C_F^0] = d(F)[F \cap H : K]$.

Proof of Theorem A. Suppose we have $H \subset F$ or $F \subset H$. Then $d(F) = 1$, cf. Remark 2. Also one may show that $d^0(F) = 1$ in this case. The proof of this claim is similar to the proof of Lemma 2. On the other hand, we have

$$\begin{aligned}
[\mathcal{O}_F^\times : \Omega_F]h^{[F:K]-1} &= [\mathcal{O}_F^\times : \Omega_F][\Omega_F : \mu_F(\Omega_F)^h] \\
&= [\mathcal{O}_F^\times : \Omega_F][\Omega_F : C_F^0] \quad \text{(cf. Proposition 4)} \\
&= [\mathcal{O}_F^\times : C_F][C_F : C_F^0] \\
&= [\mathcal{O}_F^\times : C_F][F \cap H : K] \quad \text{(cf. Proposition 9)}.
\end{aligned}$$

This gives us the formula (1) in the introduction since the index $[\mathcal{O}_F^\times : C_F]$ is already computed, cf. Theorem 1. □

5. The Case of Ray Class Fields

The index formula of Theorem 1 can be both made explicit and improved in the case of ray class fields. The aim of this section and the last one is to explain how this can be obtained. So let us assume that $F = K_{\mathfrak{m}}$ and let $L := K_{(12f_{\mathfrak{m}}^2)}$. For technical reasons, we take $\mathfrak{m}$ prime to 6. Let D_K be the discriminant of K. A ideal $\mathfrak{b}$ of $\mathcal{O}_K$ is said to be primitive if it is not of the form $t\mathfrak{b}'$ for some integer $t > 1$ and some ideal $\mathfrak{b}'$ of $\mathcal{O}_K$. We begin this section with the following lemma. It gives the exact value of $w_{K_{\mathfrak{n}}}$ for $\mathfrak{n}$ prime to 6.

LEMMA 3. *Let* $\mathfrak{n}$ *be a ideal of* $\mathcal{O}_K$ *prime to* 6 *and let us write* $\mathfrak{n} = n_1\mathfrak{n}_2$, *with* $n_1 \in \mathbb{N} - \{0\}$ *and* $\mathfrak{n}_2$ *a primitive ideal of* $\mathcal{O}_K$. *Then* $f_{\mathfrak{n}} = n_1 N(\mathfrak{n}_2)$ *and* $w_{K_{\mathfrak{n}}} | 12 f_{\mathfrak{n}}$. *More precisely,* $w_{K_{\mathfrak{n}}} = w_H n_1 n_2^*$, *where* n_2^* *is the product of* $N(\mathfrak{p})$ *where* $\mathfrak{p}$ *are those prime ideals which divide* $\mathfrak{n}_2$ *and are ramified in* $K/\mathbb{Q}$.

Proof. It is a easy consequence of the famous Lemme 5 of [Rob1]. □

DEFINITION 6. We let $X_{\mathfrak{m}}$ be the Galois submodule of $L^\times$ generated by the values $\varphi(1; \omega_1, \omega_2)$, where (ω_1, ω_2) is any positive $\mathbb{Z}$-basis of any proper ideal $\mathfrak{n}$ of $\mathcal{O}_K$ that divide $\mathfrak{m}$.

DEFINITION 7. Let $\mathfrak{a}$ and $\mathfrak{b}$ be primitive ideals of $\mathcal{O}_K$ prime to $6D_K f_{\mathfrak{m}}$. Let us write $\mathfrak{a}\mathfrak{b} = t\mathfrak{c}$, with $t \geqslant 1$ and $\mathfrak{c}$ a primitive ideal of $\mathcal{O}_K$. Then we put

$$\eta(\mathfrak{a}, \mathfrak{b}) := \left(\sqrt{\kappa(t)t}\right)^{-1} \frac{\eta(\mathfrak{a})\eta(\mathfrak{b})}{\eta(\mathcal{O}_K)\eta(\mathfrak{c})},$$

where $\mathfrak{v} \longmapsto \eta(\mathfrak{v})$ is the η-function on primitive ideals of $\mathcal{O}_K$ that are prime to 6 introduced in [H-V], Definition 8, and $\kappa\colon (\mathcal{O}_K/12\mathcal{O}_K)^\times \to \mu_H$ is the character defined in [H-V], Definition 11 and Lemma 13 (see also the remark following the proof of Lemma 13).

Remark 3. Let us recall that $K(\eta(\mathfrak{a}, \mathfrak{b}))$ is Abelian over K, cf. loc. cit., Proposition 10 (ii). Moreover, we have

$$\eta(\mathfrak{a}, \mathfrak{b})^{24} = t^{12} \frac{\Delta(\bar{\mathfrak{a}})\Delta(\bar{\mathfrak{b}})}{\Delta(\mathcal{O}_K)\Delta(\bar{\mathfrak{c}})} = \frac{\Delta(\mathfrak{a}^{-1})\Delta(\mathfrak{b}^{-1})}{\Delta(\mathcal{O}_K)\Delta(\mathfrak{a}^{-1}\mathfrak{b}^{-1})}.$$

This proves that $\eta(\mathfrak{a}, \mathfrak{b})$ is a unit. On the other hand $H(\eta(\mathfrak{a}, \mathfrak{b}))$ is a Kummer extension of H and for $x \in \mathcal{O}_K$ prime to $6D_K\mathfrak{a}\mathfrak{b}$ we have

$$\eta(\mathfrak{a}, \mathfrak{b})^{\sigma_x - 1} = \kappa(x)^{-\frac{1}{2}(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)} \left(\sqrt{x}\right)^{\sigma_t - 1} \left(\sqrt{K(t)t}\right)^{\sigma_x - 1},$$

where σ_x (resp. σ_t) is the automorphism of K^{ab}/K associated to $x\mathcal{O}_K$ (resp. $t\mathcal{O}_K$) by the Artin map, cf. [H-V] Theorem 19 (i). By the quadratic reciprocity law stated in Theorem 21 of [H-V], we know that

$$\left(\sqrt{x}\right)^{\sigma_t - 1} \left(\sqrt{t}\right)^{\sigma_x - 1} = \left(\sqrt{\kappa(t)}\right)^{\sigma_x - 1}.$$

