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Hilbert functions and generic forms

JÜRGEN HERZOG[1] and DORIN POPESCU[2]
[1] *FB 6 Mathematik und Informatik, Universität – GHS – Essen, Postfach 103764, 45117 Essen, Germany, e-mail: mat300@uni-essen.de*
[2] *Institute of Mathematics, University of Bucharest, P.O. Box 1-764, Bucharest 70700, Romania, e-mail: dpopescu@roimar.imar.ro, or: dorin@stoilow.imar.ro*

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Abstract. Let A be a homogeneous K-algebra where K is a field of characteristic 0, and $h \in A$ a generic form. We bound the Hilbert function $H(A/(h), -)$ in terms of $H(A, -)$ which extends the bound given by M. Green for generic linear forms. We apply this to some conjectures from Higher Castelnuovo Theory and Cayley–Bacharach Theory.

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0. Introduction

This paper is an attempt to understand what happens to the Hilbert function $H(R, d)$ of a homogeneous K-algebra R after reduction modulo a homogeneous generic form h of arbitrary degree. In other words, we want to compare $H(R/hR, d)$ with $H(R, d)$.

There are several motivations to study this question. The same problem was studied by M. Green for a generic linear form h, and applied to give short and nice proofs of Macaulay's theorem and the Gotzmann theorems; see [14] and [5]. Green's theorem says that $H(R/hR, d) \leqslant H(R, d)_{\langle d\rangle}$ for all d. Here the operator $a_{\langle d\rangle}$ is defined for integers a as follows: consider the dth Macaulay expansion of a, that is, the expansion

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(j)}{j}$$

with $k(d) > k(d-1) > \cdots > k(j) \geqslant j \geqslant 1$. Such an expansion exists and is unique. Then one defines

$$a_{\langle d\rangle} = \binom{k(d)-1}{d} + \binom{k(d-1)-1}{d-1} + \cdots + \binom{k(j)-1}{j}.$$

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There are similar operators defined using Macaulay expansions, and indeed, the theorem of Macaulay as well as the Kruskal–Katona theorem on the possible f-vectors of a simplicial complex are described by such operators, cf. [1].

In order to formulate the main result of the paper we have to introduce still some more of these operators. So for $d > i \geqslant 0$ we define

$$a_{\langle d,i\rangle} = \binom{k(d)-i-1}{d-i} + \binom{k(d-1)-i-1}{d-i-1} + \cdots + \binom{k(t)-i-1}{t-i},$$

where $t = j$ if $j > i$ and $t = i+1$ if $j \leqslant i$. Finally we set

$$a_{\langle\langle d,i\rangle\rangle} = a_{\langle d,i\rangle} + \binom{k(i)-i}{0},$$

where as usual $\binom{k(i)-i}{0} = 1$ if $k(i) \geqslant i$ and 0 otherwise.

Now the main result of this paper is the following

THEOREM. *Let R be a homogeneous K-algebra where K is a field of characteristic* 0, *and let $h \in R_s$ be a generic homogeneous form. Then*

$$H(R/hR, d) \leqslant \sum_{i=0}^{s-1} H(R,d)_{\langle\langle d,i\rangle\rangle}$$

for all $d \geqslant s$.

Unfortunately we have to require in our theorem that K is a field of characteristic 0. We believe however that the theorem is true in all characteristics. The reason for this unpleasant hypothesis is the method of our proof. It follows very closely the arguments of Bigatti [3] who, in her thesis, proved among other theorems, Green's theorem along the following lines: One first proves it for lexsegment ideals, then compares with strongly stable ideals which, and this is only true in characteristic 0, are generic initial ideals of arbitrary ideals. A direct generalization of Green's original proof seems to be not possible since the numerical identities which we would need are not valid, as shown in Remark 3.9.

Another remark concerning the theorem is that the right hand sum of the inequality could as well be expressed as a difference of two terms. But in our opinion sums are easier to control, so that we prefer this presentation.

As applications we have some results about Gotzmann spaces in a polynomial ring R which are related to results of Gasharov [12]. In particular we show that if I is an ideal generated by a Gotzmann space and h is a generic linear form for R/I, then all the powers of h are generic for R/I, a fact which fails badly for arbitrary homogeneous ideals.

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Even more interesting may be the application to the Eisenbud–Green–Harris Conjecture [7] and [8] which in algebraic terms states that if A is the quotient ring of the polynomial ring $K[x_1, \ldots, x_n]$ by m quadratic forms which are K-linearly independent and such that $\dim_K A < \infty$, and if $\binom{a}{2} + b$ is the dth Macaulay expansion of $H(A, 2) = \binom{n+1}{2} - m$, then $\dim_K A \leqslant 2^a + 2^b + n - a - 1$. We prove the conjecture in case K is of characteristic 0, the first n quadrics form a regular sequence and remaining quadrics are generic. The proof is an easy consequence of the following slightly more general result which is a special case of Conjecture (V_m) of [7]:

PROPOSITION. *Let B a zero dimensional complete intersection defined by quadric forms, A a factor ring of B defined by generic quadratic forms of A, and let $H(A, 2) = \binom{a}{2} + b$ with $a > b$. Then*

$$H(A, d) \leqslant \binom{a}{d} + \binom{b}{d-1} \quad \textit{for all } d \geqslant 2.$$

With the same methods we prove Conjecture $(\mathrm{III}_{k,r})$ of [7] which is a generalized Cayley–Bacharach theorem, again only under some assumption on genericness. This result would also follow from a theorem of Stanley, but he uses the Hard Lefshetz theorem whose proof is non-algebraic.

Our theorem however does not give very good results for the Fröberg Conjecture [10] on generic algebras. For example, the generic algebra defined by 6 quadrics in $K[x_1, x_2, \ldots, x_5]$ should have the Hilbert function $1 + 5t + 9t^2 + 5t^3$. But from our theorem it follows only that the Hilbert function of the algebra is coefficientwise bounded by the polynomial $1 + 5t + 9t^2 + 7t^3 + 2t^4$. Nevertheless we hope that the methods presented in our paper can be used in the future to give some contribution to Fröberg's conjecture.

1. Stable ideals and homogeneous generic forms

In this introductionary section we recall the notion of stable, strongly stable and lexsegment ideals, and prove some basic facts needed in later sections.

Let K be an infinite field and $R = K[x_1, \ldots, x_n]$ the polynomial ring with the standard grading $R = \oplus_{d \geqslant 0} R_d$ where R_d is the linear space spanned by all monomials of degree d. Throughout the paper we consider the deglex order $<$ on the set of monomials of R.

Given a monomial x^a, $a = (a_1, \ldots, a_n)$, we denote by $m(x^a)$, following Bigatti (cf. [3], or [4]) the greatest i, $1 \leqslant i \leqslant n$ such that $a_i > 0$.

