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STRUCTURAL PROPERTIES OF THE STABLE CORE

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Abstract. The stable core, an inner model of the form $\langle L[S], \in, S\rangle$ for a simply definable predicate S, was introduced by the first author in [8], where he showed that V is a class forcing extension of its stable core. We study the structural properties of the stable core and its interactions with large cardinals. We show that the GCH can fail at all regular cardinals in the stable core, that the stable core can have a discrete proper class of measurable cardinals, but that measurable cardinals need not be downward absolute to the stable core. Moreover, we show that, if large cardinals exist in V, then the stable core has inner models with a proper class of measurable limits of measurables, with a proper class of measurable limits of measurable limits of measurables, and so forth. We show this by providing a characterization of natural inner models $L[C_1, \ldots, C_n]$ for specially nested class clubs $C_1, \ldots, C_n$, like those arising in the stable core, generalizing recent results of Welch [29].

§1. Introduction. The first author introduced the inner model *stable core* while investigating under what circumstances the universe V is a class forcing extension of the inner model HOD, the collection of all hereditarily ordinal definable sets [8, 9]. He showed in [8] that there is a robust Δ_2-definable class S contained in HOD such that V is a class-forcing extension of the structure $\langle L[S], \in, S\rangle$, which he called the stable core, by an Ord-cc class partial order $\mathbb{P}$ definable from S. Indeed, for any inner model M, V is a $\mathbb{P}$-forcing extension of $\langle M[S], \in, S\rangle$, so that in particular, since $\text{HOD}[S] = \text{HOD}$, V is a $\mathbb{P}$-forcing extension of $\langle \text{HOD}, \in, S\rangle$.

Let's explain the result in more detail for the stable core $L[S]$, noting that exactly the same analysis applies to HOD. The partial order $\mathbb{P}$ is definable in $\langle L[S], \in, S\rangle$ and there is a generic filter G, meeting all dense sub-classes of $\mathbb{P}$ definable in $\langle L[S], \in, S\rangle$, such that $V = L[S][G]$. All standard forcing theorems hold for $\mathbb{P}$ since it has the Ord-cc. Thus, we get that the forcing relation for $\mathbb{P}$ is definable in $\langle L[S], \in, S\rangle$ and the forcing extension $\langle V, \in, G\rangle \models \text{ZFC}$. However, this particular generic filter G is not definable in V. To obtain G, we first force with an auxiliary forcing $\mathbb{Q}$ to add a particular class F, without adding sets, such that $V = L[F]$. We then show that G is definable from F and F is in turn definable in the structure $\langle L[S][G], \in, S, G\rangle$, so that $L[S][G] = V$. This gives a formulation of the result as a ZFC-theorem because we can say (using the definitions of $\mathbb{P}$ and $\mathbb{Q}$) that it is forced by $\mathbb{Q}$ that $V = L[F]$, where F is V-generic for $\mathbb{Q}$, and (the definition of) G is $\langle L[S], \in, S\rangle$-generic, and finally that F is definable in $\langle L[S][G], \in, S, G\rangle$. Of course, a careful formulation would say that the result holds for all sufficiently large natural numbers n, where n bounds the complexity of the formulas used.

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Without the niceness requirement on $\mathbb{P}$ that it has the Ord-cc, there is a much easier construction of a class forcing notion $\mathbb{P}$, suggested by Woodin, such that V is a class forcing extension of $\langle \mathrm{HOD}, \in, \mathbb{P} \rangle$ (see the end of Section 2). At the same time, some additional predicate must be added to HOD in order to realize all of V as a class-forcing extension because, as Hamkins and Reitz observed in [13], it is consistent that V is not a class-forcing extension of HOD. To construct such a counterexample, we suppose that κ is inaccessible in L and force over the Kelley–Morse model $\mathcal{L} = \langle V_\kappa^L, \in, V_{\kappa+1}^L \rangle$ to code the truth predicate of V_κ^L (which is an element of $V_{\kappa+1}^L$) into the continuum pattern below κ. The first-order part $V_\kappa^L[G]$ of this extension cannot be a forcing extension of $\mathrm{HOD}^{V_\kappa^L[G]} = V_\kappa^L$ (by the weak homogeneity of the coding forcing), because the truth predicate of V_κ^L is definable there and this can be recovered via the forcing relation.

While the definition of the partial order $\mathbb{P}$ is fairly involved, the *stability predicate* S simply codes the elementarity relations between sufficiently nice initial segments H_α (the collection of all sets with transitive closure of size less than α) of V. Given a natural number $n \geq 1$, call a cardinal α *n-good* if it is a strong limit cardinal and H_α satisfies Σ_n-collection. The predicate S consists of triples (n, α, β) such that $n \geq 1$, α and β are n-good cardinals and $H_\alpha \prec_{\Sigma_n} H_\beta$. We will denote by S_n the nth slice of the stability predicate S, namely $S_n = \{(\alpha, \beta) \mid (n, \alpha, \beta) \in S\}$.[1]

Clearly the stable core $L[S] \subseteq \mathrm{HOD}$, and the first author showed in [8] that it is consistent that $L[S]$ is smaller than HOD. The stable core is much more forcing absolute than HOD. The model $L[S]$ is clearly unaffected by forcings of size less than the ωth strong limit cardinal (because S is unaffected), and, assuming the GCH, is preserved by forcing to code the universe into a real [8]. Jensen showed that we can force over L to add a Π^1_2-singleton r with a forcing of size continuum (a subposet of Sacks forcing) [16]. The real r is obviously in $\mathrm{HOD}^{L[r]}$, and thus, we can already change HOD with a forcing of size continuum. Also, the forcing to code the universe into a real fails to preserve HOD whenever $V \neq L[a]$ for a set a by Vopenka's theorem that every set of ordinals is set-generic over HOD.

In order to motivate the many questions which arise about the stable core, let us briefly discuss the set-theoretic goals of studying inner models.

The study of canonical inner models has proved to be one of the most fruitful directions of modern set-theoretic research. The canonical inner models, of which Gödel's constructible universe L was the first example, are built bottom-up by a canonical procedure. The resulting fine structure of the models leads to regularity properties, such as the GCH and $\square$, and sometimes even absoluteness properties. But all known canonical inner models are incompatible with sufficiently large large cardinals, and indeed each such inner model is very far from the universe in the presence of sufficiently large large cardinals in the sense, for example, that covering fails and the large cardinals are not downward absolute.

[1] Here we simplify the definition of S originally given in [8] to make it easier to work with. We do not claim that the definitions are equivalent or that they produce the same model $L[S]$, only that it is not difficult to check that all the results from [8] still hold with the definition given here.

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The inner model HOD was introduced by Gödel, who showed that in a universe of ZF it is always a model of ZFC. But unlike the constructible universe which also shares this property, HOD has turned out to be highly non-canonical. While L cannot be modified by forcing, HOD can be easily changed by forcing because we can use forcing to code information into HOD. For instance, any subset of the ordinals from V can be made ordinal definable in a set-forcing extension by coding its characteristic function into the continuum pattern, so that it becomes an element of the HOD of the extension. Indeed, by coding all of V into the continuum pattern of a class-forcing extension, Roguski showed that every universe V is the HOD of one of its class-forcing extensions [28]. Thus, any consistent set-theoretic property, including all known large cardinals, consistently holds in HOD. At the same time, the HOD of a given universe can be very far from it. It is consistent that a universe can have measurable cardinals none of which are even weakly compact in HOD, and that a universe can have a supercompact cardinal which is not even weakly compact in HOD [2]. It is also consistent that HOD is wrong about all successor cardinals [4].

Does the stable core behave more like the canonical inner models or more like HOD? Is there a fine structural version of the stable core, does it satisfy regularity properties such as the GCH? Is there a bound on the large cardinals that are compatible with the stable core? Or, on the other hand, are the large cardinals downward absolute to the stable core? Can we code information into the stable core using forcing?

In this article, we show the following results about the structure of the stable core, which answer some of the aforementioned questions as well as motivate further questions about the structure of the stable core in the presence of sufficiently large large cardinals.

Measurable cardinals are consistent with the stable core.

Theorem 1.1.

(1) *The stable core of* $L[\mu]$, *the canonical model for one measurable cardinal, is* $L[\mu]$. *In particular, the stable core can have a measurable cardinal.*
(2) *Suppose that* $\langle\kappa_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ *is an increasing discrete sequence of measurable cardinals. If* $\vec{U} = \langle U_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, *where* U_α *is a normal measure on* κ_α, *then the stable core of* $L[\vec{U}]$ *is* $L[\vec{U}]$. *In particular, the stable core can have a discrete proper class of measurable cardinals.*

Theorem 1.1(1) is Corollary 4.3(1) and Theorem 1.1(2) is Theorem 4.6.

We can code information into the stable core over L or $L[\mu]$ using forcing.

Theorem 1.2. *Suppose* $\mathbb{P} \in L$ *is a forcing notion and* $G \subseteq \mathbb{P}$ *is* L*-generic. Then there is a further forcing extension* $L[G][H]$ *such that* $G \in L[S^{L[G][H]}]$ (*the universe of the stable core*). *An analogous result holds for* $L[\mu]$.

Theorem 1.2 is Theorems 3.1 and 4.7.

An extension of the coding results shows that the GCH can fail badly in the stable core.

Theorem 1.3.

(1) *There is a class-forcing extension of* L *such that in its stable core the* GCH *fails at every regular cardinal.*

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(2) *There is a class-forcing extension of* $L[\mu]$ *such that in its stable core there is a measurable cardinal and the* GCH *fails on a tail of regular cardinals.*

Theorem 1.3(1) is Theorem 3.3 and Theorem 1.3(2) is Theorem 4.8.

Measurable cardinals need not be downward absolute to the stable core.

THEOREM 1.4. *There is a forcing extension of* $L[\mu]$ *in which the measurable cardinal* κ *of* $L[\mu]$ *remains measurable, but it is not even weakly compact in the stable core.*

Theorem 1.4 is Theorem 5.1.

Although we don't know whether the stable core can have a measurable limit of measurables, the stable core has inner models with measurable limits of measurables, and much more. Say that a cardinal κ is 1-*measurable* if it is measurable, and, for $n < \omega$, $(n+1)$-*measurable* if it is measurable and a limit of n-measurable cardinals. Write $m_0^\#$ for $0^\#$ and $m_n^\#$ for the minimal mouse which is a sharp for a proper class of n-measurable cardinals, namely, an active mouse $\mathcal{M}$ such that the critical point of the top extender is a limit of n-measurable cardinals in $\mathcal{M}$. Here we mean *mouse* in the sense of [26, Sections 1 and 2], i.e., a mouse has only total measures on its sequence. The mouse $m_n^\#$ can also be construed as a fine structural mouse with both total and partial extenders (see [31], Section 4).

THEOREM 1.5. *For all* $n < \omega$, *if* $m_{n+1}^\#$ *exists, then* $m_n^\#$ *is in the stable core.*

Theorem 1.5 is Theorem 6.8.

Moreover, we obtain the following characterization of natural inner models of the stable core. Consider for $n < \omega$, the following class clubs:

$$C_n = \{\alpha \mid \alpha \text{ is a strong limit cardinal and } H_\alpha \prec_{\Sigma_n} V\}.$$

We show in Proposition 2.3 that the clubs C_n are definable in $L[S]$. It is not difficult then to see that they satisfy the hypothesis of the theorem below.

THEOREM 1.6. *Let* $n < \omega$ *and suppose that* $m_n^\#$ *exists. Then whenever*

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$$

are class clubs of uncountable cardinals such that for every $1 < i \leq n$ *and every* $\gamma \in C_i$,

$$\langle H_\gamma, \in, C_1, \dots, C_{i-1}\rangle \prec_{\Sigma_1} \langle V, \in, C_1, \dots, C_{i-1}\rangle,$$

then $L[C_1, \dots, C_n]$ *is a hyperclass-forcing extension of a (truncated) iterate of* $m_n^\#$.

An Ord-length iteration of the mouse $m_n^\#$ produces a model M satisfying ZFC without powerset whose largest cardinal is Ord. By truncating the model M at Ord, we obtain the model $V^M_{\mathrm{Ord}} \models$ ZFC. The structure $(V^M_{\mathrm{Ord}}, \in, V^M_{\mathrm{Ord}+1})$ is a model of the strong second-order set theory Kelley–Morse (with the Class Choice Principle). In second-order set theory, hyperclass-forcing notions are definable partial orders whose elements are classes (third-order objects). A forcing construction with hyperclass-forcing notions can be made sense of over models of Kelley–Morse (with the Class Choice Principle) for a certain class of nice enough partial orders. To obtain Theorem 1.6, we will force over the structure $(V^M_{\mathrm{Ord}}, \in, V^M_{\mathrm{Ord}+1})$ with an n-length iteration of Ord-length products of Prikry forcing. Since the Ord-length product of Prikry forcing uses full support, conditions in this forcing are classes in

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the structure $(V^M_{\mathrm{Ord}}, \in, V^M_{\mathrm{Ord}+1})$, making it a hyperclass-forcing notion. Details of the construction are provided in Section 7. Theorem 1.6 is Theorem 7.10.

§2. Preliminaries. Recall that, for a cardinal α, H_α is the collection of all sets x with transitive closure of size less than α. If α is regular, then H_α satisfies ZFC$^-$ (ZFC without the powerset axiom). But for singular α, H_α may fail to satisfy even Σ_2-collection.

The following proposition is standard.