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As a consequence we obtain the formula

$$\eta(\mathfrak{a},\mathfrak{b})^{\sigma_x-1}=\kappa(x)^{-\frac{1}{2}(N(\mathfrak{a})-1)(N(\mathfrak{b})-1)}, \tag{5.1}$$

from which we deduce easily that $\eta(\mathfrak{a},\mathfrak{b})\in K_{(12)}$. If $\mathfrak{c}$ is a ideal of $\mathcal{O}_K$ prime to 6 we have

$$\eta(\mathfrak{a},\mathfrak{b})^{N(\mathfrak{c})-(\mathfrak{c},K_{(12)}/K)}\in H. \tag{5.2}$$

DEFINITION 8. We let $\tilde{V}_{\mathfrak{m}}$ be the Galois submodule of $L^{\times}$ generated by μ_L, $X_{\mathfrak{m}}$ and by all the quotients $\eta(\mathfrak{a},\mathfrak{b})$

Our goal now is to determine the index $[C^0_{K_{\mathfrak{m}}}:\mu_{K_{\mathfrak{m}}}(\mathcal{E}_{\mathfrak{m}})^{12w_Kf_{\mathfrak{m}}h}]$. But first we need some preleminary results.

PROPOSITION 10. *The group $V_{\mathfrak{m}}$ of Theorem B is such that $V_{\mathfrak{m}}=\tilde{V}_{\mathfrak{m}}\cap\mathcal{O}_L^{\times}$. Moreover, we have $\mu_L\tilde{V}_{\mathfrak{m}}^{12w_Kf_{\mathfrak{m}}h}=\mu_LP^0_{K_{\mathfrak{m}}}$.*

Proof. Obvious. □

PROPOSITION 11. *Let $\mathfrak{n}|\mathfrak{m}$ and $\mathfrak{n}\neq(1)$. Let $\omega:=(\omega_1,\omega_2)$ (resp. $\omega':=(\omega'_1,\omega'_2)$) be a positive $\mathbb{Z}$-basis of $\mathfrak{m}$ (resp. $\mathfrak{n}$). Then*

$$\frac{\varphi(1;\omega_1,\omega_2)^{\frac{N\mathfrak{m}}{N\mathfrak{n}}}}{\varphi(1;\omega'_1,\omega'_2)}\in\mu_{12}K_{\mathfrak{m}}.$$

Proof. The claim may be deduced from Satz (1.2) of [Sch]. It is also possible to prove it as follows. Since $\mathfrak{m}$ is prime to 6 we can consider the elliptic function $z\longmapsto\Psi(z;\mathfrak{m},\mathfrak{n})$ introduced by G. Robert ([Rob2] [Rob3]). We have

$$\Psi(z;\mathfrak{m},\mathfrak{n})=\frac{1}{C(\omega,\omega')}\frac{\varphi(1;\omega_1,\omega_2)^{\frac{N\mathfrak{m}}{N\mathfrak{n}}}}{\varphi(1;\omega'_1,\omega'_2)},$$

where $C(\omega,\omega')$ is a 12th root of unity depending on ω and ω' ([Rob3] Théorème 1(c) and Théorème 3(b). Moreover, $\Psi(1;\mathfrak{m},\mathfrak{n})\in K_{\mathfrak{m}}$ thanks to Théorème 5 of loc. cit. □

PROPOSITION 12. *Suppose $\mathfrak{m}\neq(1)$ and let $\omega:=(\omega_1,\omega_2)$ be a positive $\mathbb{Z}$-basis of $\mathfrak{m}$ and let $\mathfrak{b}$ be a ideal of $\mathcal{O}_K$ prime to $6\mathfrak{m}$. Then*

$$\varphi(1;\omega_1,\omega_2)^{N(\mathfrak{b})-(\mathfrak{b},L/K)}\in K_{\mathfrak{m}}.$$

Proof. By Théorème 1(c) and the corollaire of Section 6 of [Rob3] the elliptic function $z\longrightarrow\Psi(1;\mathfrak{m},\mathfrak{b}^{-1}\mathfrak{m})$ is such that

$$\varphi(1;\omega_1,\omega_2)^{N(\mathfrak{b})-(\mathfrak{b},L/K)}=\Psi(1;\mathfrak{m},\mathfrak{b}^{-1}\mathfrak{m}).$$

Now $\Psi(1;\mathfrak{m},\mathfrak{b}^{-1}\mathfrak{m})\in K_{\mathfrak{m}}$ is an immediate consequence of Théorème 5 of [Rob3].

PROPOSITION 13. *Suppose* $\mathfrak{m} \neq (1)$. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be primitive ideals of* $\mathcal{O}_K$*, prime to* $6D_K f_{\mathfrak{m}}$. *Let* $e(\mathfrak{a},\mathfrak{b}) := N(\mathfrak{m})(N(\mathfrak{a})-1/2)(\sigma_{\mathfrak{b}}-1)$, *where* $\sigma_{\mathfrak{b}} := (\mathfrak{b}, L/K)$. *Let* (ω_1,ω_2) *be a positive* $\mathbb{Z}$*-basis of* $\mathfrak{m}$*, then*

$$\eta(\mathfrak{a},\mathfrak{b})\varphi(1;\omega_1,\omega_2)^{e(\mathfrak{a},\mathfrak{b})} \in \mu_{12}K_{\mathfrak{m}}.$$