Let $J \subset R_d$ be a set of monomials, and let $\langle J \rangle$ be the K-subspace of R_d spanned by J. Recall the following

DEFINITION 1.1. The set J of monomials is called

(1) *stable*, if $x_i(u/x_{m(u)}) \in J$ for any $u \in J$ and $1 \leqslant i \leqslant m(u)$;
(2) *strongly stable*, if $x_i(u/x_j) \in J$ for any $u \in J$ and $1 \leqslant i < j \leqslant n$ such that x_j divides u;
(3) *lexsegment* if for all $u \in J$ and all $v \in R_d$ with $v > u$ it follows that $v \in J$.

We will say that $\langle J \rangle$ has one of the above properties if J does. Finally, a monomial ideal I is said to be (strongly) stable or lexsegment, if all its homogeneous components I_d have this property.

LEMMA 1.2. *Let $I \subset R$ be a stable ideal and s a positive integer. Then $(I : x_n^s) = (I : R_s)$.*

Proof. Let $d \geqslant s$, $d \in \mathbb{N}$, $u \in R_{d-s}$ be such that $x_n^s u \in I_d$. Then for each i, $1 \leqslant i < n$, we have $x_i x_n^{s-1} u \in I_d$ because I is stable. Thus, $R_1 x_n^{s-1} u \subset I_d$. Using induction on s we obtain $R_s u \subset I_d$, that is, $(I : x_n^s) \subset (I : R_s)$. The other inclusion is obvious. □

LEMMA 1.3. *Let $I \subset R$ be a graded ideal, d and s two positive integers with $d \geqslant s$, and h a homogeneous form of R_s. Then*

(i) $\dim_K(I_d : h)_{d-s} = \dim_K(I_d \cap hR_{d-s})$;
(ii) $\dim_K(I_d \cap hR_{d-s}) \geqslant \dim_K(I_d \cap x_n^s R_{d-s})$, *if I is stable.*

Proof. (i) The map $(I_d : h)_{d-s} \to I_d \cap hR_{d-s}$ given by $v \to hv$ is clearly a linear K-isomorphism.

(ii) By Lemma 1.2 we have $(I_d : x_n^s)_{d-s} \subset (I_d : h)_{d-s}$. Thus, using (i) we get

$$\begin{aligned}\dim_K(I_d \cap x_n^s R_{d-s}) &= \dim_K(I_d : x_n^s)_{d-s} \leqslant \dim_K(I_d : h)_{d-s} \\ &= \dim_K(I_d \cap hR_{d-s}).\end{aligned}$$

PROPOSITION 1.4. *Let I be a stable ideal. Then with the assumptions and the notation of 1.3 we have*

$$\dim_K(I_d + hR_{d-s}) \leqslant \dim_K(I_d + x_n^s R_{d-s}).$$

In particular, x_n^s is generic for I and

$$H(R/(I, x_n^s), d) \leqslant H(R/(I, h), d).$$

Here $H(R/I, -)$ denotes the Hilbert function associated with the graded K-algebra R/I.

Proof. By Lemma 1.3 (ii) we have

$$\begin{aligned}\dim_K(I_d + hR_{d-s}) &= \dim_K I_d + \dim_K R_{d-s} - \dim_K(I_d \cap hR_{d-s}) \\ &\leqslant \dim_K I_d + \dim_K R_{d-s} - \dim_K(I_d \cap x_n^s R_{d-s}) \\ &= \dim_K(I_d + x_n^s R_{d-s}).\end{aligned}$$

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Let $J \subset R_d$ be a set of monomials. Following [4] we set □

$$J_{x_n^i} = (J : x_n^i) \cap \{x_1, \dots, x_{n-1}\}^{d-i}$$

for all $0 \leqslant i \leqslant d$, where $\{x_1, \dots, x_{n-1}\}^{d-i}$ denotes the set of monomials in $x_1, \dots, x_{n-1}$ of degree $d - i$. By abuse of notation we set also $\langle J \rangle_{x_n^i} = J_{x_n^i}$.

Note that $J = \bigcup_{i=0}^{d} x_n^i J_{x_n^i}$. If J is stable then $J_{x_n^0}$ is stable and if J is strongly stable (resp. lexsegment) then $J_{x_n^i}$ is strongly stable (resp. lexsegment) for all i, $0 \leqslant i \leqslant d$ (see [3, Prop. 1.4], or [4, (6.2.3)]).

The following result will be needed in Section 3.

COROLLARY 1.5. *Let $I \subset R$ be a stable ideal, s a positive integer and h a homogeneous generic form of R_s. Then*

$$H(R/(I,h),d) = H(R/(I,x_n^s),d) = \binom{n+d-1}{d} - \binom{n+d-s-1}{d-s} - \sum_{i=0}^{s-1} |(I_d)_{x_n^i}|$$

for all integers $d \geqslant s$.

Proof. By Proposition 1.4 we may suppose $h = x_n^s$. But

$$(I, x_n^s)_d = I_d + x_n^s R_{d-s} = \sum_{i=0}^{s-1} x_n^i \langle (I_d)_{x_n^i} \rangle + x_n^s R_{d-s},$$

which yields the desired equality. □

Remark 1.6. If I is monomial but not stable, then there may exist no generic monomial of degree s for I. Indeed, let $n = 3$, $d = 3$, $s = 1$ and $I = (x_1x_2x_3)$. Then $H(R/(I,x_i),3) \geqslant 4$ for any i, $1 \leqslant i \leqslant 3$, because $x_1x_2x_3 \in x_iR_2$. On the other hand, by Green's Theorem (see [14], or [5, (4.2.12)]), $H(R/(I,h),3) \leqslant H(R/I,3)_{\langle 3 \rangle} = 9_{\langle 3 \rangle} = 3$ for any generic linear form; see the next section for notation. Indeed, choosing for example $h = x_1 + x_2$ we see that $H(R/(I,h),3) = 3$.

2. Some numerical Lemmas

In this section we introduce some numerical functions. Given positive integers a, $d > i$, let

$$a = \binom{k(d)}{d} + \cdots + \binom{k(1)}{1} \quad \text{with } k(d) > \cdots > k(1) \geqslant 0$$

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be the dth Macaulay expansion of a. Then set

$$a^{\langle d,i\rangle} = \binom{k(d)+i}{d+i} + \cdots + \binom{k(1)+i}{i+1},$$

$$a^{\langle d,-i\rangle} = \binom{k(d)-i}{d-i} + \cdots + \binom{k(i+1)-i}{1},$$

and $a^{\langle\langle d,-1\rangle\rangle} = a - a_{\langle d\rangle}$, where as usual

$$a_{\langle d\rangle} = \binom{k(d)-1}{d} + \cdots + \binom{k(1)-1}{1}.$$

For $i > 1$, we then define inductively

$$a^{\langle\langle d,-i\rangle\rangle} = (a^{\langle\langle d,-1\rangle\rangle})^{\langle\langle d-1,-i+1\rangle\rangle}.$$

We will need a formula for $a^{\langle\langle d,-i\rangle\rangle}$, and for this purpose we will compare $a^{\langle\langle d,-i\rangle\rangle}$ with $a^{\langle d,-i\rangle}$. We adopt the usual convention: $\binom{k}{0} = 1$ if $k \geqslant 0$, and $\binom{k}{d} = 0$ if $k < d$.