Proposition 2.1. *Suppose α and β are uncountable cardinals.*

(1) $H_\alpha \prec_{\Sigma_1} V$.
(2) *If $H_\alpha \prec_{\Sigma_m} V$, then Σ_m-collection holds in H_α. In particular, every H_α satisfies Σ_1-collection.*
(3) *If $H_\alpha \prec_{\Sigma_m} H_\beta$ and Σ_m-collection holds in H_β, then it also holds in H_α.*

Proof. Let's prove (1), which is a classical fact attributed to Lévy. Suppose $\exists x\, \varphi(x, a)$ holds in V, where $\varphi(x, a)$ is a Δ_0-formula and $a \in H_\alpha$. We can assume without loss that a is transitive and has size at least ω. Let $X \prec_{\Sigma_1} V$ be a Σ_1-elementary substructure of size $|a|$ with $a \cup \{a\} \subseteq X$, and let M be the Mostowski collapse of X. Since M is transitive and has size $|a|$, it is in H_α. Also, by elementarity, M satisfies $\exists x\, \varphi(x, a)$. So there is $b \in M$ such that $M \models \varphi(b, a)$. But since $M \subseteq H_\alpha$ is transitive and $\varphi(x, y)$ is a Δ_0-assertion, it follows that H_α satisfies $\varphi(b, a)$ as well.

Next, let's prove (2). Fix a Σ_m-formula $\varphi(x, y, z)$ and sets $a, c \in H_\alpha$. Suppose that $H_\alpha \models \forall x \in a\, \exists y\, \varphi(x, y, c)$. Then, by Σ_m-elementarity, for every $\bar{a} \in a$, $\exists y\, \varphi(\bar{a}, y, c)$ holds in V. Thus, V satisfies $\forall x \in a\, \exists y\, \varphi(x, y, c)$. In V, by collection, there is a set b such that $\forall x \in a\, \exists y \in b\, \varphi(x, y, c)$ holds. So V satisfies

$$\psi(c) := \exists z\, \forall x \in a\, \exists y \in z\, \varphi(x, y, c).$$

If $m = 1$, then $\psi(c)$ is a Σ_1-assertion. Hence $H_\alpha \models \psi(c)$ by elementarity. Thus, we have verified Σ_1-collection in H_α. If $m > 1$, we can suppose inductively that we have verified Σ_{m-1}-collection in H_α. In this case, the formula $\psi(c)$ is equivalent by Σ_{m-1}-collection to a Σ_m-formula $\bar{\psi}(c)$. By Σ_m-elementarity, $H_\alpha \models \bar{\psi}(c)$. But then $H_\alpha \models \psi(c)$ since it satisfies Σ_{m-1}-collection by assumption. An analogous argument shows (3). $\dashv$

It follows immediately from Proposition 2.1(1) that the strong limit cardinals of V are definable in the stable core.

Corollary 2.2. *The class of strong limit cardinals of V is definable in the stable core $\langle L[S], \in, S\rangle$. Indeed, α is a strong limit cardinal if and only if there is a cardinal β such that $(\alpha, \beta) \in S_1$.*

The stable core can also define, for each n, the class club C_n (introduced in the introduction) of all strong limit cardinals α such that $H_\alpha \prec_{\Sigma_n} V$.

Proposition 2.3. *For every $n < \omega$, the class club C_n is definable in the stable core.*

Proof. The class club C_1 is definable because it is precisely the class of all strong limit cardinals. Now suppose inductively that the club C_i is definable for some $i \geq 1$. Let's argue that C_{i+1} is precisely the collection of all $\alpha \in C_i$ such that for

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cofinally many $\beta \in C_i$, we have $\langle \alpha, \beta \rangle \in S_{i+1}$. If α is a strong limit cardinal such that $H_\alpha \prec_{\Sigma_{i+1}} V$, then clearly $\alpha \in C_i$ and there are cofinally many $\beta \in C_i$ for which $H_\alpha \prec_{\Sigma_{i+1}} H_\beta$. Next, suppose that $\alpha \in C_i$ and for cofinally many $\beta \in C_i$, $H_\alpha \prec_{\Sigma_{i+1}} H_\beta$. Suppose V satisfies $\exists x\, \varphi(x, a)$, where φ is a Π_i-formula. Then there is a set b such that $\varphi(b, a)$ holds in V. Choose a large enough $\beta \in C_i$ with $H_\alpha \prec_{\Sigma_{i+1}} H_\beta$ such that $b \in H_\beta$. Thus, $H_\beta \models \varphi(b, a)$, and hence $H_\beta \models \exists x\, \varphi(x, a)$. Since $H_\alpha \prec_{\Sigma_{i+1}} H_\beta$, $H_\alpha \models \exists x, \varphi(x, a)$ as well. This completes our verification that $H_\alpha \prec_{\Sigma_{i+1}} V$. ⊣

Given a cardinal α, let $H^{<\alpha}$ denote the relation consisting of pairs $\langle \beta, H_\beta \rangle$ for $\beta < \alpha$.

Proposition 2.4. *For $m \geq 1$ and strong limit cardinals α and β, $H_\alpha \prec_{\Sigma_{m+1}} H_\beta$ if and only if $\langle H_\alpha, \in, H^{<\alpha} \rangle \prec_{\Sigma_m} \langle H_\beta, \in, H^{<\beta} \rangle$.*

Proof. For the forward direction, observe that the relation $H^{<\alpha}$ is Π_1-definable and amenable over H_α, which implies that predicates which are Σ_m-definable over $\langle H_\alpha, \in, H^{<\alpha} \rangle$ are Σ_{m+1}-definable over H_α. So let's focus on the backward direction. First, observe that a Σ_2-formula $\exists x\, \forall y\, \varphi(x, y, a)$ holds in H_α if and only if the Σ_1-formula

$$\exists z\, [z = (\beta, H_\beta) \wedge \exists x \in H_\beta\, \forall y \in H_\beta\, \varphi(x, y, a)]$$

holds in $\langle H_\alpha, \in, H^{<\alpha} \rangle$, and a Π_2-formula $\forall x\, \exists y\, \varphi(x, y, a)$ holds in H_α if and only if the Π_1-formula

$$\forall z [z = (\beta, H_\beta) \rightarrow \forall x \in H_\beta\, \exists y \in H_\beta\, \varphi(x, y, a)]$$

holds in $\langle H_\alpha, \in, H^{<\alpha} \rangle$. Both equivalences follow from Proposition 2.1(1) and the fact that α and β are strong limits. Thus, the complexity of any assertion is reduced by 1. ⊣

Proposition 2.5. *Suppose $1 \leq m < \omega$, α and β are strong limit cardinals, $\mathbb{P} \in H_\alpha$ is a partial order, and $G \subseteq \mathbb{P}$ is V-generic. For (1) and (2), suppose additionally that $H_\alpha \models \Sigma_m$-collection.*

(1) *The Definability Lemma and Truth Lemma for Σ_m-formulas hold for $\mathbb{P}$ in H_α. Indeed, if $\varphi(\bar{x})$ is a Σ_m-formula, then the relation $p \Vdash \varphi(\bar{x})$ is also Σ_m in H_α.*
(2) *$H_\alpha \prec_{\Sigma_m} H_\beta$ if and only if*

$$H_\alpha^{V[G]} = H_\alpha[G] \prec_{\Sigma_m} H_\beta[G] = H_\beta^{V[G]}.$$

(3) *H_α satisfies Σ_m-collection if and only if $H_\alpha[G]$ satisfies Σ_m-collection.*

Proof. The argument for (1) actually works for all cardinals α and β, not just strong limits. We argue that the standard definition of the forcing relation works in H_α. Suppose, for instance, that H_α satisfies $p \Vdash \sigma = \tau$ for $\mathbb{P}$-names $\sigma, \tau \in H_\alpha$ and let $H \subseteq \mathbb{P}$ be V-generic with $p \in H$. The relation $p \Vdash \sigma = \tau$ is a Σ_1-assertion stating that a tree exists witnessing the recursive definition of $\sigma = \tau$ in terms of names of lower rank (in fact, the assertion is Δ_1 because we can say "for every tree obeying the recursive definition..."). So by Σ_1-elementarity, $p \Vdash \sigma = \tau$ holds in V, and hence $\sigma_H = \tau_H$. Conversely, suppose that $\sigma_H = \tau_H$ for some V-generic filter $H \subseteq \mathbb{P}$. Then there is $p \in H$ such that $p \Vdash \sigma = \tau$, and hence, by Σ_1-elementarity, $p \Vdash \sigma = \tau$ holds in H_α as well. The remainder of the argument is by induction on

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the complexity of formulas. For instance, let's argue for negations. Suppose that the standard definition of the forcing relation holds in H_α for a formula φ. By definition of the forcing relation, $p \Vdash \neg\varphi$ if for every $q \leq p$, q does not force φ, but clearly this holds in H_α if and only if it holds V provided that they agree on what it means for q to force φ, which is the inductive assumption.

The argument that the definition of the forcing relation for a Σ_m-formula is itself Σ_m is also standard. The collection assumption is required to make sure that a formula is equivalent to its normal form where all the bounded quantifiers are pushed to the back. The argument above already shows that for formulae of the form "$\sigma = \tau$" the forcing relation is Δ_1. Let's argue for instance that for Δ_0-formulas, the complexity of the forcing relation is Δ_1. Say $p \Vdash \exists x \in \sigma\, \varphi(x, \sigma)$, where $\varphi(x, y)$ is a Δ_0-formula and by induction $q \Vdash \varphi(x, y)$ is a Δ_1-relation. Then $p \Vdash \exists x \in \sigma\, \varphi(x, \sigma)$ holds if and only if for every $q \leq p$, there is $r \leq q$ and $\tau \in \mathrm{dom}(\sigma)$ such that $r \Vdash \varphi(\tau, \sigma)$, and of course, quantification over elements of $\mathbb{P}$ is obviously bounded.

Now let's prove (2). We start with the forward direction, which is standard. Suppose that $H_\alpha \prec_{\Sigma_m} H_\beta$. Clearly, since $\mathbb{P} \in H_\alpha$, we have $H_\alpha[G] = H_\alpha^{V[G]}$ and similarly for H_β. If a Σ_m-assertion φ holds in $H_\alpha[G]$, then there is some $p \in G$ such that $p \Vdash \varphi$ holds in H_α, which is also a Σ_m-assertion by (1), and so $p \Vdash \varphi$ holds in H_β, meaning that $H_\beta[G]$ satisfies φ.

Next, let's prove the backward direction. Suppose that $H_\alpha[G] \prec_{\Sigma_m} H_\beta[G]$. The argument for $m = 1$ is trivial since if α and β are cardinals in $V[G]$, then they are also obviously cardinals in V, and so the result follows by Proposition 2.1(1). So suppose that $m \geq 2$. Since $\mathbb{P} \in H_\alpha$, α remains a strong limit in $V[G]$. Thus, $H_\alpha[G] = H_\alpha^{V[G]}$ has a definable hierarchy consisting of $H_\beta^{V[G]}$ for regular $\beta < \alpha$. The existence of such a hierarchy suffices for the standard Δ_2-definition of the ground model in a forcing extension (due independently to Woodin [30] and Laver [22]) to go through, so that H_α is Δ_2-definable in $H_\alpha[G]$. Indeed, examining the definition shows that $\langle H_\alpha, \in, H^{<\alpha}\rangle$ is Δ_1-definable in $\langle H_\alpha[G], \in, (H^{<\alpha})^{V[G]}\rangle$. Now suppose that H_α satisfies a Π_m-assertion $\varphi(a)$, and let $\varphi^*(a)$ be the equivalent Π_{m-1}-assertion which holds in $\langle H_\alpha, \in, H^{<\alpha}\rangle$. Since $\langle H_\alpha, \in, H^{<\alpha}\rangle$ is Δ_1-definable in $\langle H_\alpha[G], \in, (H^{<\alpha})^{V[G]}\rangle$, there is a Π_{m-1}-assertion $\varphi^{**}(a)$ expressing in $\langle H_\alpha[G], \in, (H^{<\alpha})^{V[G]}\rangle$ that $\varphi^*(a)$ holds in $\langle H_\alpha, \in, H^{<\alpha}\rangle$. By Proposition 2.4,

$$\langle H_\alpha[G], \in, (H^{<\alpha})^{V[G]}\rangle \prec_{\Sigma_{m-1}} \langle H_\beta[G], \in, (H^{<\beta})^{V[G]}\rangle.$$

Thus, $\langle H_\beta[G], \in, (H^{<\beta})^{V[G]}\rangle$ satisfies $\varphi^{**}(a)$, and therefore $\varphi^*(a)$ holds in $\langle H_\beta, \in, H^{<\beta}\rangle$. So finally, $\varphi(a)$ holds in H_β.

Finally, let's prove (3). Again, we start with the standard forward direction. Suppose that H_α satisfies Σ_m-collection. Let $\varphi(x, y)$ and a be such that

$$H_\alpha[G] \models \forall x \in a\, \exists y\, \varphi(x, y).$$

So there is some $p \in G$ and a name $\dot{a}$ for a such that $p \Vdash \forall x \in \dot{a}\, \exists y\, \varphi(x, y)$. Fix a name $\sigma \in \mathrm{dom}\, \dot{a}$ and apply Σ_m-collection in H_α to the statement

$$\forall q \leq p\, \exists y\, (q \Vdash \sigma \in \dot{a} \rightarrow q \Vdash \varphi(\sigma, y))$$

to obtain a collecting set y_σ. Next, apply Σ_m-collection in H_α, to the statement

$$\forall x \in \mathrm{dom}\, \dot{a}\, \exists z\, \exists q \leq p\, (q \Vdash x \in \dot{a} \rightarrow \exists y \in z\, q \Vdash \varphi(x, y)),$$

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which holds by the previous step because y_x witnesses it for x, to obtain a collecting set B. We can assume without loss that B consists only of $\mathbb{P}$-names and let $\dot{b} = \{(y, p) \mid y \in B\}$. It is not difficult to see that $\dot{b}_G$ gives the collecting set in $H_\alpha[G]$.