Proof. It suffices to prove the claim for a particular choice of (ω_1,ω_2), thanks to the formula (2.1) above. So, let us write $\mathfrak{m} = m_1\mathfrak{m}_2$, where $\mathfrak{m}_2$ is a primitive ideal of $\mathcal{O}_K$ and $m_1 \in \mathbb{N}-\{0\}$. Let $u \in \mathbb{Z}$ be such that $u \equiv -\sqrt{D_K}$ modulo $\bar{\mathfrak{b}}\mathfrak{m}_2$ and put $\alpha = (u+\sqrt{D_K})/2$. We have $f_{\mathfrak{m}} = m_1 N(\mathfrak{m}_2)$ and

$$\mathcal{O}_K = \mathbb{Z}\alpha+\mathbb{Z},\ \mathfrak{m}_2 = \mathbb{Z}\alpha+\mathbb{Z}N(\mathfrak{m}_2),\ \bar{\mathfrak{b}} = \mathbb{Z}\alpha+\mathbb{Z}N(\mathfrak{b}),\ \mathfrak{b}^{-1}\mathfrak{m} = \mathbb{Z}\frac{m_1\alpha}{N(\mathfrak{b})}+\mathbb{Z}f_{\mathfrak{m}}.$$

In particular $(m_1\alpha,\ f_{\mathfrak{m}})$ is a positive $\mathbb{Z}$-basis of $\mathfrak{m}$. Hence, Satz (1.1) of [Sch] gives us the identity

$$[i\varphi(1;m_1\alpha,f_{\mathfrak{m}})]^{\sigma_{\mathfrak{b}}} = i\varphi\left(1;\frac{m_1\alpha}{N(\mathfrak{b})},f_{\mathfrak{m}}\right)$$

which implies

$$\varphi(1;m_1\alpha,f_{\mathfrak{m}})^{\sigma_{\mathfrak{b}}-1} = \frac{\kappa(1,\mathfrak{b}^{-1}\mathfrak{m})\eta\left(\frac{\alpha}{N(\mathfrak{b}\mathfrak{m}_2)}\right)^2}{\kappa(1,\mathfrak{m})\eta\left(\frac{\alpha}{N(\mathfrak{m}_2)}\right)^2} i^{1-\sigma_{\mathfrak{b}}}.$$

On the other hand we have

$$\eta(\overline{\mathfrak{m}}_2) = \xi_1^{N(\mathfrak{m}_2)}\eta\left(\frac{\alpha}{N(\mathfrak{m}_2)}\right) \quad\text{and}\quad \eta(\mathfrak{b}\overline{\mathfrak{m}}_2) = \xi_2^{N(\mathfrak{b}\mathfrak{m}_2)}\eta\left(\frac{\alpha}{N(\mathfrak{b}\mathfrak{m}_2)}\right),$$

where $\xi_1,\xi_2 \in \mu_{48}$ ([H-V] Definition 8). Thus we obtain the decomposition

$$\eta(\mathfrak{a},\mathfrak{b})\varphi(1;m_1\alpha,f_{\mathfrak{m}})^{e(\mathfrak{a},\mathfrak{b})} = M_1M_2M_3,$$

with

$$M_1 = \eta(\mathfrak{a},\mathfrak{b})\left(\frac{\eta(\mathfrak{b}\overline{\mathfrak{m}}_2)}{\eta(\overline{\mathfrak{m}}_2)}\right)^{N(\mathfrak{m})(N(\mathfrak{a})-1)} \quad\text{and}\quad M_2 = \left(\frac{\kappa(1,\mathfrak{b}^{-1}\mathfrak{m})}{\kappa(1,\mathfrak{m})}\right)^{N(\mathfrak{m})\frac{(N(\mathfrak{a})-1)}{2}}.$$

While M_3 is the following 12th root of unity

$$M_3 = [i(\xi_1\xi_2)^2]^{-N(\mathfrak{m}_2)(\frac{N(\mathfrak{a})-1}{2})(N(\mathfrak{b})-1)}.$$

Now we claim that $M_2 \in K_{\mathfrak{m}}$. Indeed, this is a consequence of Theorem 2.1 in Chapter 12 of [K-L]. As for M_1 one may use Theorem 19 of [H-V] and the formula (5.1) above to show that $M_1 \in H$. The proposition is now proved. □

COROLLARY 3. *Suppose* $\mathfrak{m} \neq (1)$ *and let* $\pi := [C^0_{K_{\mathfrak{m}}} : \mu_{K_{\mathfrak{m}}}(\mathcal{E}_{\mathfrak{m}})^{12w_K f_{\mathfrak{m}} h}]$. *Then* π *divides* $w_{K_{\mathfrak{m}}}$. *If* $s=1$ *then* π *divides* $w_{K_{\mathfrak{m}}}/w_K$

Proof. Let us consider the two factor groups

$$\Pi := C^0_{K_{\mathfrak{m}}}/\mu_{K_{\mathfrak{m}}}(\mathcal{E}_{\mathfrak{m}})^{12w_K f_{\mathfrak{m}} h} \quad \text{and} \quad \Pi' := P^0_{K_{\mathfrak{m}}}/\mu_{K_{\mathfrak{m}}}(\tilde{V}_{\mathfrak{m}} \cap K_{\mathfrak{m}})^{12w_K f_{\mathfrak{m}} h}.$$

The inclusion $C^0_{K_{\mathfrak{m}}} \subset P^0_{K_{\mathfrak{m}}}$ induces a injective map $\Pi \to \Pi'$. On the other hand, the three Propositions 11, 12 and 13 show that Π' is generated as an Abelian group by the class of $\varphi_{\mathfrak{m}}^{w_K h}$. Recall that $w_{K_{\mathfrak{m}}}$ may be written as a finite sum $\sum n_{\mathfrak{a}}(N(\mathfrak{a})-1)$, $n_{\mathfrak{a}} \in \mathbb{Z}$, for some ideals $\mathfrak{a}$ of $\mathcal{O}_K$ prime to $6\mathfrak{m}$ and such that $(\mathfrak{a}, K_{\mathfrak{m}}/K) = 1$. Therefore, Proposition 12 implies that

$$\varphi_{\mathfrak{m}}^{w_K w_{K_{\mathfrak{m}}} h} \in \mu_{K_{\mathfrak{m}}}(\tilde{V}_{\mathfrak{m}} \cap K_{\mathfrak{m}})^{12w_K f_{\mathfrak{m}} h}.$$