LEMMA 2.1. *With the notation introduced one has*

$$(a+1)^{\langle\langle d,-1\rangle\rangle} = a^{\langle d,-1\rangle} + 1.$$

Proof. We have $(a+1)^{\langle\langle d,-1\rangle\rangle} = (a+1) - (a+1)_{\langle d\rangle}$. [9, Property 1.3] says that $(a+1)_{\langle d\rangle} = a_{\langle d\rangle} + \binom{k(1)-1}{0}$, so that $(a+1)^{\langle\langle d,-1\rangle\rangle} = a - a_{\langle d\rangle} + 1 - \binom{k(1)-1}{0} = a^{\langle d,-1\rangle} + 1$ since

$$\begin{aligned} a - a_{\langle d\rangle} &= \binom{k(d)-1}{d-1} + \cdots + \binom{k(1)-1}{0} \\ &= a^{\langle d,-1\rangle} + \binom{k(1)-1}{0}. \end{aligned} \tag{1}$$

□

Now we are in the position to prove the desired formula for $a^{\langle\langle d,-i\rangle\rangle}$.

LEMMA 2.2. *With the notation introduced one has*

$$a^{\langle\langle d,-i\rangle\rangle} = a^{\langle d,-i\rangle} + \binom{k(i)-i}{0} = \sum_{j=0}^{d-i} \binom{k(i+j)-i}{j}.$$

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Proof. We prove the assertion by induction on i. The case $i = 1$ is just equation (1) in the proof of Lemma 2.1.

Suppose now that $i > 1$. Then

$$a^{\langle\langle d,-i\rangle\rangle} = (a - a_{\langle d\rangle})^{\langle\langle d-1,-i+1\rangle\rangle} = \left(a^{\langle d,-1\rangle} + \binom{k(1)-1}{0}\right)^{\langle\langle d-1,-i+1\rangle\rangle}.$$

If $k(1) = 0$, then by induction hypothesis it follows that

$$\begin{aligned} a^{\langle\langle d,-i\rangle\rangle} &= (a^{\langle d,-1\rangle})^{\langle\langle d-1,-i+1\rangle\rangle} \\ &= \left(\binom{k(d)-1}{d-1} + \cdots + \binom{k(2)-1}{1}\right)^{\langle\langle d-1,-i+1\rangle\rangle} \\ &= a^{\langle d,-i\rangle} + \binom{(k(i)-1)-(i-1)}{0} = a^{\langle d,-i\rangle} + \binom{k(i)-i}{0}. \end{aligned}$$

If $k(1) \geqslant 1$, then $k(j) \geqslant j$ for all $j \geqslant 1$, and by the case $i = 1$, Lemma 2.1 and our induction hypothesis we obtain

$$\begin{aligned} a^{\langle\langle d,-i\rangle\rangle} &= \left(a^{\langle d,-1\rangle} + \binom{k(1)-1}{0}\right)^{\langle\langle d-1,-i+1\rangle\rangle} \\ &= ((a^{\langle d,-1\rangle} + 1)^{\langle\langle d-1,-1\rangle\rangle})^{\langle\langle d-2,-i+2\rangle\rangle} \\ &= (a^{\langle d,-2\rangle} + 1)^{\langle\langle d-2,-i+2\rangle\rangle} \\ &= a^{\langle d,-i\rangle} + 1 = a^{\langle d,-i\rangle} + \binom{k(i)-i}{0}. \end{aligned}$$ □

Now we set $a_{\langle\langle d,i\rangle\rangle} = (a^{\langle\langle d,-i\rangle\rangle})_{\langle d-i\rangle}$, and

$$a_{\langle d,i\rangle} = (a^{\langle d,-i\rangle})_{\langle d-i\rangle} = \binom{k(d)-i-1}{d-i} + \cdots + \binom{k(i+1)-i-1}{1}.$$

Remark 2.3. By definition one has

(i) $(a^{\langle\langle d,-1\rangle\rangle})_{\langle\langle d-1,i-1\rangle\rangle} = a_{\langle\langle d,i\rangle\rangle}$,
(ii) $a_{\langle\langle d,0\rangle\rangle} = a_{\langle d\rangle}$,
(iii) if $a = b^{\langle d-1\rangle}$ for a certain positive integer b, then $a_{\langle\langle d,1\rangle\rangle} = a_{\langle d,1\rangle}$.

There is a formula for $a_{\langle\langle d,i\rangle\rangle}$ similar to that of $a^{\langle\langle d,-i\rangle\rangle}$.

LEMMA 2.4. *With the notation introduced one has*

$$a_{\langle\langle d,i\rangle\rangle} = a_{\langle d,i\rangle} + \binom{k(i)-i}{0} \quad \text{for } i > 0.$$

Proof. By Lemma 2.2 we have $a_{\langle\langle d,i\rangle\rangle} = (a^{\langle d,-i\rangle} + \binom{k(i)-i}{0})_{\langle d-i\rangle}$.
If $k(i) < i$, then $a_{\langle\langle d,i\rangle\rangle} = (a^{\langle d,-i\rangle})_{\langle d-i\rangle} = a_{\langle d,i\rangle}$, and if $k(i) \geqslant i$, then

$$\begin{aligned} a_{\langle\langle d,i\rangle\rangle} &= (a^{\langle d,-i\rangle}+1)_{\langle d-i\rangle} = (a^{\langle d,-i\rangle})_{\langle d-i\rangle} + \binom{k(i+1)-i-1}{0} \\ &= a_{\langle d,i\rangle} + 1, \end{aligned}$$

by [9, Prop. 1.3]. Here we used that $k(i+1) \geqslant i+1$ since $k(i) \geqslant i$. □

3. An extension of Green's Theorem

In this section we study the behaviour of the Hilbert function of a homogeneous graded K-algebra after reduction modulo a generic homogeneous form of arbitrary degree. In the first step we consider the special case that the defining ideal of the algebra is lexsegment.

PROPOSITION 3.1. *Let $L \subset R$ be a lexsegment ideal and s a positive integer. Then*

$$H(R/(L,x_n^s),d) = H(R/(L,h),d) = \sum_{i=0}^{s-1} H(R/L,d)_{\langle\langle d,i\rangle\rangle}$$

for all integers $d \geqslant s$ and every homogeneous generic form h of R_s.