For the backward direction, assume that $H_\alpha[G]$ satisfies Σ_m-collection and let $\varphi(x, y)$ and a be such that H_α satisfies $\forall x \in a\, \exists y\, \varphi(x, y)$. Again, the case $m = 1$ is trivial since cardinals are downward absolute, so we can assume $m \geq 2$ and use the Δ_1-definability of $\langle H_\alpha, \in, H^{<\alpha}\rangle$ in $\langle H_\alpha[G], \in, (H^{<\alpha})^{V[G]}\rangle$. Thus, we can apply Σ_m-collection in $H_\alpha[G]$ to obtain a set b collecting witnesses for $\varphi(x, y)$. Since $\mathbb{P}$ can be assumed to have size less than α, we can cover $b \cap V$ with a set $\bar{b}$ of size less than α in V. So $\bar{b} \in H_\alpha$. ⊣

It follows from Proposition 2.5(2) and (3) that only an initial segment of the stability predicate can be changed by set forcing. So the stable core is at least partially forcing absolute.

Corollary 2.6. *If $\mathbb{P} \in H_\gamma$ is a forcing notion and $G \subseteq \mathbb{P}$ is V-generic, then $(n, \alpha, \beta) \in S$ if and only if $(n, \alpha, \beta) \in S^{V[G]}$ for all $\alpha, \beta \geq \gamma$. So, in particular, S and $S^{V[G]}$ agree above the size of the forcing.*

Next, let's give an argument that consistently the stable core can be a proper submodel of HOD. The fact follows from results in [8], but here we give a simplified argument suggested to the second author by Woodin.

Proposition 2.7. *It is consistent that $L[S] \subsetneq$ HOD.*

Proof. Start in L and force to add a Cohen real r. Next, force to code r into the continuum pattern on the $\aleph_n$'s and let H be $L[r]$-generic for the coding forcing $\mathbb{P}$ (the full support ω-length product forcing on coordinate n with $\mathrm{Add}(\aleph_n, \aleph_{n+2})$ whenever $n \in r$ and with trivial forcing otherwise). Observe that $\mathrm{HOD}^{L[r][H]} = L[r]$ because it has r, which the forcing $\mathbb{P}$ made definable, and it must be contained in $L[r]$ because $\mathbb{P}$ is weakly homogeneous. We would like to argue that the stable core of $L[r][H]$ is L. By Corollary 2.6, the stable core of $L[r]$ is L. So it remains to argue that forcing with $\mathbb{P}$ does not change the stable core. The forcing $\mathbb{P}$ preserves that $\aleph_\omega$ is a strong limit cardinal because it forces $2^{\aleph_n} \leq \aleph_{n+2}$ for all $n < \omega$, and it preserves all larger strong limit cardinals because it is small in size relative to them. So the strong limit cardinals of $L[r]$ are the same as in $L[r][H]$. By Corollary 2.6, only triples $(n, \aleph_\omega, \gamma)$ with $n \geq 2$ in S can be affected by $\mathbb{P}$. But for $n \geq 2$, $(n, \aleph_\omega, \gamma)$ can never make it into any stability predicate because $H_{\aleph_\omega}$ believes that there are no limit cardinals and H_γ sees $\aleph_\omega$. ⊣

We end the section with a brief description of a class forcing notion $\mathbb{P}$ making no use of the stability predicate such that V is a class generic extension of $\langle \mathrm{HOD}, \in, \mathbb{P}\rangle$ (this possibility was first suggested by Woodin). Conditions in $\mathbb{P}$ are triples $(\alpha, \varphi, \gamma)$, where $\alpha < \gamma$ are ordinals, φ is a formula with ordinal parameters below γ which defines in V_γ a non-empty subset $X(\alpha, \varphi, \gamma)$ of $P(\alpha)$. The ordering is given by $(\alpha^*, \varphi^*, \gamma^*) \leq (\alpha, \varphi, \gamma)$ whenever $\alpha \leq \alpha^*$ and for all $y \in X(\alpha^*, \varphi^*, \gamma^*)$, $y \cap \alpha \in X(\alpha, \varphi, \gamma)$. Observe that $\mathbb{P}$ is a V-definable class contained in HOD, and hence $\langle \mathrm{HOD}, \in, \mathbb{P}\rangle \models$ ZFC. It is not difficult to see that if A is an Ord-Cohen generic class of ordinals, then the collection $G(A) = \{(\alpha, \varphi, \gamma) \in \mathbb{P} \mid A \cap \alpha \in X(\alpha, \varphi, \gamma)\}$ is $\mathbb{P}$-generic over V. But since we can easily recover A from $G(A)$ and clearly $V = L[A]$,

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we have that $V = L[G(A)]$. In particular, we get that $G(A)$ is $\langle \mathrm{HOD}, \in, \mathbb{P}\rangle$-generic and $\mathrm{HOD}[G(A)] = L[G(A)] = L[A] = V$. However, unlike the forcing in [8], $\mathbb{P}$ does not have the Ord-cc.

§3. Coding into the stable core over L. We will argue that any set added generically over L can be coded into the stable core of a further forcing extension. It is easiest to code into the strong limit cardinals (because these are always definable in the stable core), but we will show that we can actually code into any mth slice S_m of the stability predicate.

THEOREM 3.1. *Suppose $\mathbb{P} \in L$ is a forcing notion and $G \subseteq \mathbb{P}$ is L-generic. Then for every $m \geq 1$, there is a further forcing extension $L[G][H]$ such that $G \in L[S_m^{L[G][H]}]$.*

PROOF. We can assume via coding that $G \subseteq \kappa$ for some cardinal κ. Also, since $\mathbb{P}$ is a set forcing, GCH holds on a tail of the cardinals in $L[G]$, and so on a tail, the strong limit cardinals coincide with the limit cardinals. Also, on a tail, S^L agrees with $S^{L[G]}$ by Corollary 2.6.

We work in L. High above κ, we will define a sequence $\langle(\beta_\xi, \beta_\xi^*) \mid \xi < \kappa\rangle$ of *coding pairs* such that $(\beta_\xi, \beta_\xi^*) \in S_m^L$. The coding forcing $\mathbb{C}$ will be defined so that if $H \subseteq \mathbb{C}$ is $L[G]$-generic, then we will have $\xi \in G$ if and only if $(\beta_\xi, \beta_\xi^*) \in S_m^{L[G][H]}$. Since $L[S_m^{L[G][H]}]$ can construct L, it will have the sequence of the coding pairs as well as $S_m^{L[G][H]}$, so that all the information put together will allow it to recover G.

Call a strong limit cardinal α *m-stable* if $H_\alpha \prec_{\Sigma_m} L$. Observe that there is a proper class of m-stable cardinals and if α and β are both m-stable, then the pair $(\alpha, \beta) \in S_m^L$. Let δ_0 be the least strong limit cardinal above κ. Let β_0 be the least m-stable cardinal above δ_0 of cofinality δ_0^+ and let β_0^* be the least m-stable cardinal above β_0. Now supposing we have defined the pairs $(\beta_\eta, \beta_\eta^*)$ of m-stable cardinals for all $\eta < \xi$, let δ_ξ be the supremum of the β_η^* for $\eta < \xi$, let β_ξ be the least m-stable cardinal above δ_ξ of cofinality δ_ξ^+, and let β_ξ^* be the least m-stable cardinal above β_ξ. In particular, $\beta_\xi > \delta_\xi^+$ since, by m-stability, β_ξ is a strong limit cardinal. Note that the sequence $\langle(\beta_\xi, \beta_\xi^*) \mid \xi < \kappa\rangle$ is Σ_{m+1}-definable over L. Note also that $\beta_\eta < \beta_\eta^* < \beta_\xi < \beta_\xi^*$ for all $\eta < \xi < \kappa$ and for limit $\lambda < \kappa$, $\beta_\lambda > \bigcup_{\xi<\lambda} \beta_\xi$, so that the sequence of the β_ξ will be purposefully discontinuous. Since the forcing $\mathbb{P}$ is small relative to δ_0, by Corollary 2.6, the coding pairs $(\beta_\xi, \beta_\xi^*) \in S_m^{L[G]}$.

Now for $\xi < \kappa$, let $\mathbb{C}_\xi$ be the following forcing. If $\xi \in G$, then $\mathbb{C}_\xi$ is the trivial forcing. If $\xi \notin G$, then $\mathbb{C}_\xi = \mathrm{Coll}(\delta_\xi^+, \beta_\xi)$. Let $\mathbb{C}$ be the full support product $\Pi_{\xi<\kappa}\mathbb{C}_\xi$ and let $H \subseteq \mathbb{C}$ be $L[G]$-generic.

Let's check that $\mathbb{C}$ collapses the minimum number of cardinals, namely $\mathbb{C}$ collapses a cardinal δ if and only if there is a non-trivial forcing stage ξ such that $\delta_\xi^+ < \delta \leq \beta_\xi$. For every $\xi < \kappa$, the forcing $\mathbb{C}$ factors as $\Pi_{\eta<\xi}\mathbb{C}_\eta \times \Pi_{\xi\leq\eta<\kappa}\mathbb{C}_\eta$, where the second part is $<\!\delta_\xi^+$-closed (using full support), and so cannot collapse any cardinals $\leq \delta_\xi^+$. Observe next that the forcing $\mathrm{Coll}(\delta_\xi^+, \beta_\xi)$ has size $\beta_\xi^{\delta_\xi} = \beta_\xi$ because $\mathrm{cf}(\beta_\xi) > \delta_\xi$ by our choice of β_ξ, and so cannot collapse any cardinal $\geq \beta_\xi^+$. It follows that the forcing $\mathbb{C}$ cannot collapse any $\delta \in (\beta_\xi, \delta_{\xi+1}^+]$. It remains to show that δ_λ and δ_λ^+ for a limit λ are preserved. By what we already showed, δ_λ is a limit of cardinals in the forcing extension, and therefore remains a cardinal. Also, by what we already

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showed, if δ_λ^+ is collapsed, then it must be collapsed to δ_λ. Suppose this happens and fix a bijection $f : \delta_\lambda \to \delta_\lambda^+$ in the forcing extension. We can let $f = \bigcup_{\xi<\lambda} f_\xi$, where $f_\xi : \gamma_\xi \to \delta_\lambda^+$ and the γ_ξ are cofinal in δ_λ. Each function f_ξ must be added by some proper initial segment of $\Pi_{\xi<\lambda}\mathbb{C}_\xi$ by closure, and therefore its range must be bounded in δ_λ^+. Now build a descending sequence of conditions $\langle p_\xi \mid \xi < \lambda\rangle$ in $\Pi_{\xi<\lambda}\mathbb{C}_\xi$ such that p_ξ decides the bound on the range of f_ξ. But then any condition p below the entire sequence forces that f is bounded in δ_λ^+, which is the desired contradiction.

By the following claim, the forcing $\mathbb{C}$ also preserves the GCH where the coding forcing takes place, so the strong limit cardinals of $L[G][H]$ are precisely the limit cardinals there.

Claim 1. *The* GCH *continues to hold on the part where it holds in* $L[G]$ *in the forcing extension* $L[G][H]$ *by* $\mathbb{C}$.

Proof. By closure, it is clear that wherever the GCH held below δ_0^+, it will continue to hold. Since $G \subseteq \kappa$, GCH holds in $L[G]$ above δ_0.

If there is trivial forcing at stage 0, then the GCH holds at δ_0^+ in $L[G][H]$. So suppose that $\mathbb{C}_0 = \mathrm{Coll}(\delta_0^+, \beta_0)$ is a non-trivial stage. Recall that $\mathrm{Coll}(\delta_0^+, \beta_0)$ has size β_0 so that there are β_0^+-many nice names for subsets of δ_0^+ (and of course in $L[G][H]$, $(\delta_0^+)^+ = (\beta_0^+)^{L[G]}$), which shows that the GCH holds at δ_0^+ in $L[G][H]$ in this case as well.

Now suppose inductively that the GCH holds up to some cardinal ρ. If $\rho = \delta_\xi^+$ for a successor ordinal ξ, we repeat the argument for $\xi = 0$. If $\delta_\xi^+ < \rho < \delta_{\xi+1}^+$ and there was non-trivial forcing at stage ξ, then $\beta_\xi < \rho < \delta_{\xi+1}^+$, and so the GCH continues to hold because the initial forcing is small relative to ρ and the tail forcing is closed. Next, suppose $\rho = \delta_\lambda^+$ for a limit cardinal $\lambda < \kappa$. Since λ is a limit, the initial segment forcing $\Pi_{\xi<\lambda}\mathbb{C}_\xi$ has size at most δ_λ^+. This means that there are $(\delta_\lambda^+)^+$-many nice-names for subsets of δ_λ^+, so that the GCH holds at $\rho = \delta_\lambda^+$. Finally, suppose $\rho = \delta_\lambda$. Each $A \subseteq \delta_\lambda$ is uniquely determined by the sequence $\langle A_\xi \mid \xi < \lambda\rangle$ with $A_\xi = A \cap \beta_\xi$. Let $\dot{f}_\xi$ be a name for an injection from $P(\beta_\xi)$ into δ_λ, which exists since, by assumption, the GCH holds below δ_λ in $L[G][H]$. Let's argue that every sequence $\langle A_\xi \mid \xi < \lambda\rangle$ such that $A_\xi \subseteq \beta_\xi$ in the extension has a name of the form $\dot{A}$, where $\dot{A}(\xi) = \dot{f}_\xi^{-1}(\gamma)$ for some $\gamma \in \delta_\lambda$. Let $\dot{B}$ be any name for the sequence $\langle A_\xi \mid \xi < \lambda\rangle$ and $p' \in H$ be a condition forcing that $\dot{B}$ is a sequence of the right form. Below p', we build a descending sequence p_ξ for $\xi < \lambda$ of conditions deciding that $\dot{B}(\xi) = \dot{f}_\xi^{-1}(\gamma_\xi)$ for some fixed $\gamma_\xi < \delta_\lambda$. By closure, there is some p below the entire sequence. So by density, there is some such $p \in H$. It follows that there are at most as many subsets of δ_λ in the extension as there are functions $f : \lambda \to \delta_\lambda$ in the ground model, and there are δ_λ^+-many such functions. ⊣

Now we will argue that the pair (β_ξ, β_ξ^*) belongs in $S_m^{L[G][H]}$ if and only if $\xi \in G$. If $\xi \notin G$, then β_ξ is not even a cardinal in $L[G][H]$, and therefore certainly $(\beta_\xi, \beta_\xi^*) \notin S_m^{L[G][H]}$. Suppose that $\xi \in G$, so that there is trivial forcing at stage ξ. By what we already argued about which cardinals are collapsed in $L[G][H]$, it follows that β_ξ and β_ξ^* are limit cardinals there. Let $\mathbb{C}_{\text{small}} = \Pi_{\eta<\xi}\mathbb{C}_\eta$ and $\mathbb{C}_{\text{tail}} = \Pi_{\xi<\eta<\kappa}\mathbb{C}_\eta$, and note that since there is no forcing at stage ξ, $\mathbb{C}$ factors as $\mathbb{C}_{\text{small}} \times \mathbb{C}_{\text{tail}}$.