In the case $s = 1$, the group Π is generated by the classes of $\varphi_{\mathfrak{m}}^{w_K h(\sigma-1)}$, $\sigma \in \mathrm{Gal}(K_{\mathfrak{m}}/K)$. But Π is cyclic. Hence π is the least positive integer such that

$$\varphi_{\mathfrak{m}}^{w_K h(\sigma-1)\pi} \in \mu_{K_{\mathfrak{m}}}(\tilde{V}_{\mathfrak{m}} \cap K_{\mathfrak{m}})^{12w_K f_{\mathfrak{m}} h}, \quad \text{for all } \sigma \in \mathrm{Gal}(K_{\mathfrak{m}}/K).$$

In particular $\pi | w_{K_{\mathfrak{m}}}/w_K$. □

PROPOSITION 14. *Suppose* $\mathfrak{m} \neq (1)$ *and let* $\pi' := w_K \pi$ *if* $s = 1$ *and* $\pi' := \pi$ *if* $s \geqslant 2$, *then* $w_{K_{\mathfrak{m}}}$ *divides* π'.

Proof. We deduce from the proof of Corollary 3 that

$$\varphi(1; \omega_1, \omega_2)^{\pi'} \in \mu_L K_{\mathfrak{m}},$$

where (ω_1, ω_2) is any positive $\mathbb{Z}$-basis of $\mathfrak{m}$. In other words there is $x \in K_{\mathfrak{m}}$ such that

$$\xi := x\varphi(1; \omega_1, \omega_2)^{\pi'} \in \mu_L \subset \mu_{24 f_{\mathfrak{m}}^2}. \tag{5.3}$$

Actually we may prove that $\xi \in \mu_{24 f_{\mathfrak{m}}} \cap L$. Even $\xi \in \mu_{12 f_{\mathfrak{m}}}$ in the cases $D_K \equiv 1, 4$ or 5 modulo 8. Indeed, Let $u \in \mathbb{Z}$ (resp. $u \in 4\mathbb{Z}$ if $2 | D_K$) be such that $u \equiv -\sqrt{D_K}$ modulo $\mathfrak{m}_2$, then put $\alpha := (u + \sqrt{D_K})/2$. We have $\mathcal{O}_K = \mathbb{Z}\alpha + \mathbb{Z}$ and $\mathfrak{m}_2 = \mathbb{Z}\alpha + \mathbb{Z}N(\mathfrak{m}_2)$. Now take $v := 1 + 12 f_{\mathfrak{m}}$ and let $\sigma_v = (v\mathcal{O}_K, L/K)$. We have

$$\xi^{\sigma_v - 1} = \xi^{N(v\mathcal{O}_K)-1} = \xi^{24 f_{\mathfrak{m}}} \quad \text{and} \quad [x\varphi(1; \omega_1, \omega_2)^{\pi'}]^{\sigma_v - 1} = 1$$

([Sch] Satz (1.2)). This makes it clear that $\xi \in \mu_{24 f_{\mathfrak{m}}} \cap L$. If $D_K \equiv 1$ or 5 modulo 8 then $w_L = 12 f_{\mathfrak{m}}^2$. If $D_K \equiv 4$ modulo 8 we have $w_L = 24 f_{\mathfrak{m}}^2$, but if we take $\lambda := 1 + 6 f_{\mathfrak{m}}^2 \alpha$ then

$$\xi^{\sigma_\lambda - 1} = \xi^{12 f_{\mathfrak{m}}} \quad \text{and} \quad [x\varphi(1; \omega_1, \omega_2)^{\pi'}]^{\sigma_\lambda - 1} = 1.$$

The last equality is a application of Satz (1.2) of [Sch]. Thus we may conclude that $\xi \in \mu_{12 f_{\mathfrak{m}}}$ for all $D_K \equiv 1, 4$ or 5 modulo 8. In such a situation (5.3) is possible only if $w_{K_{\mathfrak{m}}}$divides π', [Sch] Satz (1.3). It remains to prove the proposition in the case $D_K \equiv 0$ modulo 8. Since $\xi^2 \in \mu_{12 f_{\mathfrak{m}}}$, formula (5.3) and Satz (1.3) of [Sch] give $w_{K_{\mathfrak{m}}} | 2\pi'$. Let us

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remark that $w_{K_{\mathfrak{m}}}/2$ is odd ($D_K \equiv 0$ modulo 8). Hence we only need to prove that $2|\pi'$. Let λ be as above, we have

$$\xi^{\sigma_\lambda - 1} = 1 \quad \text{and} \quad [x\varphi(1;\omega_1,\omega_2)^{\pi'}]^{\sigma_\lambda - 1} = i^{2\pi'},$$

cf. [Sch] Satz (1.2). The proof of Proposition 14 is now complete.