Proof. By Corollary 1.5 we have

$$\begin{aligned} H(R/(L,x_n^s),d) &= H(R/(L,h),d) \\ &= \binom{n+d-1}{d} - \binom{n+d-s-1}{d-s} - \sum_{i=0}^{s-1} |(L_d)_{x_n^i}| \\ &= \sum_{i=0}^{s-1}\left(\binom{n+d-i-2}{d-i} - |(L_d)_{x_n^i}|\right) \\ &= \sum_{i=0}^{s-1}(|\{x_1,\ldots,x_{n-1}\}^{d-i}| - |(L_d)_{x_n^i}|) \\ &= \sum_{i=0}^{s-1} H(R/L,d)_{\langle\langle d,i\rangle\rangle}. \end{aligned}$$

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The last equality follows from the next Lemma. □

LEMMA 3.2. *Let $L \subset R$ be a lexsegment ideal and d and i be two integers with $d \geqslant i \geqslant 0$ and $d > 0$. Then*

$$|\{x_1, \ldots, x_{n-1}\}^{d-i} \setminus (L_d)_{x_n^i}| = H(R/L, d)_{\langle\langle d,i\rangle\rangle}.$$

Proof. We apply induction on i. The case $i = 0$ has been done in [4, (7.3.1)] (or use Green's theorem [14] in connection with 1.4). Suppose now that $i > 0$. The set of monomials

$$\bigcup_{i=1}^{d} x_n^{i-1}(L_d)_{x_n^i}$$

is a lexsegment in $\{x_1, \ldots, x_n\}^{d-1}$. We denote by L'_{d-1} the K-vector space spanned by this set. Notice that $L' = \sum_{j \geqslant 0} L'_j$ is a lexsegment ideal, and we have

$$(L'_{d-1})_{x_n^{i-1}} = (L_d)_{x_n^i}.$$

By induction hypothesis we obtain

$$\begin{aligned} |\{x_1, \ldots, x_{n-1}\}^{d-i} \setminus (L_d)_{x_n^i}| &= |\{x_1, \ldots, x_{n-1}\}^{d-i} \setminus (L'_{d-1})_{x_n^{i-1}}| \qquad (2) \\ &= H(R/L', d-1)_{\langle\langle d-1, i-1\rangle\rangle}. \end{aligned}$$

But $\dim_K L_d = \dim_K L'_{d-1} + |(L_d)_{x_n^0}|$ and $|(L_d)_{x_n^0}| = |\{x_1, \ldots, x_{n-1}\}^d| - H(R/L, d)_{\langle d\rangle}$,by the case $i = 0$. It follows that $\dim_K L_d = \dim_K L'_{d-1} + \binom{n+d-2}{d} - H(R/L, d)_{\langle d\rangle}$, and so

$$\begin{aligned} H(R/L', d-1) &= \dim_K R_{d-1} - \dim_K L'_{d-1} \qquad (3) \\ &= \binom{n+d-2}{d-1} - \dim_K L_d \qquad (4) \\ &\quad + \binom{n+d-2}{d} - H(R/L, d)_{\langle d\rangle} \\ &= H(R/L, d) - H(R/L, d)_{\langle d\rangle} = H(R/L, d)^{\langle\langle d,-1\rangle\rangle}. \end{aligned}$$

Substituting (3) in (2) we obtain

$$\begin{aligned} |\{x_1, \ldots, x_{n-1}\}^{d-i} \setminus (L_d)_{x_n^i}| &= (H(R/L, d)^{\langle\langle d,-1\rangle\rangle})_{\langle\langle d-1, i-1\rangle\rangle} \\ &= H(R/L, d)_{\langle\langle d,i\rangle\rangle}, \end{aligned}$$

by Remark 2.3. □

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COROLLARY 3.3. *Let $L \subset R$ be a lexsegment ideal, $d \geqslant s$ two positive integers and*

$$H(R/L,d) = \binom{k(d)}{d} + \cdots + \binom{k(j)}{j}, \quad k(d) > \cdots > k(j) \geqslant j$$

be the dth Macaulay expansion of $H(R/L,d)$. Then

$$H(R/(L,x_n^s),d) = \sum_{i=0}^{s-1} H(R/L,d)_{\langle d,i\rangle} + \binom{s-j}{1}.$$

The proof follows from Proposition 3.1 and Lemma 2.4.

The next result yields the crucial comparision between a lexsegment ideal and a strongly stable ideal.

PROPOSITION 3.4. *Let L be a lexsegment ideal and $I \subset R$ a strongly stable ideal such that* $\dim_K L_d \leqslant \dim_K I_d$ *for a certain positive integer d. Then*

$$\sum_{i=0}^{t} |(L_d)_{x_n^i}| \leqslant \sum_{i=0}^{t} |(I_d)_{x_n^i}|$$

for any integer t, $0 \leqslant t \leqslant d$.

Proof. By [2], [3, Th. 2.1]B or [4, Th. (6.3.2)] we see that the inequality holds for $t = 0$. Suppose the inequality does not hold for a certain t, $1 \leqslant t \leqslant d$, which we may consider to be the smallest possible. Set

$$a_j = \sum_{i=0}^{j} |(I_d)_{x_n^i}| - \sum_{i=0}^{j} |(L_d)_{x_n^i}|, \quad 0 \leqslant j \leqslant d.$$

Then a_{t-1} is not negative but a_t is. Set

$$L^{(t)} = \bigcup_{i\geqslant t}^{d} x_n^{i-t}(L_d)_{x_n^i}.$$

Note that $L^{(t)}$ is a lexsegment of $\{x_1,\ldots,x_n\}^{d-t}$, because $L^{(t)}$ generates $(L : x_n^t)_{d-t}$. Let $J_i \subset \{x_1,\ldots,x_{n-1}\}^{d-i}$ be the unique lexsegment such that $|J_i| = |(I_d)_{x_n^i}|$, $0 \leqslant i \leqslant d$. By [3, Lem. 1.6 and its Remark, Th. 2.1], or [4, (6.2.9), (6.2.10), (6.3.2)] the set $J^{(t)} = \bigcup_{i\geqslant t}^{d} x_n^{i-t} J_i$ is strongly stable since $I^{(t)} = \bigcup_{i\geqslant t}^{d} x_n^{i-t}(I_d)_{x_n^i}$ generates $(I : x_n^t)_{d-t}$ which is strongly stable. (In fact, $J^{(t)}$ is the lexsegment of $I^{(t)}$ with respect to x_n in the terminology of [3], or [4]). Let J_t' be the unique lexsegment of $\{x_1,\ldots,x_{n-1}\}^{d-t}$ such that $|J_t'| = |(I_d)_{x_n^t}| + a_{t-1}$. Such a lexsegment exists because $|(L_d)_{x_n^t}| - |(I_d)_{x_n^t}| - a_{t-1} = -a_t > 0$ – in fact, even $J_t' \subset (L_d)_{x_n^t}$. Using