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Let $H_{\text{small}} \times H_{\text{tail}}$ be the corresponding factoring of the generic filter H. Since $\mathbb{C}_{\text{tail}}$ is $\leq \beta^*_\xi$-closed, we have that $H^{L[G][H]}_{\beta_\xi} = H^{L[G][H_{\text{small}}]}_{\beta_\xi}$ and $H^{L[G][H]}_{\beta^*_\xi} = H^{L[G][H_{\text{small}}]}_{\beta^*_\xi}$. By Proposition 2.5(3), $H^{L[G][H_{\text{small}}]}_{\beta_\xi}$ satisfies Σ_m-collection and by Proposition 2.5(2), $H^{L[G][H_{\text{small}}]}_{\beta_\xi} \prec_{\Sigma_m} H^{L[G][H_{\text{small}}]}_{\beta^*_\xi}$. ⊣

It follows from Theorem 3.1 that (consistently) the stable core is not a fine-structural or in any sense canonical inner model. Among the numerous corollaries of Theorem 3.1 are the following.

Corollary 3.2.

(1) *The* GCH *can fail on an arbitrarily large initial segment of the regular cardinals in the stable core.*

(2) *An arbitrarily large ordinal of L can be countable in the stable core.*

(3) MA + ¬CH *can hold in the stable core.*

Proof. For (1), we force over L to violate the GCH on an initial segment of the regular cardinals, and then code all the subsets we add into the stable core of the forcing extension by the coding forcing $\mathbb{C}$. For (2), we force over L to collapse the ordinal, and then code the collapsing map into the stable core of the forcing extension by the coding forcing $\mathbb{C}$. For (3), we force Martin's Axiom with $2^\omega = \kappa$, where κ is uncountable and regular, to hold over L, and let $L[G]$ be the forcing extension. We then code the G into the stable core of the forcing extension high above κ. Any ccc partial order $\mathbb{P}$ on κ in the stable core of the coding extension already exists in $L[G]$, and therefore G will have added a partial generic filter (meeting some less than continuum many dense sets) for it. ⊣

Theorem 3.3. *It is consistent that the* GCH *fails at all regular cardinals in the stable core.*

Proof. The idea will be to force the GCH to fail at all regular cardinals over L, and then use Ord-many coding pairs to code all the added subsets into the stable core of a forcing extension. In this argument, we will code into the limit cardinals, namely S_1, by using generalized Cohen forcing instead of the collapse forcing.

In L, let $\mathbb{P}$ be the Easton support Ord-length product forcing with $\text{Add}(\kappa, \kappa^{++})$ at every regular cardinal κ, and let $G \subseteq \mathbb{P}$ be L-generic. Standard arguments show that in $L[G]$, $2^\kappa = \kappa^{++}$ for every regular cardinal κ, while the GCH continues to hold at singular cardinals (see, for example, [15]). Since $\langle L[G], \in, G\rangle$ has a definable global well-order, we can assume via coding that $G \subseteq \text{Ord}$ (and define the coding forcing in this expanded structure).

We first work in L. Let δ_0 be the least strong limit cardinal. Above δ_0, we will define a sequence $\langle(\beta_\xi, \beta^*_\xi) \mid \xi \in \text{Ord}\rangle$ of *coding pairs* of strong limit cardinals. Let $\beta_0 < \beta^*_0$ be the next two strong limit cardinals above δ_0. Now supposing we have defined the pairs $(\beta_\eta, \beta^*_\eta)$ of strong limit cardinals for all $\eta < \xi$, let δ_ξ be the supremum of the β^*_η for $\eta < \xi$ and let $\beta_\xi < \beta^*_\xi$ be the next two strong limit cardinals above δ_ξ. Observe that every strong limit cardinal of L remains a strong limit in $L[G]$, and so in particular, the elements β_ξ and β^*_ξ of the coding pairs are strong limits in $L[G]$.

For each ordinal ξ, let $\mathbb{C}_\xi$ be the following forcing. If $\xi \in G$, then $\mathbb{C}_\xi$ is the trivial forcing. So suppose that $\xi \notin G$. If δ_ξ is singular, we let $\mathbb{C}_\xi = \text{Add}(\delta^+_\xi, \beta_\xi)$ (the partial order to add β_ξ-many Cohen subsets to δ^+_ξ with bounded conditions),

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and otherwise, we let $\mathbb{C}_\xi = \mathrm{Add}(\delta_\xi, \beta_\xi)$. Let's argue that all forcing notions $\mathbb{C}_\xi$ are cardinal preserving. If δ_ξ is singular, then the GCH holds at δ_ξ, and therefore $\mathrm{Add}(\delta_\xi^+, \beta_\xi)$ has the $(2^{<\delta_\xi^+})^+ = (2^{\delta_\xi})^+ = \delta_\xi^{++}$ chain condition, which means that it preserves all cardinals. If δ_ξ is regular, then it is inaccessible because it is always a limit cardinal, and therefore $\mathrm{Add}(\delta_\xi, \beta_\xi)$ preserves all cardinals. Obviously, every nontrivial forcing $\mathbb{C}_\xi$ destroys the strong limit property of β_ξ in the forcing extension.

Let $\mathbb{C}$ be the Ord-length Easton support product $\Pi_{\xi\in\mathrm{Ord}}\mathbb{C}_\xi$. Let's argue that the forcing notion $\mathbb{C}$ is also cardinal preserving. Observe first that if δ_ξ is singular, then the initial segment $\Pi_{\eta<\xi}\mathbb{C}_\eta$ has size $\delta_\xi^{\delta_\xi} = \delta_\xi^+$ since the GCH holds at δ_ξ. If δ_ξ is regular, then δ_ξ is inaccessible, so that conditions in $\Pi_{\eta<\xi}\mathbb{C}_\eta$ are bounded, and hence $\Pi_{\eta<\xi}\mathbb{C}_\eta$ has size $\delta_\xi^{<\delta_\xi} = \delta_\xi$. Now we can argue that if $\delta_\xi^+ < \gamma < \delta_{\xi+1}$ is a cardinal, then it remains a cardinal in the forcing extension by $\mathbb{C}$ because by previous calculations, the initial segment $\Pi_{\eta<\xi}\mathbb{C}_\eta \times \mathbb{C}_\xi$ cannot collapse γ, and the tail forcing is highly closed. Cardinals of the form $\delta_{\xi+1}$ cannot be collapsed because the successor stage forcings are cardinal preserving. It remains to consider cardinals of the form δ_λ and δ_λ^+ for a limit cardinal λ. By what we already showed, δ_λ is a limit of cardinals in the forcing extension, and hence must be a cardinal itself. If δ_λ is regular, then it is inaccessible, and hence the initial segment $\Pi_{\xi<\lambda}\mathbb{C}_\xi$ is too small to collapse δ_λ^+. So suppose that δ_λ is singular with $\mathrm{cof}(\delta_\lambda) = \mu < \delta_\lambda$. By regrouping the product, we can view the forcing $\Pi_{\xi<\lambda}\mathbb{C}_\xi$ as a product of length μ, which is $<\mu$-closed on a tail. Thus, an analogous argument to the one given in the proof of Theorem 3.1 shows that δ_λ^+ cannot be collapsed to δ_λ in this case, completing the proof that $\mathbb{C}$ is cardinal preserving. In particular, this implies that the GCH continues to fail at all regular cardinals in any forcing extension by $\mathbb{C}$.

Let $H \subseteq \mathbb{C}$ be $L[G]$-generic. For each $\xi \in \mathrm{Ord}$, we can factor $\mathbb{C}$ as the product $\Pi_{\eta<\xi}C_\eta \times \Pi_{\xi\leq\eta}C_\eta$, where the tail forcing $\Pi_{\xi\leq\eta}C_\eta$ is $<\delta_\xi$-closed since we used Easton support. Note that since $\mathbb{C}$ is a progressively closed class product, it preserves ZFC to the forcing extension $L[G][H]$ [27].

Suppose $\xi < \kappa$ is a trivial stage of forcing in $\mathbb{C}$. Let $\mathbb{C}_{\mathrm{small}} = \Pi_{\eta<\xi}\mathbb{C}_\eta$ and $\mathbb{C}_{\mathrm{tail}} = \Pi_{\xi<\eta}\mathbb{C}_\eta$. The forcing $\mathbb{C}_{\mathrm{small}}$ has size at most δ_ξ^+, and therefore cannot destroy the strong limit property of β_ξ and β_ξ^*, and neither can $\mathbb{C}_{\mathrm{tail}}$, which is $<\beta_\xi^*$-closed. It follows that β_ξ and β_ξ^* remain strong limits in $L[G][H]$. $\dashv$

In the above result, we coded the subsets added by G into S_1. Let's see what it would take to code subsets added by G into the mth slice S_m of the stability predicate for $m \geq 2$. The main problem is that if α is a singular cardinal, then $\mathbb{P} \restriction \alpha$ has unbounded support in α, and therefore $\mathbb{P} \restriction \alpha$ is not a class forcing over H_α, which prevents us from using standard lifting arguments to go from $H_\alpha^L \prec_{\Sigma_m} H_\beta^L$ to $H_\alpha^{L[G]} \prec_{\Sigma_m} H_\beta^{L[G]}$. The construction would go through for m, if we assume that L has a proper class of inaccessible cardinals α such that $H_\alpha^L \prec_{\Sigma_{m+1}} L$. The class forcing $\mathbb{P}$ is Δ_2-definable, so the forcing relation for Σ_m-formulas is Σ_{m+1}-definable. Using this, we can argue that if $H_\alpha^L \prec_{\Sigma_{m+1}} H_\beta^L$, then $H_\alpha^{L[G]} = H_\alpha^L[G] \prec_{\Sigma_m} H_\beta^L[G] = H_\beta^{L[G]}$.

Finally, let's note that if we only wanted the GCH to fail cofinally, then we could force in a single step to add κ^{++}-many subsets to some κ, followed by the forcing to code the sets into the stable core, and do this for cofinally many cardinals, spacing them out enough to prevent interference.

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§4. Measurable cardinals in the stable core. In [18], Kennedy, Magidor, and Väänänen studied properties of the model $\langle L[\text{Card}], \in, \text{Card}\rangle$ for the class Card of cardinals of V. They showed that if there is a measurable cardinal, then $L[\mu]$, the canonical model for a single measurable cardinal, is contained in $L[\text{Card}]$. In particular, $L[\text{Card}]^{L[\mu]} = L[\mu]$, which shows that $L[\text{Card}]$ can have a measurable cardinal. Recently, Welch showed that if $m_1^\#$ exists, then $L[\text{Card}]$ is a certain Prikry-type forcing extension of an iterate of $m_1^\#$ adding Prikry sequences to all measurable cardinals in it [29]. It follows from this that, in the presence of sufficiently large large cardinals, the model $L[\text{Card}]$ satisfies the GCH and has no measurable cardinals, although it does have an inner model with a proper class of measurables.

We adapt techniques of [18] to show that if there is a measurable cardinal, then, for every $m \geq 1$, $L[\mu]$ is contained in $L[S_m]$. In particular, $L[S^{L[\mu]}] = L[\mu]$, showing that the stable core can have a measurable cardinal. Indeed, we improve this result to show that the stable core can have a discrete proper class of measurable cardinals.

Let's start with the following easy proposition showing that if $0^\#$ exists, then it is in the stable core.

Proposition 4.1. *If $0^\#$ exists, then $0^\# \in L[S_m]$ for every $m \geq 1$.*

Proof. Every $L[S_m]$ has many increasing ω-sequences of V-cardinals, so fix some such sequence $\langle \alpha_n \mid n < \omega\rangle$. We have that $\varphi(x_0, \dots, x_{n-1}) \in 0^\#$ if and only if $L_{\alpha_n} \models \varphi(\alpha_0, \dots, \alpha_{n-1})$. $\dashv$

Theorem 4.2. *Suppose that κ is a measurable cardinal and $L[\mu]$ is the canonical inner model with a normal measure μ on κ. Then $L[\mu] \subseteq L[S_m]$ for every $m \geq 1$.*

The proof of this theorem uses techniques from the proof of Kunen's Uniqueness Theorem ([19], [20], for a modern account, see, for example, [15, Theorem 19.14]) and is following the idea of Theorem 9.1 in [18].

Proof of Theorem 4.2. We will first argue that for some sufficiently large λ, the normal measure μ_λ on the λth iterated ultrapower of $L[\mu]$ by μ is in $L[S_m]$, and then find in $L[S_m]$ an elementary substructure of size $(\kappa^+)^V$ of an initial segment $L_\theta[\mu_\lambda]$, for some very large θ (in particular, ensuring that $\mu_\lambda \in L_\theta[\mu_\lambda]$), of the iterate that will collapse to $L_{\bar\theta}[\mu]$. Since we were able to choose the substructure to be of size $(\kappa^+)^V$, $\bar\theta \geq (\kappa^+)^{L[\mu]}$, ensuring that $L_{\bar\theta}[\mu]$ contains μ.