THEOREM 2. *We have*

$$[C^0_{K_{\mathfrak{m}}} : \mu_{K_{\mathfrak{m}}}\mathcal{E}_{\mathfrak{m}}^{12 w_K f_{\mathfrak{m}} h}] = \begin{cases} \frac{w_{K_{\mathfrak{m}}}}{w_K} & \text{if } s = 0 \text{ or } s = 1 \\ w_{K_{\mathfrak{m}}} & \text{if } s \geqslant 2. \end{cases}$$

Proof. If $\mathfrak{m} \neq (1)$ the theorem is equivalent to Corollary 3 and Proposition 14. If $\mathfrak{m} = (1)$ and $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ all these groups are equal to μ_K. Now suppose $\mathfrak{m} = (1)$ and $w_K = 2$. Let $\mathfrak{c}_1, \ldots, \mathfrak{c}_r$ be primitive ideals of $\mathcal{O}_K$ prime to 6 and such that the gcd of $N(\mathfrak{c}_i) - 1$, $i = 1, \ldots, r$ is w_K. Then the factor group $C^0_H/\mu_H\mathcal{E}_{(1)}^{12 w_K h}$ is cyclic generated by the image of ϕ^{24h}, where

$$\phi := \prod_{i,j} \eta(\mathfrak{c}_i, \mathfrak{c}_j)^{n_i n_j},$$

the integers $n_1, \ldots, n_r$ satisfying $2 = \sum n_i(N(\mathfrak{c}_i) - 1)$. Indeed, this follows from (5.2) and the property (5.1) which implies

$$\eta(\mathfrak{a}, \mathfrak{b})\phi^{\frac{(N(\mathfrak{a})-1)}{2}\frac{(N(\mathfrak{b})-1)}{2}} \in H,$$

for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{O}_K$ prime to 6. Now let $d \in \mathbb{Z}$ be such that $\phi^{24d} \in \mu_H H^{24h}$. In particular there is $x \in H$ such that $\xi := x\phi^d$ is a root of unity, say of order n. By Lemma 14 (ii) of [H-V] there is a $\lambda \in \mathcal{O}_K$ prime to 6 such that $N(\lambda\mathcal{O}_K) \equiv 1$ modulo n and $\kappa(\lambda)$ is a primitive w_Hth root of unity. In particular, $\xi^{\sigma_\lambda - 1} = 1$. On the other hand we have $\xi^{\sigma_\lambda - 1} = \left[x\phi^d\right]^{\sigma_\lambda - 1} = \kappa(\lambda)^{-2d}$, thanks to (5.1). This proves that $w_H/2|d$. The theorem is now proved. □

6. Some General Properties of the Index $(R : U)$

Let $\widehat{\mathfrak{m}} := \mathfrak{p}_1 \cdots \mathfrak{p}_s$. For each ideal $\mathfrak{r}$ of $\mathcal{O}_K$ such that $\mathfrak{r}\,|\widehat{\mathfrak{m}}$ we denote by $T_{\mathfrak{r}}$ the compositum in G_F of the inertia groups $T_{\mathfrak{p}}$ as $\mathfrak{p}$ varies through the maximal ideals dividing $\mathfrak{r}$. In particular we have $T_{(1)} = \{1\}$, $T_{\widehat{\mathfrak{m}}} = G_1$; and in general $T_{\mathfrak{r}} = \mathrm{Gal}(F/F_{\mathfrak{n}(\mathfrak{r})})$, where $\mathfrak{n}(\mathfrak{r})$ is the largest divisor of $\mathfrak{m}$ coprime with $\mathfrak{r}$. This implies

PROPOSITION 15. *U is generated as an R-module by the elements $s(T_{\mathfrak{r}})\prod_{\mathfrak{p}|\mathfrak{n}(\mathfrak{r})}(1 - (\mathfrak{p}, F)) = \alpha_{\mathfrak{n}(\mathfrak{r})}$, where $\mathfrak{r}$ varies over the divisors of $\widehat{\mathfrak{m}}$.*

Let $\mathfrak{s}$ be a divisor of $\widehat{\mathfrak{m}}$. We denote by $U_{\mathfrak{s}}$ the R-module generated in $\mathbb{Q}[G_F]$ by the elements $s(T_{\mathfrak{r}})\prod_{\mathfrak{p}|\mathfrak{s}/\mathfrak{r}}(1 - (\mathfrak{p}, F))$, where $\mathfrak{r}$ varies over the divisors of $\mathfrak{s}$. Hence, $U_{(1)} = R$ and $U_{\widehat{\mathfrak{m}}} = U$. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_K$ such that $\mathfrak{p}$ divides $\widehat{\mathfrak{m}}$ but not $\mathfrak{S}$. Then we have $U_{\mathfrak{S}\mathfrak{p}} = U_{\mathfrak{S}}(T_{\mathfrak{p}}) + (1 - (\mathfrak{p}, F))U_{\mathfrak{S}}$, where $U_{\mathfrak{S}}(T_{\mathfrak{p}})$ is the R-module generated by the elements $s(T_{\mathfrak{r}\mathfrak{p}}) \prod_{\mathfrak{q}|\mathfrak{S}/\mathfrak{r}}(1 - (\mathfrak{q}, F))$. As in [Sin2] Lemma 5.1 one may prove that $U_{\mathfrak{S}}$ is a lattice of $\mathbb{Q}[G_F]$ and that the index $(U_{\mathfrak{S}} : U_{\mathfrak{S}\mathfrak{p}})$ is an integer divisible only by the primes dividing $|T_{\mathfrak{p}}|$. On the other hand the expression

$$(R : U) = \prod_{i=0}^{s-1} (U_{\mathfrak{r}_i} : U_{\mathfrak{r}_{i+1}}), \tag{6.1}$$

of $(R : U)$ as a product of indices of the form $(U_{\mathfrak{r}} : U_{\mathfrak{r}\mathfrak{p}})$, with $\mathfrak{p} \nmid \mathfrak{r}$; where $\mathfrak{r}_0 := (1)$ and $\mathfrak{r}_i := \mathfrak{r}_{i-1}\mathfrak{p}_i$ for $i = 1, \ldots, s$, implies the following

PROPOSITION 16. *The index* $(R : U)$ *is an integer divisible only by the primes dividing* $|G_1|$. *Moreover, if at most two ideals ramify in* F/K, *or if* G_1 *is the direct product of the inertia groups, then* $(R : U) = 1$.

7. The Index $(R : U)$ in the Case of Ray Class Fields

In this Section we are interested in computing the index $(R : U)$ for ray class Fields. Recall that $(R : U) = 1$ if $s = 1$ or 2, cf. Proposition 16. Thus we can assume that $s \geqslant 3$. On the other hand, we are able to make this computation only when $hf_{\mathfrak{m}}$ is prime to w_K. So, throughout this section we suppose that $F = K_{\mathfrak{m}}$, $s \geqslant 3$ and $\gcd(f_{\mathfrak{m}}h, w_K) = 1$.