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[3, Lem. 1.5], or [4, (6.2.6)] we see also that $J' = J'_t \cup \bigcup_{i>t}^{d} x_n^{i-t} J_i$ is strongly stable because $J^{(t)}$ is so. But we have

$$\begin{aligned} |J'| &= |J'_t| + \dim_K I_d - \sum_{i=0}^{t} |(J_i)| \\ &= |(I_d)_{x_n^t}| + a_{t-1} + \dim_K I_d - \sum_{i=0}^{t} |(I_d)_{x_n^i}| \\ &= a_{t-1} + \dim_K I_d - \sum_{i=0}^{t-1} |(I_d)_{x_n^i}| \\ &\geqslant \dim_K L_d - \sum_{i=0}^{t-1} |(L_d)_{x_n^i}| = |L^{(t)}|. \end{aligned}$$

Using again the case $t = 0$, it follows that

$$|(L_d)_{x_n^t}| = |L^{(t)}_{x_n^0}| \leqslant |J'_{x_n^0}| = |J'_t| = |(I_d)_{x_n^t}| + a_{t-1},$$

that is $a_t \geqslant 0$, a contradiction. □

We would like to remark that the inequality of 3.4 is only valid for the whole sum and not necessarily for the single summands. In other words, it does not hold in general that $|(L_d)_{x_n^i}| \leqslant |(I_d)_{x_n^i}|$ for $i > 0$. Indeed, let $n = 3$, $d = 2$, $I = (x_1^2, x_1x_2, x_2^2)$, $L = (x_1^2, x_1x_2, x_1x_3)$. Then I is strongly stable, L is a lexsegment ideal, $\dim_K L_2 = \dim_K I_2$, $L_{x_3} = \{x_1\}$ and $I_{x_3} = \emptyset$, that is $|L_{x_3}| > |I_{x_3}|$.

In the next steps we prove the desired inequality of Hilbert functions by a comparison of the general graded ideals with their initial ideals.

LEMMA 3.5. *Let $I \subset R$ be a graded ideal, s a positive integer and h a homogeneous generic form of R_s. Then*

$$H(R/(I,h),d) \leqslant H(R/(\mathrm{in}(I),x_n^s),d)$$

for all integers $d \geqslant 0$.

Proof. If h is generic, then $H(R/(I,h),d) \leqslant H(R/(I,h'),d)$ for any $h' \in R_s$ and $d \in \mathbb{N}$. In particular, this inequality holds for $h' = x_n^s$. Thus

$$\begin{aligned} H(R/(I,h),d) &\leqslant H(R/(I,x_n^s),d) = H(R/\mathrm{in}(I,x_n^s),d) \\ &\leqslant H(R/(\mathrm{in}(I),x_n^s),d), \end{aligned}$$

by Macaulay's Theorem (see [15], [6, Th. 15.3], or [5, (4.2.10)]) and because $(\mathrm{in}(I),x_n^s) \subset \mathrm{in}(I,x_n^s)$. □

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PROPOSITION 3.6. *Let $I \subset R$ be a graded ideal such that* $\mathrm{in}(I)$ *is strongly stable, s a positive integer and h a homogeneous generic form of R_s. Then*

$$H(R/(I,h),d) \leqslant \sum_{i=0}^{s-1} H(R/I,d)_{\langle\langle d,i\rangle\rangle}$$

for all integers $d \geqslant s$.

Proof. Set $J = \mathrm{in}(I)$. By Lemma 3.5 we have

$$H(R/(I,h),d) \leqslant H(R/(J,x_n^s),d) \quad \text{for all integers } d \geqslant 0.$$

Let $L \subset R$ be a lexsegment ideal such that $\dim_K L_d = \dim_K J_d$ for all $d \in \mathbb{N}$. Fix $d \geqslant s$. Then

$$L_d + x_n^s R_{d-s} = \left\langle \bigcup_{i=0}^{s-1} x_n^i (L_d)_{x_n^i} \right\rangle + x_n^s R_{d-s},$$

$$J_d + x_n^s R_{d-s} = \left\langle \bigcup_{i=0}^{s-1} x_n^i (J_d)_{x_n^i} \right\rangle + x_n^s R_{d-s}, \tag{5}$$

the unions being disjoint. By Lemma 3.4 we have $\Sigma_{i=0}^{s-1}|(L_d)_{x_n^i}| \leqslant \Sigma_{i=0}^{s-1}|(J_d)_{x_n^i}|$ and using (5) it follows

$$H(R/(J,x_n^s),d) \leqslant H(R/(L,x_n^s),d). \tag{6}$$

By Proposition 3.1 we have

$$\begin{aligned} H(R/(L,x_n^s),d) &= \sum_{i=0}^{s-1} H(R/L,d)_{\langle\langle d,i\rangle\rangle} \\ &= \sum_{i=0}^{s-1} H(R/J,d)_{\langle\langle d,i\rangle\rangle} \\ &= \sum_{i=0}^{s-1} H(R/I,d)_{\langle\langle d,i\rangle\rangle}. \end{aligned} \tag{7}$$

The last equality follows from Macaulay's Theorem. The proof ends combining (4), (6) and (7). □

We are now ready to prove the main result of this paper.

THEOREM 3.7. *Suppose* $\mathrm{char}\, K = 0$. *Let $I \subset R$ be a graded ideal, s a positive integer and h a homogeneous generic form of degree s. Then*

$$H(R/(I,h),d) \leqslant \sum_{i=0}^{s-1} H(R/I,d)_{\langle\langle d,i\rangle\rangle}$$

for all integers $d \geqslant s$.

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Proof. By Galligo's Theorem (see [11], or [6, Th. 15.20]) the generic initial ideal $\mathrm{in}(I)$ is Borel-fixed, and so strongly stable because $\mathrm{char}\, K = 0$ (see [6, Th. 15.23]). Now we may apply Proposition 3.6. □

COROLLARY 3.8. *Suppose* $\mathrm{char}\, K = 0$. *Let* $I \subset R$ *be a graded ideal,* $d \geqslant s$ *two positive integers,* h *a homogeneous generic form of* R_s *and*

$$H(R/I, d) = \binom{k(d)}{d} + \cdots + \binom{k(j)}{j}, \quad k(d) > \cdots > k(j) \geqslant j$$

the d*th Macaulay expansion of* $H(R/I, d)$. *Then*

$$H(R/(I,h), d) \leqslant \sum_{i=0}^{s-1} H(R/I, d)_{\langle d,i\rangle} + \binom{s-j}{1}.$$

For the proof apply Theorem 3.7 and Lemma 2.4.

Remark 3.9. (i) The bound given by Theorem 3.7 and Corollary 3.8 is reached when I is a lexsegment ideal (see Proposition 3.1).