We can assume that $\mu \in L[\mu]$. We work in V and fix $m \geq 1$. Let $\lambda > \kappa^+$ be a strong limit cardinal with unboundedly many α in λ such that $(\alpha, \lambda) \in S_m$. Let $j_\lambda : L[\mu] \to L[\mu_\lambda]$ be the embedding given by the λth iterated ultrapower of $L[\mu]$ by μ, so that in $L[\mu_\lambda]$, μ_λ is a normal measure on the cardinal $\lambda = j_\lambda(\kappa)$ (by [17, Corollary 19.7], for all cardinals $\lambda > \kappa^+$, the λth element of the critical sequence is λ). Let $\langle \kappa_\xi \mid \xi < \lambda\rangle$ be the critical sequence of the iteration by μ. Finally, let $\mathcal{F}$ denote the filter generated by the tails

$$A_\xi = \{\eta < \lambda \mid \xi \leq \eta \text{ such that } (\eta, \lambda) \in S_m\}$$

for $\xi < \lambda$. We will argue that $L[\mu_\lambda] = L[\mathcal{F}]$. It will follow that $L[\mu_\lambda] \subseteq L[S_m]$ since $L[S_m]$ can compute $L[\mathcal{F}]$ from S_m.

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First, let's argue that $\mu_\lambda \subseteq \mathcal{F}$. Suppose $X \in \mu_\lambda$. Then there must be a $\zeta < \lambda$ such that $\{\kappa_\xi \mid \zeta \leq \xi < \lambda\} \subseteq X$ (see [17, Lemma 19.5]). As $\kappa_\eta = \eta$ for every sufficiently large cardinal $\eta < \lambda$ (see [17, Corollary 19.7]), it follows that

$$\{\eta < \lambda \mid \zeta' \leq \eta \text{ and } \eta \text{ is a cardinal}\} \subseteq X$$

for some $\zeta' < \lambda$. In particular, $A_{\zeta'} \subseteq X$, and thus $X \in \mathcal{F}$. But now since μ_λ is an ultrafilter in $L[\mu_\lambda]$ and $\mathcal{F}$ is a filter, it follows that $\mathcal{F} \cap L[\mu_\lambda] \subseteq \mu_\lambda$ and hence $\mathcal{F} \cap L[\mu_\lambda] = \mu_\lambda$. From here it is not difficult to see that $L[\mu_\lambda] = L[\mathcal{F}]$, and hence $L[\mu_\lambda] \subseteq L[S_m]$.

Now we will define in $L[S_m]$, a sequence of length $(\kappa^+)^V$ whose elements will generate the desired elementary substructure. Recall that if η is a strong limit cardinal of cofinality greater than κ and moreover $\eta > \lambda$ (the length of the iteration), then $j_\lambda(\eta) = \eta$ (see [17, Corollary 19.7]). Let $\lambda^* \gg \lambda$ be a strong limit cardinal of cofinality greater than $(\kappa^+)^V$ such that the set $S_m^{\lambda^*} = \{\eta \mid (\eta, \lambda^*) \in S_m\}$ is unbounded in λ^*. Let η_0 be the $(\kappa^+)^V$th element of $S_m^{\lambda^*}$ above λ. Inductively, let $\eta_{\xi+1}$ be the $(\kappa^+)^V$th element of $S_m^{\lambda^*}$ above η_ξ and $\eta_\delta = \bigcup_{\xi<\delta} \eta_\xi$ for limit ordinals δ. Let $A = \{\eta_{\xi+1} \mid \xi < (\kappa^+)^V\}$. As $(\kappa^+)^V$ is regular (in V), it follows that $\mathrm{cf}^V(\eta_{\xi+1}) = (\kappa^+)^V$ for all $\eta_{\xi+1} \in A$. Therefore each element of A is fixed by the iteration embedding j_λ.

Fix θ above the supremum of A. Let $X \prec L_\theta[\mu_\lambda]$ be the Skolem hull of $\kappa \cup A$ in $L_\theta[\mu_\lambda]$, and note that $X \in L[S_m]$. Let N denote the Mostowski collapse of X, and let

$$\sigma : N \to X \prec L_\theta[\mu_\lambda]$$

be the inverse of the collapse embedding. Note that λ is in X as it is definable as the unique measurable cardinal in $L_\theta[\mu_\lambda]$. In fact, $\sigma(\kappa) = \lambda$ by the following argument. As X is generated by elements from j_λ"$L[\mu]$, it is contained in j_λ"$L[\mu] \prec L[\mu_\lambda]$. But there is no $\gamma \in j_\lambda$"$L[\mu]$ with $\kappa < \gamma < \lambda$, so λ has to collapse to κ. Finally, since $|A| = (\kappa^+)^V$ and $\sigma(\kappa) = \lambda$, the collapse N has the form $L_{\bar\theta}[\nu]$ with ν a normal measure on κ and $\bar\theta$ an ordinal of size $(\kappa^+)^V$. By Kunen's Uniqueness Theorem (see, for example, [15, Theorem 19.14]), $N = L_{\bar\theta}[\mu]$, and thus $L_{\bar\theta}[\mu] \in L[S_m]$. So $L[\mu] \subseteq L[S_m]$, as desired. ⊣

COROLLARY 4.3.

(1) *We have* $L[S^{L[\mu]}] = L[\mu]$. *In particular, it is consistent that the stable core has a measurable cardinal.*

(2) *Let* K^{DJ} *denote the Dodd–Jensen core model below a measurable cardinal. Then* $K^{DJ} \subseteq L[S]$, *and hence* $L[S^{K^{DJ}}] = K^{DJ}$.

(3) *If* $0^\dagger$ *exists, then* $0^\dagger \in L[S]$.

PROOF. (1) follows immediately from Theorem 4.2 by applying it inside $V = L[\mu]$.

For (2) we first recall the definition of the Dodd–Jensen core model K^{DJ} from [5]. We call a transitive model M of the form $M = \langle J_\alpha[U], \in, U\rangle$ a *Dodd–Jensen mouse* if M satisfies that U is a normal measure on some $\kappa < \alpha$, all of the iterated ultrapowers of M by U are well-founded, and M has a fine structural property implying that $M = \mathrm{Hull}_1^M(\rho \cup p)$ (the Σ_1-Skolem closure of M) for some ordinal $\rho < \kappa$ and some

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finite set of parameters $p \subseteq \alpha$ (see [5, Definition 5.4]). The Dodd–Jensen core model $K^{DJ} = L[\mathcal{M}]$, where $\mathcal{M}$ is the class of all such Dodd–Jensen mice (see [5, Definition 6.3] or, for a modern account, [26]). So we need to argue that every such mouse M is in $L[S]$. We essentially follow the proof of Theorem 4.2 to show that some λth iterate M_λ of M is in $L[S]$. Then we argue that $M \in L[M_\lambda]$ as, by Σ_1-elementarity of j_λ, M is isomorphic to $\text{Hull}_1^{M_\lambda}(\rho \cup j_\lambda(p))$. Hence, $M \in L[S]$.

For (3), since the strong limit cardinals of V are definable in $L[S]$, the result for $0^\dagger$ follows from Theorem 4.2 as in the proof of Proposition 4.1. ⊣

By analyzing the proof of Theorem 4.2, we see that what it really used was not S_m, but the class club C_m (of all strong limit cardinals α such that $H_\alpha \prec_{\Sigma_m} V$) which is, of course, definable from S_m. Thus, abstracting away the argument, we see that the proof of Theorem 4.2 relied on the fact that for a certain club C of cardinals (namely C_m), $L[\mu] \subseteq L[C]$ and $L[C] \subseteq L[S]$. The following result shows that this argument with one club C cannot be pushed further to show that stronger large cardinals are in the stable core. Given a club C, we will denote by $\hat{C}$, the collection of all successor elements of C together with its least element.

Theorem 4.4 [29]. *Suppose that C is a class club of uncountable cardinals. Then there is an* Ord*-length iteration of the mouse $m_1^\#$ such that in the direct limit model M_C (truncated at* Ord*), the measurable cardinals are precisely the elements of $\hat{C}$.*

Proof. Let $\bar{\kappa}$ be the critical point of the top measure of $m_1^\#$. Let $\langle \alpha_\xi \mid \xi \in \text{Ord}\rangle$ be the increasing enumeration of C.

Iterate the first measurable cardinal κ_0 of $m_1^\#$ α_0-many times, so that κ_0 iterates to α_0, and let M_{α_0} be the iterate. Since $m_1^\#$ is countable, M_{α_0} has cardinality α_0, and hence the critical point of the top measure $\bar{\kappa}_{\alpha_0}$, the image of $\bar{\kappa}$ in M_{α_0}, is below α_1. In particular, the next measurable cardinal κ_1 above α_0 in M_{α_0} is below α_1 and we can iterate it to α_1 by iterating it α_1-many times. Repeat this for all successor ordinals ξ and take direct limits along the iteration embeddings at limit stages with the following exception.

Suppose we have carried out the construction for a limit ξ-many steps resulting in the model M_ξ, where $\bar{\kappa}_\xi$ is the critical point of the top measure, in which the measurable cardinals limit up to $\bar{\kappa}_\xi$. In this case, we must have $\xi = \alpha_\xi$. When this happens we have run out of room and don't have a measurable cardinal to iterate to the next element $\alpha_{\xi+1}$ of $\hat{C}$. To make more space, in the next step, we iterate up the top measure to obtain the model $M_{\xi+1}$ with more measurable cardinals. By cardinality considerations, the critical point $\bar{\kappa}_{\xi+1}$ of the top measure is obviously below $\alpha_{\xi+1}$. Hence, we can continue the construction, iterating the least measurable cardinal $\kappa_{\xi+1}$ above ξ in $M_{\xi+1}$ to $\alpha_{\xi+1}$.

Let M be the resulting model obtained as the direct limit along the iteration embeddings and let M_C be M truncated at Ord, which is the cardinal on which the top measure of M lives. The construction ensures that we hit every element of $\hat{C}$ along the way, so that the measurable cardinals in M_C are exactly the elements $\hat{C}$. ⊣

A more elaborate version of this iteration argument is going to be used in Section 6 to generalize the results of [29] to a finite number of specially nested clubs.

Corollary 4.5. *If C is a class club of uncountable V-cardinals, then $m_1^\# \notin L[C]$.*

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PROOF. Suppose towards a contradiction that $m_1^\# \in L[C]$. Iterate $m_1^\#$ inside $L[C]$ to a model M as in the proof of Theorem 4.4, and let M_C be the truncation of M at Ord. In particular, C is definable in M_C by considering the closure of its measurable cardinals. It follows that $L[C]$ is a definable sub-class of M_C, so that $L[C] = M_C$. But this is impossible because $m_1^\#$ is a countable model (in $L[C]$), which means, in particular, that $\omega_1^{m_1^\#} = \omega_1^{M_C}$ is countable in $L[C]$. ⊣

In the last section of the article we will show that, unlike L[Card], the stable core, given sufficiently large large cardinals, can have inner models with a proper class of n-measurable cardinals for any $n < \omega$.

Now we can say more precisely what Welch showed about the model L[Card] in [29]. Let M_C, for the class club C of limit cardinals, be the iterate of $m_1^\#$ (truncated at Ord) obtained as in the proof of Theorem 4.4 in which the measurable cardinals are precisely the cardinals of V of the form $\aleph_{\omega\cdot\alpha+\omega}$ (namely elements of $\hat{C}$). Let $U_\alpha \subseteq P^{M_C}(\aleph_{\omega\cdot\alpha+\omega})$ be the iteration measures on $\aleph_{\omega\cdot\alpha+\omega}$ in M_C, and note that a subset of $\aleph_{\omega\cdot\alpha+\omega}$ in M_C is in U_α if and only if it contains some tail of the cardinals. Let $\vec{U} = \langle U_\alpha \mid \alpha \in \mathrm{Ord}\rangle$. The model M_C has the form $L[\vec{U}]$ because it is an iterate of the mouse $m_1^\#$. Let $W_\alpha \subseteq P^{L[\mathrm{Card}]}(\aleph_{\omega\cdot\alpha+\omega})$ consist of all subsets of $\aleph_{\omega\cdot\alpha+\omega}$ in L[Card] containing some tail of cardinals $\aleph_{\omega\cdot\alpha+n}$, $n < \omega$, and let $\vec{W}$ be the sequence of the W_α. Now it is easy to see that $L[\vec{W}] = L[\vec{U}]$, and hence, since $\vec{W}$ is definable in L[Card], M_C is contained in L[Card]. Let f be a function on $\hat{C}$ such that

$$f(\omega\cdot\alpha+\omega) = \langle \omega\cdot\alpha+n \mid n < \omega\rangle.$$

We clearly have that $L[\mathrm{Card}] = L[f]$, and also $L[f] = L[\vec{W}][f] = M_C[f]$ because the sequence $\vec{W}$ can be recovered from f. Thus, $L[\mathrm{Card}] = M_C[f]$, and it turns out that in some sense which we will explain in detail in Section 7, $M_C[f]$ is a Prikry-type forcing extension of M_C adding Prikry sequences to all its measurable cardinals.

Note that in this construction we iterated the measurable cardinals to elements of $\hat{C}$, where C is the club of limit cardinals, instead of to all successor cardinals, because we need to have enough cardinals in between to be able to use them to define the measures in $\vec{U}$, so that $L[\vec{U}]$ is contained in $L[C]$. If on the other hand, we iterate the measurable cardinals to all successor cardinals, then we can get an inclusion in the other direction: $L[C]$ is contained in the iterate M_C.

We will now generalize the result that the stable core of $L[\mu]$ is equal to $L[\mu]$ to show that if $\vec{U}$ is a discrete proper class sequence of normal measures, then the stable core of $L[\vec{U}]$ is $L[\vec{U}]$. It follows that the stable core can have a proper class of measurable cardinals.

THEOREM 4.6. *If $\vec{U}$ is a discrete proper class sequence of normal measures, then $L[S]^{L[\vec{U}]} = L[\vec{U}]$. In particular, it is consistent that the stable core has a proper class of measurable cardinals.*

PROOF. Let $\vec{U}$ be a discrete proper class sequence of normal measures and work in $V = L[\vec{U}]$. Consider the stable core $L[S]$ and the corresponding core models $K_0 = K^{L[S]}$ and $K = K^V$. Recall that all measurable cardinals in V are measurable in K as witnessed by restrictions of the measures in $\vec{U}$, and therefore V and K have the same universes. Compare K_0 and K in V. As both are proper class models they have a common iterate K^*.