Remark 4. Let $\mathfrak{n}$ be a proper ideal of $\mathcal{O}_K$ prime to w_K. The global class field theory gives the exact sequence

$$1 \longrightarrow \mu_K \longrightarrow (\mathcal{O}_K/\mathfrak{n})^{\times} \longrightarrow \mathrm{Gal}(K_{\mathfrak{n}}/H) \longrightarrow 1.$$

The order of $(\mathcal{O}_K/\mathfrak{n})^{\times}$ is usually denoted $\varphi(\mathfrak{n})$. We have $\varphi(\mathfrak{n}) = \prod_{\mathfrak{p}^e||\mathfrak{n}} \varphi(\mathfrak{p}^e)$ and $\varphi(\mathfrak{p}^e) = N(\mathfrak{p})^{e-1}(N(\mathfrak{p}) - 1)$. This enables us to make the following deductions.

(1) The ramification index at $\mathfrak{p}_i$ in $K_{\mathfrak{m}}/K$ is equal to $\varphi(\mathfrak{p}_i^{e_i})$ because we have $T_{\mathfrak{p}_i} = \mathrm{Gal}(K_{\mathfrak{m}}/K_{\mathfrak{m}_i})$, where $\mathfrak{m}_i := \mathfrak{m}\mathfrak{p}_i^{-e_i}$.
(2) Let $\mathfrak{r}$ be a divisor of $\hat{\mathfrak{m}}$ such that $\mathfrak{r} \neq \hat{\mathfrak{m}}$. Since $T_{\mathfrak{r}} = \mathrm{Gal}(K_{\mathfrak{m}}/K_{\mathfrak{n}(\mathfrak{r})})$ we have

$$\#T_{\mathfrak{r}} = \frac{\varphi(\mathfrak{m})}{\varphi(\mathfrak{n}(\mathfrak{r}))} = \prod_{\mathfrak{p}_i|\mathfrak{r}} \varphi(\mathfrak{p}_i^{e_i}).$$

This proves that $T_{\mathfrak{r}}$ is a direct product of $T_{\mathfrak{p}_i}$, $\mathfrak{p}_i|\mathfrak{r}$.

In particular if $\mathfrak{r}|\widehat{\mathfrak{m}}$ and $\mathfrak{r}'|\widehat{\mathfrak{m}}$, are coprime such that $\mathfrak{r}\mathfrak{r}' \neq \widehat{\mathfrak{m}}$, then we have $T_{\mathfrak{r}} \cap T_{\mathfrak{r}'} = \{1\}$. This may be used to prove that $U_{\mathfrak{r}}$ is free over $T_{\mathfrak{r}'}$, cf. [Sin1] Proposition 5.2. Let us suppose that $\mathfrak{r}' = \mathfrak{p}$ is a maximal ideal of K. Then we have

$$(U_{\mathfrak{r}} : U_{\mathfrak{r}\mathfrak{p}}) = \#B/(1 - \mathcal{F}_{\mathfrak{p}}^{-1})B, \tag{7.1}$$

where $B := B(\mathfrak{r}, \mathfrak{p}) := U_{\mathfrak{r}}^{T_{\mathfrak{p}}}/U_{\mathfrak{r}}(T_{\mathfrak{p}})$ ([Sin2], Lemma 5.1). But if $\mathfrak{r}\mathfrak{p} \neq \widehat{\mathfrak{m}}$ the intersection $T_{\mathfrak{r}} \cap T_{\mathfrak{p}} = \{1\}$ and then $U_{\mathfrak{r}}$ is a free $T_{\mathfrak{p}}$-module. In particular $U_{\mathfrak{r}}^{T_{\mathfrak{p}}} = s(T_{\mathfrak{p}})U_{\mathfrak{r}} = U_{\mathfrak{r}}(T_{\mathfrak{p}})$ and consequently the index $(U_{\mathfrak{r}} : U_{\mathfrak{r}\mathfrak{p}}) = 1$. The formula (6.1) becomes

$$(R : U) = (U_{\mathfrak{r}_{s-1}} : U) = \left[U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}} : (1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}} + U_{\mathfrak{r}_{s-1}}(T_{\mathfrak{p}_s})\right].$$

The last equality is a application of (7.1). Let X and Y be the R-modules defined as follows

$$X := U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}/(1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}} + s(T_{\mathfrak{p}_s})U_{\mathfrak{r}_{s-1}},$$
$$Y := U_{\mathfrak{r}_{s-1}}(T_{\mathfrak{p}_s}) + (1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}/s(T_{\mathfrak{p}_s})U_{\mathfrak{r}_{s-1}} + (1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}.$$

Then X/Y is obviously isomorphic to $U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}/(1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}} + U_{\mathfrak{r}_{s-1}}(T_{\mathfrak{p}_s})$. Thus we have $(R : U) = [X : Y]$. On the other hand

$$X \simeq N/(1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})N \quad \text{with} \quad N := U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}/s(T_{\mathfrak{p}_s})U_{\mathfrak{r}_{s-1}}.$$

It is not difficult to determine the structure of Y as an R-module. Indeed, we have

LEMMA 4. *We have*

$$U_{\mathfrak{r}_{s-1}}(T_{\mathfrak{p}_s}) = s(T_{\mathfrak{p}_s})U_{\mathfrak{r}_{s-1}} + s(G_1)R. \tag{7.2}$$

Moreover, let $\tilde{D}$ be the subgroup of $G := \mathrm{Gal}(K_{\mathfrak{m}}/K)$ generated by $G_1 := \mathrm{Gal}(K_{\mathfrak{m}}/H)$ and by $\mathcal{F}_{\mathfrak{p}_i}, i = 1, \ldots, s$. Then we have

$$s(G_1)R \cap (s(T_{\mathfrak{p}_s})U_{\mathfrak{r}_{s-1}} + (1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}) = s(G_1)(I + w_K R), \tag{7.3}$$

where I is the augmentation ideal of the group ring $\mathbb{Z}[\tilde{D}]$.