(ii) The proof of Green's theorem in the case $s = 1$ is much easier than our proof (see [14], or [5, Th. 4.2.12]. One could hope to give a proof of 3.7 along those lines. Setting $a = H(R/I, d)$ and $b = H(R/(I,h), d)$ one can derive, using Green's arguments, the following inequality

$$b \leqslant \sum_{i=0}^{s-1} b_{\langle\langle d,i\rangle\rangle} + \sum_{i=0}^{s-1} (a-b)_{\langle\langle n-s,i\rangle\rangle}.$$

We now would need that this inequality implies $b \leqslant \Sigma_{i=0}^{s-1} a_{\langle\langle d,i\rangle\rangle}$. For $s = 1$, this was shown by Green. However, if $s = 2$ the second inequality does not follow from the first. Take for example $d = 4$, $a = 44 = \binom{7}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1}$, $b = 32 = \binom{6}{4} + \binom{5}{3} + \binom{4}{2} + \binom{1}{1}$. Then $a_{\langle 4\rangle} = 18$, $a_{\langle\langle 4,1\rangle\rangle} = 13$, $(a-b)_{\langle 2\rangle} = 12_{\langle 2\rangle} = 7$, $(a-b)_{\langle\langle 2,1\rangle\rangle} = 4$, $b_{\langle 4\rangle} = 12$, $b_{\langle\langle 4,1\rangle\rangle} = 10$. Thus

$$b < 33 = b_{\langle 4\rangle} + b_{\langle\langle 4,1\rangle\rangle} + (a-b)_{\langle 2\rangle} + (a-b)_{\langle\langle 2,1\rangle\rangle},$$

but $b > 31 = a_{\langle 4\rangle} + a_{\langle\langle 4,1\rangle\rangle}$.

COROLLARY 3.10. *Let* $(A, \mathfrak{m})$ *be a Noetherian local domain of characteristic* 0, $R = A[x_1, \ldots, x_n]$ *the polynomial ring,* $I \subset R$ *a homogeneous ideal, and* h *a homogeneous form of degree* s *which is generic in* $Q(A)[x_1, \ldots, x_n]$. *Then*

$$e(R/(I,h), d) \leqslant \sum_{i=0}^{s-1} e(R/I, d)_{\langle\langle d,i\rangle\rangle}.$$

For the proof we apply Theorem 3.7 for the quotient field of A, and use that $e_A((R/I,d) = e(A)\text{rank}_A(R/I)_d$.

4. Some applications

Throughout this section K will be a field of characteristic 0, and $R = K[x_1,\ldots,x_n]$ the polynomial ring over K.

The first applications we have in mind are related to the Eisenbud–Green–Harris Conjecture. We shall need the following

LEMMA 4.1. *Let $b \geqslant a$, then $b_{\langle\langle d,i\rangle\rangle} \geqslant a_{\langle\langle d,i\rangle\rangle}$ for all $d > i \geqslant 0$.*

Proof. It suffices to show the assertion for $b = a+1$. Let $a = \Sigma_{j=1}^{d}\binom{k(j)}{j}$ be the dth Macaulay expansion of a. We will use induction on i in order to prove the lemma. For $i = 0$ we have

$$\begin{aligned}(a+1)_{\langle\langle d,0\rangle\rangle} &= (a+1)_{\langle d\rangle} = a_{\langle d\rangle} + \binom{k(1)-1}{0}\\ &\geqslant a_{\langle d\rangle} = a_{\langle\langle d,0\rangle\rangle}.\end{aligned}$$

Let $i > 0$. We have

$$(a+1)_{\langle\langle d,i\rangle\rangle} = ((a+1)^{\langle\langle d,-1\rangle\rangle})_{\langle\langle d-1,i-1\rangle\rangle}.$$

Applying the induction hypothesis it suffices to show that

$$(a+1)^{\langle\langle d,-1\rangle\rangle} \geqslant a^{\langle\langle d,-1\rangle\rangle}.$$

But

$$\begin{aligned}(a+1)^{\langle\langle d,-1\rangle\rangle} &= a+1-(a+1)_{\langle d\rangle}\\ &= a - a_{\langle d\rangle} + 1 - \binom{k(1)-1}{0} \geqslant a - a_{\langle d\rangle} = a^{\langle\langle d,-1\rangle\rangle}.\end{aligned}$$

We have the following result about algebras defined by generic quadrics

PROPOSITION 4.2. *Let B be a zero dimensional complete intersection defined by quadratic forms, A a factor ring of B defined by generic quadratic forms of B, and let $\binom{a}{2}+\binom{b}{1}$ be the 2-Macaulay expansion of $H(A,2)$. Then*

$$H(A,d) \leqslant \binom{a}{d} + \binom{b}{d-1} \quad \textit{for } d \geqslant 2.$$

Proof. We denote by A_m the factor ring of B which is defined by m quadrics. If $m = 0$, then A_m is a complete intersection, and so $H(A_m, d) = \binom{a}{d}$, as required.

Now let $m \geqslant 0$ and assume the inequality is already shown for A_m. Let $\binom{a}{2} + b$ be the 2-Macaulay expansion of $H(A_m, 2)$. We will distinguish two cases.

First assume that $b \geqslant 1$. Then $\binom{a}{2} + b - 1$ is the 2-Macaulay expansion of $H(A_{m+1}, 2)$. For $d \geqslant 3$ we have

$$\begin{aligned} H(A_{m+1}, d) &\leqslant H(A_m, d)_{\langle\langle d,0\rangle\rangle} + H(A_m, d)_{\langle\langle d,1\rangle\rangle} \\ &\leqslant \binom{a-1}{d} + \binom{b-1}{d-1} + \binom{a-2}{d-1} + \binom{b-2}{d-2}. \end{aligned}$$

This sum should be less than

$$\binom{a}{d} + \binom{b-1}{d-1} = \binom{a-1}{d} + \binom{a-1}{d-1} + \binom{b-1}{d-1},$$

which is obviously true.

In the second case we have $H(A_m, 2) = \binom{a}{2}$. Then for $d \geqslant 3$,

$$\begin{aligned} H(A_{m+1}, d) &\leqslant H(A_m, d)_{\langle\langle d,0\rangle\rangle} + H(A_m, d)_{\langle\langle d,1\rangle\rangle} \\ &\leqslant \binom{a-1}{d} + \binom{a-2}{d-1}. \end{aligned}$$

This is what we wanted to show, since $H(A_{m+1}, 2) = \binom{a-1}{2} + \binom{a-2}{1}$. □

Remark 4.3. Using the same ideas as in the proof of 4.2 we get the following stronger form which is very much connected with Conjecture (V_m) from [7]. Let B be a zero dimensional complete intersection of quadrics, A a factor ring of B defined by generic s-forms of A, and let

$$H(A, s) = \binom{k(s)}{s} + \cdots + \binom{k(1)}{1}$$

be the s-Macaulay expansion of $H(A, s)$. Then

$$H(A, d) \leqslant \binom{k(s)}{d} + \cdots + \binom{k(1)}{d-s+1}$$

for $d \geqslant s$.