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Case 1. *The K-side of the coiteration drops.*

Then K^* is the Ord-length iterate of some mouse[2] $\mathcal{M}$ which appears after the last drop on the K-side of the coiteration such that K^* is the result of hitting a measure on some κ in $\mathcal{M}$ and its images (truncated at Ord). The successive images of κ form a V-definable club D_0 of ordinals which are regular cardinals in K^*. The K_0-side of the coiteration does not drop, so there is an iteration map $\pi_0\colon K_0 \to K^*$ and the ordinals α such that $\pi_0``\alpha \subseteq \alpha$ form a V-definable club D_1. Let $D = D_0 \cap D_1$. Note that the iteration of K_0 has set-length, since the measures on the K-side, and therefore also those on the K_0-side, are bounded by the measurable κ which is sent to Ord on the K-side of the iteration (by the discreteness of the measure sequence). It follows that for some δ, all elements of D of cofinality at least δ are fixed by the iteration map π_0.

Let $n < \omega$ be large enough such that D is Σ_n-definable in V. Recall from Proposition 2.3 that the class club C_n consisting of all strong limit β such that $H_\beta \prec_{\Sigma_n} V$ is definable from S. Let $\beta \in C_n$ be sufficiently large. In V, D is cofinal in Ord. Therefore, in H_β, $D \cap H_\beta$ is cofinal in β, and hence $\beta \in D$. So a tail of C_n is contained in D and there is a δ-sequence of adjacent elements of C_n contained in D such that its limit λ is singular of uncountable cofinality in $L[S]$. But $\lambda \in D$ and all elements of D are regular in K^*. As $\pi_0(\lambda) = \lambda$, λ is also regular in K_0, contradicting the covering lemma for sequences of measures in $L[S]$ (see [24], [25]).

Case 2. *The K_0-side of the coiteration drops.*

Let $\mathcal{N}$ be the model on the K_0-side of the coiteration after the last drop. Then $\mathcal{N} \cap \mathrm{Ord} < \mathrm{Ord}$, but the coiteration of $\mathcal{N}$ and an iterate K' of K results in the common proper class iterate K^*. The iteration from K to K' is non-dropping and hence K' is universal. But this contradicts the fact that the coiteration of $\mathcal{N}$ and K' does not terminate after set-many steps.

Case 3. *Both sides of the coiteration do not drop, i.e., there are elementary embeddings $\pi_0\colon K_0 \to K^*$ and $\pi\colon K \to K^*$.*

As K^*, and hence K_0, has a proper class discrete sequence of measures it is universal in $V = L[\vec{U}]$. Therefore, in fact, $K_0 = K^*$ by the proof of Theorem 7.4.8 in [31]. Finally, we argue that K cannot move in the iteration to K_0. Suppose this is not the case and let U on κ be the first measure that is used. Let κ^* be the image of κ in K_0. For some large enough $n < \omega$, C_n can define a proper class C^* of fixed points of π as follows. There is a V-definable club C of ordinals α such that $\pi``\alpha \subseteq \alpha$. As in Case 1, a tail of C_n is contained in C. Let $\beta \in C_n$ be an arbitrary element of that tail and let γ be the ω-th element of C_n above β. Then γ is a closure point of π and $\mathrm{cf}(\gamma) = \omega$ in V and hence in K, since the universes of V and K agree. So the iteration map is continuous at γ and therefore $\pi(\gamma) = \gamma$.

Let $\bar{K}_0$ be the transitive collapse of $\mathrm{Hull}^{K_0}(\kappa \cup \{\kappa^*\} \cup C^*)$. Then $\bar{K}_0$ has a proper class of measurable cardinals including κ. In particular, $\bar{K}_0$ is a universal weasel, and hence an iterate of K, where the first measure used in the iteration has critical

[2]Note that when we say *mouse* here we mean a fine structural iterable premouse which has partial measures on its sequence as, for example, in [31, Section 4].

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point above κ. Therefore $\bar{K}_0$ and hence $L[S]$ and K_0 contain the measure U on κ, contradicting the fact that this measure was used in the iteration.

Therefore, we obtain that $K_0 = K$. As $L[\vec{U}]$ can be reconstructed from K it follows that $L[S] = L[\vec{U}]$. ⊣

The arguments of Section 3 generalize directly to coding sets added generically over $L[\mu]$ into the stable core of a further forcing extension. If the forcing adding the generic sets is either small relative to the measurable cardinal κ of $L[\mu]$ or $\leq \kappa$-closed, and the coding is done high above κ, then the stable core of the coding extension will continue to think that κ is measurable.

Theorem 4.7. *Suppose $V = L[\mu]$. If $\mathbb{P}$ is a forcing notion of size less than κ or $\mathbb{P}$ is $\leq \kappa$-closed and $G \subseteq \mathbb{P}$ is V-generic, then there is a further forcing extension $V[G][H]$ such that $G \in L[S^{V[G][H]}]$, and κ remains a measurable cardinal there.*

Proof. Suppose $\mathbb{P}$ is a small forcing. By the Lévy–Solovay theorem, κ remains measurable in $V[G]$, as witnessed by the normal measure ν on κ such that $A \in \nu$ if and only if there is $\bar{A} \in \mu$ with $\bar{A} \subseteq A$. Since the coding forcing defined in the proof of Theorem 3.1 is $\leq \kappa$-closed, ν continues to be a normal measure on κ in $V[G][H]$. Since $L[\mu] \subseteq L[S^{V[G][H]}]$ by Theorem 4.2, in $L[S^{V[G][H]}]$, we can define ν^* such that $A \in \nu^*$ if and only if there is $\bar{A} \in \mu$ with $\bar{A} \subseteq A$, and clearly ν^* must be a normal measure on κ in $L[S^{V[G][H]}]$.

The argument for $\leq \kappa$-closed $\mathbb{P}$ is even easier because μ remains a normal measure on κ in $V[G][H]$. ⊣

Moreover, we get the following variant of Theorem 3.3 in the presence of a measurable cardinal.

Theorem 4.8. *It is consistent relative to the existence of a measurable cardinal that the stable core has a measurable cardinal above which the* GCH *fails at all regular cardinals.*

§5. Measurable cardinals are not downward absolute to the stable core. In this section, we show that it is consistent that measurable cardinals are not downward absolute to the stable core. We will use a modification of Kunen's classical argument that weakly compact cardinals are not downward absolute [21].

Theorem 5.1. *Measurable cardinals are not downward absolute to the stable core. Indeed, it is possible to have a measurable cardinal in V which is not even weakly compact in the stable core.*

Proof. Suppose $V = L[\mu]$, where μ is a normal measure on a measurable cardinal κ.

Let $\mathbb{P}_\kappa$ be the Easton support iteration of length κ forcing with $\mathrm{Add}(\alpha, 1)$ at every stage α such that α is a regular cardinal in $V^{\mathbb{P}_\alpha}$. It is a standard fact that whenever the GCH holds, which is the case in $V = L[\mu]$, $\mathbb{P}_\kappa$ preserves all cardinals, cofinalities, and the GCH (see, for example, [3]). Let $G \subseteq \mathbb{P}_\kappa$ be V-generic. In $V[G]$, let $\mathbb{Q}$ be the forcing to add a homogeneous κ-Souslin tree and let T be the $V[G]$-generic tree added by $\mathbb{Q}$ (the forcing originally appeared in [21] and a more modern version is written up in [12]). In $V[G][T]$, let $\mathbb{C}$ be the forcing, as in the proof of Theorem 3.1

for the case $m = 1$, to code T, using the strong limit cardinals, into the stable core of a forcing extension high above κ and let $H \subseteq \mathbb{C}$ be $V[G][T]$-generic. Finally, in $V[G][T][H]$, we force with T to add a branch to it, and let $b \subseteq T$ be a $V[G][T][H]$-generic branch of T. Note that, as a notion of forcing, T is $<\kappa$-distributive and κ-cc. As Kunen showed, the combined forcing $\mathbb{Q} * \dot{T}$, where $\dot{T}$ is the canonical name for T, has a $<\kappa$-closed dense subset, and therefore is forcing equivalent to $\mathrm{Add}(\kappa, 1)$ [21]. Thus, the iteration $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{T}$ is forcing equivalent to $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$.

Obviously, κ is not even weakly compact in $V[G][T]$, and hence also not in $V[G][T][H]$ because the coding forcing $\mathbb{C}$ is highly closed, and so cannot add a branch to T. Thus, κ is also not weakly compact in the stable core of $V[G][T][H]$ because, by our coding, $L[S^{V[G][T][H]}]$ has the κ-Souslin tree T.

Next, let's argue that the measurability of κ is resurrected in the final model $V[G][T][H][b]$. Standard lifting arguments show that κ is measurable in $V[G][T][b]$, which is a forcing extension by $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$. But $V[G][T][H][b]$ and $V[G][T][b]$ have the same subsets of κ, which means that κ is also measurable in $V[G][T][H][b]$.

For ease of reference, let $W = V[G][T][H]$. We will argue that the stable core of $W[b]$ is the same as the stable core of W, so that κ is not weakly compact there.

Note, using Claim 1 in the proof of Theorem 3.1, that all forcing extensions in this argument satisfy the GCH. Observe now that since W and $W[b]$ have the same cardinals (since forcing with T is $<\kappa$-distributive and κ-cc), and the GCH holds in both models, they have the same strong limit cardinals (namely the limit cardinals). Thus, we have that $S_1^W = S_1^{W[b]}$. The remaining arguments will therefore assume that $n \geq 2$ in the triple (n, α, β).

It is easy to see that for $\alpha < \beta \leq \kappa$, a triple $(n, \alpha, \beta) \in S^W$ if and only if it is in $S^{W[b]}$ because forcing with the tree T does not add small subsets to κ.

The case $\kappa < \alpha < \beta$ follows by Proposition 2.5 because α is above the size of the forcing T.

Finally, we consider the remaining case $\alpha \leq \kappa < \beta$. Observe that for every strong limit cardinal $\alpha < \kappa$, H_α^W satisfies the assertion that for every successor cardinal γ^+, there is an $L_\alpha(H_{\gamma^+})$-generic filter for $\mathrm{Add}(\gamma^+, 1)^{L_\alpha(H_{\gamma^+})}$. The reason is that $H_{\gamma^+}^{V[G]} = H_{\gamma^+}^{V[G_\gamma]}$, where we factor $\mathbb{P}_\kappa \cong \mathbb{P}_{\gamma+1} * \mathbb{P}_{\mathrm{tail}}$ and correspondingly factor $G \cong G_{\gamma+1} * G_{\mathrm{tail}}$. The complexity of the assertion is Π_2 because we can express it as follows:

$$\forall \bar{\gamma} \forall H \exists \gamma < \bar{\gamma}\, [(\gamma \text{ is regular and } \gamma^{++} = \bar{\gamma} \text{ and } H = H_{\gamma^+}) \to$$

$$\exists g \exists Y\, (Y = L_{\bar{\gamma}}(H) \text{ and } g \text{ is } Y\text{-generic for } \mathrm{Add}(\gamma^+, 1)^Y].$$

However, H_β^W cannot satisfy this assertion because it obviously cannot have an $L_\beta(H_{\gamma^+})$-generic filter for $\mathrm{Add}(\gamma^+, 1)^{L_\beta(H_{\gamma^+})}$, where $\gamma = \kappa^{++}$, because the H_{γ^+} of $L_\beta(H_{\gamma^+})$ is the real H_{γ^+} of W. The same argument holds for $W[b]$ showing that no triple (n, α, β) can be in either S^W or $S^{W[b]}$ for $n \geq 2$. ⊣

§6. Separating the stable core and L[Card]. Finally, we would like to consider the possible relationships between the stable core $L[S]$ and the model $L[\mathrm{Card}]$.

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Even though the stable core can define the class of strong limit cardinals of V, there is no reason to believe that it can see the cardinals. Indeed, it is even possible to make $L[\mathrm{Card}]$ larger than the stable core.

THEOREM 6.1. *It is consistent that* $L[S] \subsetneq L[\mathrm{Card}]$.

PROOF. We start in L. Force to add an L-generic Cohen real r. Next, force with the full support product $\mathbb{P} = \Pi_{k<\omega}\mathbb{Q}_k$, where if $k \in r$, then $\mathbb{Q}_k = \mathrm{Coll}(\aleph_{2k}, \aleph_{2k+1})$, and otherwise $\mathbb{Q}_k$ is the trivial forcing. The forcing $\mathbb{P}$ codes r into the cardinals of the forcing extension. Suppose $H \subseteq \Pi_{k<\omega}\mathbb{Q}_k$ is $L[r]$-generic, and observe that $r \in L[\mathrm{Card}^{L[r][H]}]$ because it can be constructed by comparing the cardinals of L with the cardinals of $L[r][H]$. However, the stable core $L[S^{L[r][H]}] = L$ remains unchanged because we preserved the strong limit cardinals, and for the slices S_n of the stable core with $n \geq 2$, it suffices that the forcing has size smaller than the second strong limit cardinal. $\dashv$

Next, let's show that in various situations, $L[\mathrm{Card}]$ can be a proper submodel of the stable core $L[S]$.

THEOREM 6.2. *It is consistent that* $L[\mathrm{Card}] \subsetneq L[S]$.