Proof. The identity (7.2) is easy to verify using the definitions and the fact that $T_{\mathfrak{r}\mathfrak{p}_s}$ is the direct product of $T_{\mathfrak{r}}$ and $T_{\mathfrak{p}_s}$ for every proper divisor $\mathfrak{r}$ of $\mathfrak{r}_{s-1}$. Let us prove (7.3). Since G_1 is generated by all $T_{\mathfrak{p}_i}$ the trace $s(G_1) \in U_{\mathfrak{r}_{s-1}}^{T_{\mathfrak{p}_s}}$, moreover, $w_K s(G_1) = s(T_{\mathfrak{p}_s})s(T_{\mathfrak{r}_{s-1}})$ because $\#T_{\mathfrak{p}_s} \cap T_{\mathfrak{r}_{s-1}} = w_K$. On the other hand, if $i \in \{1, \ldots, s-1\}$ we have

$$s(G_1)(1 - \mathcal{F}_{\mathfrak{p}_i}^{-1}) = \gamma_i s(T_{\mathfrak{p}_s})s(T_{\mathfrak{r}_{s-1}\mathfrak{p}_i^{-1}})(1 - (\mathfrak{p}_i, F))$$

for some $\gamma_i \in R$. Thus if we denote by E the R-module on the left-hand side of (7.3), then $s(G_1)(I + w_K R) \subset E$. Conversely, E may be written in the form $E = w_K s(G_1)R + V$, where V is such that $V \subset \mathbb{Q}I \cap s(G_1)R$. But the identity $\mathbb{Q}I \cap s(G_1)R = s(G_1)I$, shows that $E \subset s(G_1)(I + w_K R)$. $\square$

COROLLARY 4. *We have the isomorphisms*

$$Y \simeq s(G_1)R/s(G_1)(I + w_K R) \simeq \mathbb{Z}[G/\tilde{D}]/w_K\mathbb{Z}[G/\tilde{D}].$$

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Let D be the subgroup of $\mathrm{Gal}(H/K)$ *generated by* $(\mathfrak{p}_i, H/K)$*,* $i = 1, \dots, s$*. Then we have*

$$\#Y = w_K^e, \quad e := [\mathrm{Gal}(H/K) : D]. \tag{7.4}$$

Proof. The first isomorphism is a consequence of the definition of Y and Lemma 4. The second one is clear. Now since $G/\tilde{D} \simeq \mathrm{Gal}(H/K)/D$ the last assertion follows.

We devote the remaining of this Section to the determination of the structure of X. Let J be the subgroup of G generated by the ℓ-parts of $T_{\mathfrak{p}_s}$, $\ell | w_K$. We have $T_{\mathfrak{r}_{s-1}} \cap T_{\mathfrak{p}_s} \subset J$. On the other hand, the group J is cyclic since

$$T_{\mathfrak{p}_s} \simeq (\mathcal{O}_K/\mathfrak{p}^e)^\times \simeq \mathbb{Z}/(N\mathfrak{p} - 1)\mathbb{Z} \times \mathcal{O}_K/\mathfrak{p}^{e-1},$$

where $\mathfrak{p} = \mathfrak{p}_s$, $e = e_s$. Let $Z_{\mathfrak{p}_s}$ be the subgroup of $T_{\mathfrak{p}_s}$ such that $T_{\mathfrak{p}_s} = J \times Z_{\mathfrak{p}_s}$. If $\mathfrak{r}$ is a divisor of $\widehat{\mathfrak{m}}$ then we let $Z_{\mathfrak{r}}$ be the subgroup of G generated by $Z_{\mathfrak{p}_s}$ if $\mathfrak{p}_s | \mathfrak{r}$ and by the inertia groups $T_{\mathfrak{p}}$, $\mathfrak{p} | \mathfrak{r}$ and $\mathfrak{p} \neq \mathfrak{p}_s$, thus

$$Z_{\mathfrak{r}} := \begin{cases} \prod_{\mathfrak{p}|\mathfrak{r}} T_{\mathfrak{p}}, & \text{if } \mathfrak{p}_s \nmid \mathfrak{r}, \\ Z_{\mathfrak{p}_s} \times \prod_{\substack{\mathfrak{p}|\mathfrak{r} \\ \mathfrak{p} \neq \mathfrak{p}_s}} T_{\mathfrak{p}}, & \text{if } \mathfrak{p}_s \mid \mathfrak{r}. \end{cases}$$

LEMMA 5. *We have* $T_{\mathfrak{r}_{s-1}} \cap Z_{\mathfrak{p}_s} = \{1\}$*. Moreover if* $\mathfrak{r}$ *and* $\mathfrak{r}'$ *are coprime such that* $\mathfrak{p}_s \nmid \mathfrak{r}$*. Then* $T_{\mathfrak{r}} \cap Z_{\mathfrak{r}'} = \{1\}$*.*