From the above proposition the following version of the Eisenbud–Green–Harris Conjecture [7] follows quite easily:

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COROLLARY 4.4. *With the notation and assumptions of Proposition* 4.2 *one has*

$$\dim_K A \leqslant 2^a + 2^b + n - a - 1.$$

Proof. It follows from 4.2 that

$$\begin{aligned}\dim_K A &= \sum_{i\geqslant 0} H(A,i) \leqslant 1 + n + \sum_{i\geqslant 2} H(A,i)\\ &= 1 + n + \sum_{i=2}^{a}\binom{a}{i} + \sum_{i=1}^{b}\binom{b}{i}\\ &= 2^a + 2^b + n - a - 1.\end{aligned}$$ □

The next application is of a similar nature. We shall prove the following version of Conjecture $(\mathrm{III}_{k,r})$ of [7].

PROPOSITION 4.5. *Let* $B = K[x_1,\dots,x_n]/I$ *be a complete intersection of dimension* 0 *defined by quadrics, and let* $f \in B$ *be a generic form of degree* j. *Set* $A = B/fB$; *then*

$$\dim_K A \leqslant 2^n - 2^{n-j}.$$

Proof. We have $\dim_K A_d = \binom{n}{d}$ for $d < j$, while for $d \geqslant j$ we have

$$\dim_K A_d \leqslant \sum_{i=0}^{j-1}\binom{n}{d}_{\langle\langle d,i\rangle\rangle} = \sum_{i=0}^{j-1}\binom{n-i-1}{d-i}.$$

Thus

$$\dim_k A \leqslant 2^n + \sum_{d=j}^{n}\left(\sum_{i=0}^{j-1}\binom{n-i-1}{d-i} - \binom{n}{d}\right).$$

The assertion follows since

$$\sum_{d=j}^{n}\left(\sum_{i=0}^{j-1}\binom{n-i-1}{d-i} - \binom{n}{d}\right) = -\sum_{d=j}^{n}\binom{n-j}{d-j} = -2^{n-j}.$$ □

The following applications concern Gotzmann spaces. Recall that a linear subspace $P \subset R_d$ is called a *Gotzmann space* if $R_1P \subset R_{d+1}$ has the smallest possible dimension, that is, if

$$\dim_K R_{d+1}/R_1P = (\dim_K R_d/P)^{\langle d\rangle}.$$

LEMMA 4.6. *Let* d, t *be two positive integers and* $P \subset R_d$ *a linear subspace. Then*

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(i) $\dim_K R_{d+t}/R_t P \leqslant (\dim_K R_d/P)^{\langle d,t \rangle}$.
(ii) *If equality holds in* (i), *then* P *is Gotzmann*.

Proof. Apply induction on t. If $t = 1$, then (i) follows from Macaulay's Theorem (see [15], or [5, (4.2.10)], and (ii) holds by definition. Suppose $t > 1$. Then

$$\begin{aligned} \dim_K R_{d+t}/R_t P &\leqslant (\dim_K R_{d+t-1}/R_{t-1}P)^{\langle d+t-1 \rangle} \\ &\leqslant ((\dim_K R_d/P)^{\langle d,t-1 \rangle})^{\langle d+t-1 \rangle} = (\dim_K R_d/P)^{\langle d,t \rangle}, \end{aligned} \tag{8}$$

where the first inequality follows from Macaulay's Theorem and the second one from the induction hypothesis and [5, (4.2.13)]. If the ends of (8) are equal then the inequalities of (8) are in fact equalities and so P must be Gotzmann by induction hypothesis. □

LEMMA 4.7. *Let* d, t *be two positive integers*, $P \subset R_d$ *a linear subspace and* I *the ideal generated by* P. *Then the following statements are equivalent:*

(i) P *is Gotzmann*.

(ii)
$$H(R/I, d+j+1) = (H(R/I, d+j))^{\langle d+j \rangle}$$
for all j, $0 \leqslant j < t$.

(iii) $\dim_K R_{d+t}/R_t P = (\dim_K R_d/P)^{\langle d,t \rangle}$.

(iv)
$$H(R/I, d+t) - H(R/I, d) = \sum_{j=1}^{t} H(R/I, d+j)_{\langle d+j \rangle}.$$

Proof. Note that (i) $\Rightarrow$ (ii), by Gotzmann's Persistence Theorem (see [13]), (ii) $\Rightarrow$ (iii) by recurrence since $I_{d+j} = R_j P$, and (iii) $\Rightarrow$ (i), by Lemma 4.6. Since we have

$$\begin{aligned} H(R/I, d+t) - H(R/I, d) &= \sum_{i=0}^{t-1} (H(R/I, d+i+1) - H(R/I, d+i)) \\ &\leqslant \sum_{i=0}^{t-1} (H(R/I, d+i)^{\langle d+i \rangle} - H(R/I, d+i)) \\ &= \sum_{i=0}^{t-1} (H(R/I, d+i)^{\langle d+i \rangle})_{\langle d+i+1 \rangle} \\ &= \sum_{j=1}^{t} H(R/I, d+j)_{\langle d+j \rangle}, \end{aligned}$$

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it is clear that (ii) is equivalent with (iv). □

PROPOSITION 4.8. *Let $q \geqslant s$ be two positive integers, $P \subset R_q$ a Gotzmann subspace, I the ideal generated by P, z a generic element of R_1 and h a homogeneous generic form of R_s. Then*

$$\begin{aligned} H(R/(I,h),d) &= H(R/(I,z^s),d) = \sum_{i=0}^{s-1} H(R/I,d)_{\langle\langle d,i\rangle\rangle} \\ &= \sum_{i=0}^{s-1} H(R/I,d)_{\langle d,i\rangle} \end{aligned}$$

for all $d \geqslant q+s$. In particular z^s is generic for I.

Proof. We have

$$\begin{aligned} H(R/I,d) - H(R/I,d-s) &\leqslant H(R/(I,h),d) \leqslant H(R/(I,z^s),d) \\ &\leqslant \sum_{i=0}^{s-1} (H(R/(I,z),d-i) \\ &\leqslant \sum_{i=0}^{s-1} H(R/I,d-i)_{\langle d-i\rangle} \end{aligned}$$

for any integer $d \geqslant s$. The last inequality follows from Green's Theorem (we may also apply Theorem 3.7 for $s=1$). For the third inequality apply induction on s, the case $s=1$ being trivial. Suppose $s>1$. Note that

$$\begin{aligned} H(R/(I,z),d) &= H(R/(I,z^s),d) - H(R/((I,z^s):z),d-1) \\ &\geqslant (R/(I,z^s),d) - H(R/(I,z^{s-1}),d-1). \end{aligned}$$

Now, it is enough to apply the induction hypothesis.

By Lemma 4.7 (iv) we see that the above inequalities are in fact equalities. Moreover we have

$$\begin{aligned} H(R/I,d)_{\langle d,i\rangle} &= (H(R/I,d-1)^{\langle d-1\rangle})_{\langle d,i\rangle} = H(R/I,d-1)_{\langle d-1,i-1\rangle} \\ &= \cdots = H(R/I,d-i)_{\langle d-i\rangle} \end{aligned}$$

for $0 \leqslant i \leqslant s$. Since by Remark 2.3 (iii),

$$(H(R/I,d-j)^{\langle d-j\rangle})_{\langle\langle d-j+1,1\rangle\rangle} = H(R/I,d-j)_{\langle d-j\rangle},$$

for $1 \leqslant j \leqslant s$, we similarly obtain

$$H(R/I,d)_{\langle\langle d,i\rangle\rangle} = H(R/I,d-i)_{\langle d-i\rangle}. \qquad \square$$

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The next result generalizes part of [12, Th. 2.1].