PROOF. We start in L and force to add a Cohen real r. We then code r into the stable core of a further forcing extension using the coding forcing from the proof of Theorem 3.3. More precisely, we let δ_0 be any sufficiently large singular strong limit cardinal, and let $\langle(\beta_n, \beta_n^*) \mid n < \omega\rangle$ be the sequence of ω-many pairs of successive strong limit cardinals above δ_0 (note that they must all be singular), which will be our coding pairs. Now define that $\mathbb{C}_n$ is trivial for $n \notin r$, and otherwise let $\mathbb{C}_n = \mathrm{Add}(\delta_n^+, \beta_n)$, where $\delta_n = \beta_{n-1}^*$ for $n > 0$. By the arguments given in the proof of Theorem 3.3, the full support forcing $\mathbb{C} = \Pi_{n<\omega}\mathbb{C}_n$ is cardinal preserving. Let $H \subseteq \mathbb{C}$ be $L[r]$-generic. Now observe that since we have $\mathrm{Card}^L = \mathrm{Card}^{L[r][H]}$, it follows that $L[\mathrm{Card}^{L[r][H]}] = L$, but $L[r] \subseteq L[S^{L[r][H]}]$. $\dashv$

THEOREM 6.3. *It is consistent that* $L[S]$ *has a measurable cardinal and* $L[\mathrm{Card}] \subsetneq L[S]$.

PROOF. We start in a model $V = L[\mu]$ with a measurable cardinal κ and force to add a Cohen subset to some $\delta \gg \kappa$. Let $G \subseteq \mathrm{Add}(\delta, 1)$ be V-generic. We then code G into the stable core using the cardinal preserving coding forcing $\mathbb{C}$ from the proof of Theorem 3.3. Let H be $V[G]$-generic for $\mathbb{C}$. Since we forced high above κ, κ remains measurable in $L[S^{V[G][H]}]$ as in Theorem 4.7. Because $\mathbb{C}$ preserves cardinals, $L[\mathrm{Card}^{V[G][H]}] = L[\mathrm{Card}^V] = V$, but $L[S^{V[G][H]}]$ contains G by construction. $\dashv$

We also separate the models $L[\mathrm{Card}]$ and $L[S]$ by showing that, for each $n < \omega$, if $m_{n+1}^{\#}$ exists, then $m_n^{\#} \in L[S]$. Recall that even $m_1^{\#}$ cannot be an element of $L[\mathrm{Card}]$ (Corollary 4.5).

Given a class A, we will say that a cardinal γ is Σ_n-*stable relative to* A if $\langle H_\gamma, \in, A\rangle \prec_{\Sigma_n} \langle V, \in, A\rangle$. We will say that a class B is Σ_n-*stable relative to* A if every $\gamma \in B$ is Σ_n-stable relative to A. Let us say that a cardinal is *strictly* n-*measurable* if

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it is n-measurable, but not $n+1$-measurable. Recall that given a club C, we denote by $\hat{C}$, the collection of all successor elements of C together with its least element.

THEOREM 6.4. *Suppose that $C_1 \supseteq C_2$ are class clubs of uncountable cardinals such that C_2 is Σ_1-stable relative to C_1. Then there is an* Ord*-length iteration of the mouse $m_2^\#$ such that in the direct limit model M_{C_1,C_2} (truncated at* Ord*) the strictly 1-measurable cardinals are precisely the elements of $\hat{C}_1$ and the 2-measurable cardinals are precisely the elements of $\hat{C}_2$.*

PROOF. Let $C_1 = \langle \alpha_\xi \mid \xi \in \mathrm{Ord}\rangle$ and let $\langle \gamma_\xi \mid \xi \in \mathrm{Ord}\rangle$ be a sequence such that α_{γ_ξ} is the ξth element of C_2 in the enumeration of C_1. The iteration will closely resemble the iteration from the proof of Theorem 4.4.

Iterate the first measurable cardinal κ_0 of $m_2^\#$ α_0-many times, so that κ_0 iterates to α_0, and let M_{α_0} be the iterate. Continue to iterate measurable cardinals onto elements of $\hat{C}_1$ until we reach for the first time a direct limit stage η_0 where in the model M_{η_0} all measurable cardinals below the first 2-measurable cardinal are elements of $\hat{C}_1$. It is not difficult to see that η_0 is the first cardinal such that $\eta_0 = \alpha_{\eta_0}$, the η_0th element of C_1, and that in M_{η_0}, η_0 is the first 2-measurable cardinal. Since C_2 is Σ_1-stable relative to C_1, η_0 must be below the first element of C_2. To achieve our goal of making the least 2-measurable the least element of C_2, at this stage, we iterate up η_0 to obtain a model M_{η_0+1} with more strictly 1-measurable cardinals and continue iterating measurable cardinals onto elements of $\hat{C}_1$. Let η_ξ be the ξth stage where we iterate up the first 2-measurable cardinal η_ξ as above. Since α_{γ_0}, the least element of C_2, is Σ_1-stable relative to C_1, it must be that $\eta_\xi < \alpha_{\gamma_0}$ for every $\xi < \alpha_{\gamma_0}$ as every iteration of a shorter length $\beta < \alpha_{\gamma_0}$ (of the kind we have been doing) has to be an element of $H_{\alpha_{\gamma_0}}$ by Σ_1-elementarity. We would like to argue that the thread t in the stage α_{γ_0} direct limit such that $t(\xi)$ is the first 2-measurable cardinal maps to α_{γ_0} as desired. It suffices to observe that every ordinal thread s below t must be constant from some stage onward. So suppose that $s < t$, which by definition of direct limit means that on a tail of stages ξ, $s(\xi) < t(\xi)$. Fix some such ξ in the tail and consider a stage $\eta_{\bar{\xi}} > \xi$ where we have $s(\eta_{\bar{\xi}}) < t(\eta_{\bar{\xi}}) = \eta_{\bar{\xi}}$. Here the equality holds by elementarity as $\eta_{\bar{\xi}}$ is the first 2-measurable cardinal in $M_{\eta_{\bar{\xi}}}$. Since the critical points of the iteration after this stage are above $\eta_{\bar{\xi}}$, the thread s remains constant from that point onward.

Thus, α_{γ_0} must be the first 2-measurable cardinal in the direct limit model $M_{\alpha_{\gamma_0}}$. Having correctly positioned the first 2-measurable cardinal, we proceed with iterating the strictly 1-measurable cardinals onto elements of $\hat{C}_1$ below the next element of C_2. As in the proof of Theorem 4.4, it will be the case that at some limit stages in C_2, we will need to use the top measure of $m_2^\#$ to create more room for the iteration to proceed.

Let M be the resulting model obtained as the direct limit along the iteration embeddings and let M_{C_1,C_2} be M truncated at Ord. The construction ensures that the strictly 1-measurable cardinals of M_{C_1,C_2} are precisely the elements of $\hat{C}_1$ and 2-measurable cardinals of M_{C_1,C_2} are precisely the elements of $\hat{C}_2$. $\dashv$

Given a club C, let C^* denote the club of all limit points of C. Next, let's argue that if C_1 and C_2 are clubs as above, then $M_{C_1^*,C_2^*}$ is contained in $L[C_1, C_2]$.

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THEOREM 6.5. *Suppose that $C_1 \supseteq C_2$ are class clubs of uncountable cardinals such that C_2 is Σ_1-stable relative to C_1. Then $M_{C_1^*, C_2^*}$ (obtained as in Theorem 6.4) is contained in $L[C_1, C_2]$.*

PROOF. Given $\alpha \in \hat{C}_1^*$, let $U_\alpha \subseteq P(\alpha)^{M_{C_1^*, C_2^*}}$ be the iteration measures in $M_{C_1^*, C_2^*}$, and note that a set from $M_{C_1^*, C_2^*}$ is in U_α if and only if it contains a tail of $C_1 \cap \alpha$. Let $\vec{U} = \langle U_\alpha \mid \alpha \in \mathrm{Ord}\rangle$. Similarly, for $\beta \in \hat{C}_2^*$, let $W_\beta \subseteq P(\beta)^{M_{C_1^*, C_2^*}}$ be the iteration measures in $M_{C_1^*, C_2^*}$, and let $\vec{W} = \langle W_\beta \mid \beta \in \mathrm{Ord}\rangle$. Here we also have that a set is in W_β if and only if it contains a tail of $C_2 \cap \beta$ because, as we noted in the proof of Theorem 6.4, at every stage in C_2 we iterate up the measure on the 2-measurable cardinal until we reach an element of $\hat{C}_2^*$. Finally, observe that $M_{C_1^*, C_2^*} = L[\vec{U}, \vec{W}]$, and thus, it is contained in $L[C_1, C_2]$. ⊣

Theorems 6.4 and 6.5 easily generalize to n nested clubs $C_1, \dots, C_n$ such that C_i is Σ_1-stable relative to $C_1, \dots, C_{i-1}$ (more precisely, relative to the class canonically coding the the sequence $\langle C_1, \dots, C_{i-1}\rangle$ of classes) for all $1 < i \leq n$.

THEOREM 6.6. *Suppose that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$ are class clubs of uncountable cardinals such that C_i is Σ_1-stable relative to $C_1, \dots, C_{i-1}$ for all $1 < i \leq n$. Then there is an* Ord*-length iteration of the mouse $m_n^\#$ such that in the direct limit model $M_{C_1, \dots, C_n}$ (truncated at* Ord*), for $1 \leq i \leq n$, the strictly i-measurable cardinals are precisely the elements of $\hat{C}_i$.*

THEOREM 6.7. *Suppose that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$ are class clubs of uncountable cardinals such that C_i is Σ_1-stable relative to $C_1, \dots, C_{i-1}$ for all $1 < i \leq n$. Then $M_{C_1^*, \dots, C_n^*}$ (obtained as in Theorem 6.6) is contained in $L[C_1, \dots, C_n]$.*

THEOREM 6.8. *For all $n < \omega$, if $m_{n+1}^\#$ exists, then $m_n^\#$ is in the stable core.*

PROOF. By Proposition 2.3, for $i \geq 1$, the stable core can define class clubs C_i of strong limit cardinals α such that $H_\alpha \prec_{\Sigma_i} V$. In particular, each C_i is Σ_1-stable relative to $C_1, \dots, C_{i-1}$. Fix some $n < \omega$. If $m_{n+1}^\#$ exists, then by Theorem 6.7, $M_{C_1^*, \dots, C_{n+1}^*}$ is contained in the stable core, and so in particular, the stable core has $m_n^\#$. ⊣

§7. Characterizing models $L[C_1, \dots, C_n]$. In this section, we will generalize Welch's arguments in [29] to show that, in the presence of many measurable cardinals, models $L[C_1, \dots, C_n]$, where

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$$

are class clubs of uncountable cardinals such that C_i is Σ_1-stable relative to $C_1, \dots, C_{i-1}$, are truncations to Ord of forcing extensions of an iterate of the mouse $m_n^\#$ via a full support product of Prikry forcings.

In [10], Fuchs defined, given a discrete set D of measurable cardinals, a Prikry-type forcing $\mathbb{P}_D$ to singularize them all, as follows. For every $\alpha \in D$, we fix a normal measure μ_α on α with respect to which the forcing $\mathbb{P}_D$ will be defined. Conditions in $\mathbb{P}_D$ are pairs $\langle h, H\rangle$ such that H is a function on D with $H(\alpha) \in \mu_\alpha$ and h is a function on D with finite support such that $h(\alpha) \in [\alpha]^{<\omega}$ is a finite sequence of elements of α below the least element of $H(\alpha)$ and above all $\beta \in D$ with $\beta < \alpha$. Extension is defined by $(h, H) \leq (f, F)$ if for all $\alpha \in D$, $H(\alpha) \subseteq F(\alpha)$, $h(\alpha)$ end-extends $f(\alpha)$, and $h(\alpha) \setminus f(\alpha) \subseteq F(\alpha)$. Note that the first coordinate of the pair has finite support while the second coordinate has full support so that the forcing is a mix of a finite support and a full support-product. It is not difficult to see that the Magidor iteration of Prikry forcing for a discrete set D of measurable cardinals is equivalent to $\mathbb{P}_D$. For the definition and properties of the Magidor iteration, see Section 6 of [11].

THEOREM 7.1 (See [10]). *The forcing $\mathbb{P}_D$ has the $|D|^+$-cc, preserves all cardinals, and preserves all cofinalities not in D.*

The forcing $\mathbb{P}_D$ also has the Prikry property, namely, given a condition $(h, H) \in \mathbb{P}_D$ and a sentence φ of the forcing language, there is a condition (h, H^*) deciding φ such that for every $\alpha \in D$, $H^*(\alpha) \subseteq H(\alpha)$ (see [11, Section 6] for details).

For an ordinal λ, let $D_{<\lambda}$ be $D \restriction \lambda$ and $D_{\geq\lambda}$ be the rest of D. The forcing $\mathbb{P}_D$ factors as $\mathbb{P}_{D_{<\lambda}} \times \mathbb{P}_{D_{\geq\lambda}}$.

PROPOSITION 7.2. *Suppose f is V-generic for $\mathbb{P}_D$. If $g : \gamma \to V_\gamma$ is a function in $V[f]$ with $\gamma < \lambda$, then g is added by $f \restriction \lambda$.*

PROOF. It suffices to see that $\mathbb{P}_{D_{\geq\lambda}}$ cannot add g over $V[f \restriction \lambda]$. Let $\dot{g}$ be a $\mathbb{P}_{D_{\geq\lambda}}$-name for g so that $\mathbb{1}_{\mathbb{P}_{D_{\geq\lambda}}} \Vdash \dot{g} : \check{\gamma} \to \check{V}_\gamma$. By the Prikry property of $\mathbb{P}_{D_{\geq\lambda}}$, for every $x \in V_\gamma$ and $\alpha < \gamma$, there is some condition $(\emptyset, H_{x,\alpha})$ deciding whether $\dot{g}(\check{\alpha}) = \check{x}$. There are less than κ many such conditions, where κ is the least measurable cardinal in D greater than or equal to λ. Thus, we can intersect all the measure one sets on each coordinate of $H_{x,\alpha}$ to obtain a condition $(\emptyset, H)$ below all of the $(\emptyset, H_{x,\alpha})$. Clearly $(\emptyset, H)$ decides $\dot{g}$. ⊣

PROPOSITION 7.3. *Suppose that κ is inaccessible and D is contained in and unbounded in κ. Then κ remains inaccessible after forcing with $\mathbb{P}_D$.*

PROOF. By Theorem 7.1, κ is regular after forcing with $\mathbb{P}_D$. So it remains to show that κ is a strong limit after forcing with $\mathbb{P}_D$. Fix a cardinal $\alpha < \kappa$ and let λ be the least measurable cardinal in D above α. Since $D \subseteq \kappa$, $\lambda < \kappa$. By Proposition 7.2, every subset of α added by $\mathbb{P}_D$ is already added by $\mathbb{P}_{D_{<\lambda}}$. But since κ is an inaccessible, $\mathbb{P}_{D_{<\lambda}}$ clearly has a chain condition less than λ, and therefore a nice-name counting argument shows that it adds less than κ-many subsets of α. ⊣

Fuchs showed that the forcing $\mathbb{P}_D$ has a Mathias-like criterion for establishing when a collection of sequences is generic for it.