Proof. Clear. □

Using Lemma 5 one may show, as in [Sin1] Proposition 5.2, that if $\mathfrak{r}$ and $\mathfrak{p}_s\mathfrak{r}'$ are coprime then $U_{\mathfrak{r}}$ is free over $Z_{\mathfrak{r}'}$. In particular, for $\mathfrak{p} = \mathfrak{p}_s$ and $\mathfrak{r} = \mathfrak{r}_{s-1}$ we have $N = U_{\mathfrak{r}}^{T_{\mathfrak{p}}}/s(T_{\mathfrak{p}})U_{\mathfrak{r}} = H^2(J, U_{\mathfrak{r}}^{Z_{\mathfrak{p}}})$. Let us put for $i = 0$ to $s-1$, $B_i^n := H^n(J, U_{\mathfrak{r}_i}^{Z_{\mathfrak{r}'_i}})$, where $\mathfrak{r}'_i$ is such that $\mathfrak{r}_i\mathfrak{r}'_i = \widehat{\mathfrak{m}}$. Since J is finite, we have $\#JB_i^n = 0$. On the other hand, we see that $s(J)s(Z_{\widehat{\mathfrak{m}}}) = w_K s(G_1)$. Hence, since R is a free $Z_{\widehat{\mathfrak{m}}}$-module we have

$$B_0^{2n} = (R^{Z_{\widehat{\mathfrak{m}}}})^J / s(J) R^{Z_{\widehat{\mathfrak{m}}}} = R^{G_1}/s(J)s(Z_{\widehat{\mathfrak{m}}})R \simeq (\mathbb{Z}/w_K\mathbb{Z})[G/G_1].$$

Moreover, $B_0^{2n+1} = H^1(J, R^{Z_{\widehat{\mathfrak{m}}}}) = 0$ by [Sin2] Lemma 5.2. Let $i \in \{1, \dots, s-1\}$, then J and $Z_{\mathfrak{r}'_i}$ act trivially on B_i^n. In fact the group $T_{\mathfrak{r}_i}$ also acts trivially on B_i^n. The proof is exactly the same as for Lemma 5.3 of [Sin1]. But G_1 is generated by J, $Z_{\mathfrak{r}'_i}$ and by $T_{\mathfrak{r}_i}$. Thus B_i^n is naturally a $\mathbb{Z}[\mathrm{Gal}(H/K)]$-module.

PROPOSITION 17. *We have an exact sequence of* R*-modules*

$$0 \longrightarrow B_i^n/(1 - \mathcal{F}_{\mathfrak{p}_{i+1}}^{-1}) \longrightarrow B_{i+1}^n \longrightarrow (B_i^{n+1})^{\mathcal{F}_{\mathfrak{p}_{i+1}}} \longrightarrow 0 \tag{7.5}$$

Proof. We refer to the proof of Proposition 6.3 of [Yin1].

The exact sequence (7.5) splits in our case because w_K and h are supposed to be coprime, see [Yin1], Lemma 6.5. Hence using induction we obtain the structure of B_i^n, $i \in \{1, \dots, s-1\}$. Indeed we have

$$B_i^n \simeq ((\mathbb{Z}/w_K\mathbb{Z})[G_{\mathfrak{m}}/D^{(i)}])^{2^{i-1}}, \tag{7.6}$$

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where $D^{(i)}$ is the subgroup of $G_{\mathfrak{m}}$ generated by G_1 and by the Frobenius automorphisms $\mathcal{F}_{\mathfrak{p}_j}, j \in \{1, \ldots, i\}$. □

COROLLARY 5. *We have*

$$X = B_{s-1}^2/(1 - \mathcal{F}_{\mathfrak{p}_s}^{-1})B_{s-1}^2 \simeq ((\mathbb{Z}/w_K\mathbb{Z})[G_{\mathfrak{m}}/D^{(s)}])^{2^{s-2}}.$$

In particular, X has order $\#X = (w_K)^{e(2^{s-2})}$.

Putting the results of Proposition 16 together with the results of Corollaries 4 and 5, we get

PROPOSITION 18. *Suppose* $F := K_{\mathfrak{m}}$, *then*

$$(R : U) = \begin{cases} 1, & \text{if } s = 0, 1 \text{ or } 2, \\ (w_K)^{e(2^{s-2}-1)}, & \text{if } s \geqslant 3 \text{ and } f_{\mathfrak{m}}h \text{ prime to } w_K. \end{cases}$$

Proof of Theorem B. The formula (2) is equivalent to Theorem 2 since $C_{K_{\mathfrak{m}}}^0 = \mu_{K_{\mathfrak{m}}}\Omega_{K_{\mathfrak{m}}}^h$. The formula (3) may be deduced from the following identities:

$$\begin{aligned}
&[\mathcal{O}_{K_{\mathfrak{m}}}^\times : \Omega_{K_{\mathfrak{m}}}][\Omega_{K_{\mathfrak{m}}} : \mu_{K_{\mathfrak{m}}}(\mathcal{E}_{\mathfrak{m}})^{12w_Kf_{\mathfrak{m}}}] \\
&\quad = [\mathcal{O}_{K_{\mathfrak{m}}}^\times : \mathcal{E}_{\mathfrak{m}}][\mathcal{E}_{\mathfrak{m}} : \mu_{K_{\mathfrak{m}}}(\mathcal{E}_{\mathfrak{m}})^{12w_Kf_{\mathfrak{m}}}] \\
&\quad = [\mathcal{O}_{K_{\mathfrak{m}}}^\times : \mathcal{E}_{\mathfrak{m}}](12w_Kf_{\mathfrak{m}})^{[K_{\mathfrak{m}}:K]-1}.
\end{aligned}$$

The index $(\mathbb{Z}[\mathrm{Gal}(K_{\mathfrak{m}}/K)] : U)$ has already been computed, (Propositions 16 and 18). (Let us remark that $w_K \neq 2$ only when $K = \mathbb{Q}(\sqrt{-1})$ or $K = \mathbb{Q}(\sqrt{-3})$, but in these two cases, we have $h = 1$ and $w_K = 4$ (resp. $w_K = 6$). In particular, h is prime to w_K if and only if h is odd.) On the other hand,

$$\frac{\prod_{\mathfrak{p}}[K_{\mathfrak{m}} \cap K_{\mathfrak{p}^\infty} : H]}{[K_{\mathfrak{m}} : H]} = \begin{cases} 1 & \text{if } \mathfrak{m} = (1) \\ w_K^{1-s} & \text{if } \mathfrak{m} \neq (1). \end{cases}$$

by Remark 4 above (recall $\mathfrak{m}$ is prime to 6). Theorem B is now proved. □

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