PROPOSITION 4.9. *Let $d \geqslant s$ be two positive integers, $P \subset R_d$ a Gotzmann subspace, I the ideal generated by P and h a homogeneous generic form of R_s. Then $(I_q : h)_{q-s} = I_{q-s}$ for any $q \geqslant d+s$.*

This follows from the next lemma and 4.7(ii).

LEMMA 4.10. *Let $I \subset R$ be a graded ideal, $d \geqslant s$ two positive integers and h a homogeneous generic form of R_s. Then*

$$\dim_K[(I_d : h)_{d-s}/I_{d-s}] \leqslant H(R/I, d-s) - (H(R/I, d))^{\langle\langle d,-s\rangle\rangle}.$$

Proof. By Theorem 3.7 we obtain

$$\begin{aligned}\dim_K[(I_d : h)_{d-s}] &= H(R/(I,h), d) - H(R/I, d) + \dim_K R_{d-s} \\ &\leqslant \sum_{i=0}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle} - H(R/I, d) + \dim_K R_{d-s}.\end{aligned}$$

It follows that

$$\begin{aligned}\dim_K[(I_d : h)_{d-s}/I_{d-s}] \leqslant\ & H(R/I, d-s) - H(R/I, d) \\ & + \sum_{i=0}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle}. \end{aligned} \tag{9}$$

Note that

$$\begin{aligned}&\sum_{i=0}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle} - H(R/I, d) \\ &\quad = \sum_{i=1}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle} - H(R/I, d)^{\langle\langle d,-1\rangle\rangle} \\ &\quad = \sum_{i=2}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle} + (H(R/I, d)^{\langle\langle d,-1\rangle\rangle})_{\langle d-1\rangle} - H(R/I, d)^{\langle\langle d,-1\rangle\rangle} \\ &\quad = \sum_{i=2}^{s-1} H(R/I, d)_{\langle\langle d,i\rangle\rangle} - H(R/I, d)^{\langle\langle d,-2\rangle\rangle} \\ &\quad = \cdots = -H(R/I, d)^{\langle\langle d,-s\rangle\rangle},\end{aligned} \tag{10}$$

by recurrence. Substituting (10) in (9) we are done. □

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We end our paper with the following

PROPOSITION 4.11. *Let d, s be two positive integers, $P \subset R_d$ a Gotzmann subspace, h a homogeneous generic form of R_s, and*

$$\dim_K R_d/P = \binom{k(d)}{d} + \cdots + \binom{k(1)}{1},$$

the dth Macaulay expansion of $\dim_K R_d/P$. Suppose that $k(j) \leqslant j$ for all $j < d$. Then $\langle R_{t-d}P, hR_{t-s}\rangle \subset R_t$ is a Gotzmann subspace for all $t \geqslant d+s$.

Proof. Let I be the ideal generated by $R_{t-d}P$ in R. By Proposition 4.8 we have

$$H(R/(I,h),t) = H(R/(I,z^s),t) = \sum_{i=0}^{s-1} H(R/I,t-i)_{\langle t-i\rangle}.$$

But $H(R/I,t-i) = (\dim_K R_d/P)^{\langle d,t-d-i\rangle}$, and so

$$H(R/I,t-i)_{\langle t-i\rangle} = \binom{k(d)+t-d-i-1}{t-i},$$

because $k(j)+t-d-i-1 < t-d-i+j$ for all $j < d$. Thus

$$H(R/(I,h),t) = \sum_{i=0}^{s-1} \binom{k(d)+t-d-i-1}{t-i}$$

is exactly the tth Macaulay expansion of $H(R/(I,h),t)$. It follows

$$H(R/(I,h),t)^{\langle t\rangle} = \sum_{i=0}^{s-1} \binom{k(d)+t-d-i}{t-i+1} = H(R/(I,h),t+1),$$

that is, $I_t + hR_{t-s}$ is Gotzmann. □

EXAMPLE 4.12. Let $P = Kx_1 \subset R_1$. Then $\dim_K R_2/R_1P = \binom{n+1}{2} - n = \binom{n}{2}$ and $\dim_K R_1/P = \binom{n-1}{1}$. Clearly P is Gotzmann. By Lemma 4.7 $(R_{t-1}x_1, hR_{t-s}) \subset R_t$ is a Gotzmann subspace for any homogeneous generic form h of R_s.

Remark 4.13. Let $P \subset R_d$ be a Gotzmann subspace. If z is a generic element of R_1 then $R_{t-d}P + zR_{t-1} \subset R_t$ is a Gotzmann subspace for any $t \geqslant d$ by [12, Th. 2.1]. Then we may ask if the conditions of Proposition 4.11 are not too restrictive. This is not the case since there are Gotzmann subspaces $P \subset R_2$ such that $\langle P, h\rangle \subset R_2$ is not Gotzmann for a homogeneous generic form $h \in R_2$ as

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shown in the following example. Roughly speaking the reason is that the operation '$\langle\ \rangle$' does not commute in general with the integer addition.

EXAMPLE 4.14. Let $P = Kx_1^2 \subset R_2$, I be the ideal generated by P and h a homogeneous generic form of R_2. We have

$$\dim_K R_2/P = \binom{n+1}{2} - 1 = \binom{n}{2} + \binom{n-1}{1}$$

and

$$\dim_K R_3/R_1P = \binom{n+2}{3} - n = \binom{n+1}{3} + \binom{n}{2} = (\dim_K R_2/P)^{\langle 2\rangle},$$

that is, P is Gotzmann. Then

$$H(R/I,t) = \binom{n+t-2}{t} + \binom{n+t-3}{t-1}$$

for all $t \geqslant 2$, and so

$$\begin{aligned} H(R/(I,h),t) &= H(R/I,t)_{\langle t\rangle} + H(R/I,t-1)_{\langle t-1\rangle} \\ &= \binom{n+t-3}{t} + 2\binom{n+t-4}{t-1} + \binom{n+t-5}{t-2} \end{aligned}$$

for all $t \geqslant 4$ by Proposition 4.8. For $n = 4$ we have

$$H(R/(I,h),5) = 20 < 22 = 16^{\langle 4\rangle} = H(R/(I,h),4)^{\langle 4\rangle}$$

(note that $16 = \binom{6}{4} + \binom{3}{3}$ is the 4-Macaulay expansion of 16). Thus if $n = 4$ then $R_2\langle P,h\rangle \subset R_4$ is not Gotzmann and so $\langle P,h\rangle$ cannot be Gotzmann, too (see Lemma 4.7).

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