THEOREM 7.4 [10]. *Suppose that M is a transitive model of* ZFC, *D is a discrete set of measurable cardinals in M, and the forcing $\mathbb{P}^M_D$ is constructed in M as above. A function f on D such that $f(\alpha) \in [\alpha]^\omega$ is an ω-sequence in α above $\beta \in D$ for every $\beta < \alpha$ is M-generic for $\mathbb{P}^M_D$ if and only if for every function H on D with $H(\alpha) \in \mu_\alpha$, $\bigcup_{\alpha \in D} f(\alpha) \setminus H(\alpha)$ is finite.*

We will now give the technical set-up for the forcing construction that we want to perform over iterates of the mice $m_n^\#$.

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Let ZFC^-_I be the theory consisting of ZFC^- together with the assertion that there is a largest cardinal κ and that it is inaccessible, namely κ is regular and for every $\alpha < \kappa$, $P(\alpha)$ exists and has size smaller than κ. Note that, in particular, V_α exists in models of ZFC^-_I for all ordinals $\alpha \leq \kappa$. Natural models of ZFC^-_I are H_{κ^+} for an inaccessible cardinal κ. The theory ZFC^-_I is bi-interpretable with the second-order class set theory $\mathrm{KM} + \mathrm{CC}$, Kelley–Morse set theory (KM) together with the Class Choice Principle (CC) [23]. Models of Kelley–Morse are two-sorted of the form $\mathcal{V} = (V, \in, \mathcal{C})$, with V consisting of the sets, $\mathcal{C}$ consisting of classes, and $\in$ being a membership relation between sets as well as between sets and classes. The axioms of Kelley–Morse are ZFC together with the following axioms for classes: extensionality, existence of a global well-order class, class replacement asserting that every class function restricted to a set is a set, and comprehension for all second-order assertions. The Class Choice Principle CC is a scheme of assertions, which asserts for every second-order formula $\varphi(x, X, Y)$ that if for every set x, there is a class X such that $\varphi(x, X, Y)$ holds, then there is a single class Z choosing witnesses for every set x, in the sense that $\varphi(x, Z_x, Y)$ holds for every set x, where $Z_x = \{y \mid \langle x, y\rangle \in Z\}$ is the xth slice of Z. If $\mathcal{V} = (V, \in, \mathcal{C})$ is a model of $\mathrm{KM} + \mathrm{CC}$, then the collection of all extensional well-founded relations in $\mathcal{C}$, modulo isomorphism and with a natural membership relation, forms a model $M_\mathcal{V}$ of ZFC^-_I, whose largest cardinal κ is (isomorphic to) Ord, such that $V_\kappa^{M_\mathcal{V}} \cong V$ and the collection of all subsets of $V_\kappa^{M_\mathcal{V}}$ in $M_\mathcal{V}$ is precisely $\mathcal{C}$ (modulo the isomorphism).[3] On the other hand, given any model $M \models \mathrm{ZFC}^-_I$, we have that $\mathcal{V} = (V_\kappa^M, \in, \mathcal{C})$, where $\mathcal{C}$ consists of all subsets of V_κ^M in M, is a model of $\mathrm{KM} + \mathrm{CC}$, and moreover, $M_\mathcal{V}$ is then precisely M.

The bi-interpretability of the two theories was used by Antos and Friedman in [1] to develop a theory of hyperclass forcing over models of $\mathrm{KM} + \mathrm{CC}$. A *hyperclass forcing* over a model $\mathcal{V} = (V, \in, \mathcal{C}) \models \mathrm{KM} + \mathrm{CC}$ is a partial order on a sub-collection of $\mathcal{C}$ that is definable over $\mathcal{V}$. Suppose that G is $\mathbb{P}$-generic for a hyperclass-forcing $\mathbb{P}$ over $\mathcal{V}$, meaning that it meets all the definable dense sub-collections of $\mathbb{P}$. We move to $M_\mathcal{V}$, over which $\mathbb{P}$ is a definable class-forcing, and consider the forcing extension $M_\mathcal{V}[G]$. The forcing $\mathbb{P}$ may not preserve ZFC^-_I, but whenever it does, we define that the hyperclass-forcing extension $\mathcal{V}[G]$ is the Kelley–Morse model consisting of $V_\kappa^{M_\mathcal{V}[G]}$ together with all the subsets of $V_\kappa^{M_\mathcal{V}[G]}$ in $M_\mathcal{V}[G]$.

An Ord-length iterate M of a mouse $m_n^\#$ is obviously a model of ZFC^-_I with largest cardinal Ord, and moreover it has a definable global well-ordering. Thus, M naturally gives rise to a model of $\mathrm{KM} + \mathrm{CC}$, namely its truncation at Ord, whose classes are the subsets of V^M_{Ord} in M.

Let M be a model of ZFC^-_I with a largest cardinal κ and a definable well-ordering of the universe. Let D be a discrete set in M of measurable cardinals below κ and suppose that D is unbounded in κ. Over M, $\mathbb{P}_D$ is a class forcing notion all of whose antichains are sets. Class forcing works the same way over models of ZFC^- as it does over models of ZFC. Pretame forcing (see [7] for definition and properties) preserves ZFC^- to forcing extensions and has definable forcing relations (this is due to Stanley and can be found in [14]). In a model with a definable global well-order, every class forcing all of whose antichains are sets is pretame. Although “mixing of names” is

[3]We will from now on ignore the isomorphism and assume we have actual equality.

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not always doable with class forcing that has proper class-sized antichains, it still works if all antichains are sets. Finally, the existence of a definable global well-order gives that the Mathias criterion of Theorem 7.4 still holds in this setting.

Forcing with $\mathbb{P}_D$ preserves the inaccessibility of κ by Proposition 7.3, while singularizing all the measurable cardinals below it. Thus, in particular, a forcing extension by $\mathbb{P}_D$ remains a model of ZFC^-_I.

Proposition 7.5. *Given a $\mathbb{P}_D$-generic f, we have that $V_\kappa^{M[f]} = V_\kappa^M[f]$.*

Proof. The one inclusion $V_\kappa^M[f] \subseteq V_\kappa^{M[f]}$ is clear. For the other inclusion suppose that $a \in V_\kappa^{M[f]}$. There is some $\beth$-fixed point cardinal $\alpha < \kappa$ in $M[f]$ such that $a \in V_\alpha^{M[f]}$, so that a is coded there by a subset A of α in $M[f]$. By Proposition 7.2, A must be added by some initial segment of the product $\mathbb{P}_D$, and therefore is an interpretation of a name in V_κ^M by an initial segment of f. $\dashv$

We can also view $\mathbb{P}_D$ as a hyperclass forcing over the model $\mathcal{V} = (V_\kappa^M, \in, \mathcal{C})$, with $\mathcal{C}$ being the collection of all subsets of V_κ^M in M. Since $M[f]$ is a model of ZFC^-_I (because κ remains inaccessible), we can form the forcing extension $\mathcal{V}[f]$, and it is (by definition) the model $(V_\kappa[f], \in, \mathcal{C}^*)$ with $\mathcal{C}^*$ being the collection of all subsets of $V_\kappa^M[f]$ in $M[f]$.

Now we go back to our specific setting in which we consider Ord-length iterates of the mice $m_n^\#$.

Let C be the class club of limit cardinals and let M be the (non-truncated) iterate model of $m_1^\#$ constructed by Welch (see Section 4). The model M satisfies ZFC^-_I with the largest cardinal Ord and has a definable well-ordering of the universe. Let $D = \hat{C}$ be the collection of all measurable cardinals of M. Let f be the function on D such that $f(\omega\cdot\alpha+\omega) = \langle \omega\cdot\alpha+n \mid n<\omega\rangle$. Welch showed that f is generic for $\mathbb{P}_D$ (defined using measures arising in the iteration) by verifying the Mathias criterion. As before, let M_C be M truncated at Ord. Let $\mathcal{C}$ be the collection of subsets of M_C in M and $\mathcal{C}^*$ be the collection of all subsets of $M_C[f]$ in $M[f]$. With this set-up, Welch proved the following theorem.

Theorem 7.6 (Welch [29]). *The model $M[\mathrm{Card}]$ is a class forcing extension of M by the class forcing $\mathbb{P}_D$. Equivalently, the second-order model $(M_C[\mathrm{Card}], \in, \mathcal{C}^*)$ is a hyperclass-forcing extension of $(M, \in, \mathcal{C})$ by the hyperclass forcing $\mathbb{P}_D$.*

Indeed, it is not difficult to see that the function f cannot be added by any set forcing over M (or equivalently, cannot be added by any class forcing over M_C).

Theorem 7.7. *In the notation of Theorem 7.6, f is not set-generic over M. Equivalently, f is not class-generic over the second-order model $(M_C[f], \in, \mathcal{C})$.*

Proof. Consider the regressive function g with domain D defined by $g(\alpha) = \min(f(\alpha))$. If g_0 is any regressive function in M on D then by genericity, g dominates g_0 at all but finitely many elements of D. But if $\mathbb{P}$ is any set-forcing of M, $\mathbb{P}$ has size at most Ord in M, and therefore we will argue that $\mathbb{P}$ cannot add such a dominating function. Let $\{p_\xi \mid \xi \in \mathrm{Ord}\}$ be a listing of the elements of $\mathbb{P}$ in which every element of $\mathbb{P}$ appears cofinally often. Let $\dot{g}$ be a $\mathbb{P}$-name for a regressive function on D. For every $\xi \in \mathrm{Ord}$, choose a condition p_ξ^* extending p_ξ that decides $\dot{g}(\alpha_\xi) = \beta_\xi$, where α_ξ is the ξth element of D. Define $g_0(\alpha_\xi) = \beta_\xi + 1$. Then any condition p_ξ in $\mathbb{P}$

has an extension forcing $g_0(\alpha_\xi) > \dot{g}(\alpha_\xi)$. Since every $p \in \mathbb{P}$ appears in our listing cofinally often, for cofinally many $\xi < \kappa$, every $p \in \mathbb{P}$ has an extension forcing $g_0(\alpha_\xi) > \dot{g}(\alpha_\xi)$, which means that $\dot{g}$ cannot be forced to dominate all regressive g_0 on D in M on a final segment of D. ⊣

We will need to make use of the following theorem.

THEOREM 7.8. *Forcing with $\mathbb{P}_D$ preserves all measurable cardinals not in D. Indeed, if $\kappa \notin D$ is a measurable cardinal in M and μ is a normal measure on κ that does not concentrate on measurable cardinals, then κ has a normal measure $\bar{\mu} \in M[f]$, a forcing extension by $\mathbb{P}_D$, such that $\bar{A} \in \bar{\mu}$ if and only if there is $A \in \mu$ with $A \subseteq \bar{A}$.*

See Section 6 of [11] for a proof.

Suppose that $m_2^\#$ exists and $C_1 \supseteq C_2$ are class clubs of uncountable cardinals such that C_2 is Σ_1-stable relative to C_1. Let M be the (untruncated) iterate of $m_2^\#$ constructed as in the proof of Theorem 6.4 for the clubs C_1^* and C_2^* consisting of the limit points of C_1 and C_2 respectively. With this set-up, we get the following generalization of Welch's theorem. Recall that we denote by $M_{C_1^*,C_2*}$ the truncation at Ord of the model M obtained by iterating the mouse $m_2^\#$ so that its strictly 1-measurable cardinals are precisely the elements of C_1^* and its 2-measurable cardinals are the precisely the elements of C_2^*.

Let D_i for $i = 1, 2$, be the class of strictly i-measurable cardinals in M. For every measurable cardinal $\alpha \in M$, let μ_α be the normal measure on α arising from the iteration. Let $\mathbb{P}_{D_1}$ and $\mathbb{P}_{D_2}$ be the product Prikry forcings defined with respect to the measures μ_α in M. Let f be M-generic for $\mathbb{P}_{D_2}$. In $M[f]$, every measurable cardinal in D_1 remains measurable by Theorem 7.8. Since no μ_α concentrates on measurable cardinals (otherwise α would have Mitchell order 2), by Theorem 7.8, every measurable cardinal $\alpha \in D_1$ has, in $M[f]$, a normal measure $\bar{\mu}_\alpha$ generated by μ_α. Thus, we can define in $M[f]$, the Prikry product forcing $\bar{\mathbb{P}}_{D_1}$ with respect to the measures $\bar{\mu}_\alpha$. Let $\dot{\mathbb{P}}_{D_1}$ be the $\mathbb{P}_{D_2}$-name for this Prikry forcing product. Notice that $\mathbb{P}_{D_1}$ is the product Prikry forcing defined in M, while $\bar{\mathbb{P}}_{D_1}$ is the product Prikry forcing defined in a forcing extension of M by $\mathbb{P}_{D_2}$. Although these are potentially different forcing notions, we will show below that they are forcing equivalent.

THEOREM 7.9. *The model $M[C_1, C_2]$ is a forcing extension of M by the class forcing iteration $\mathbb{P}_{D_2} * \dot{\mathbb{P}}_{D_1}$. The iteration $\mathbb{P}_{D_2} * \dot{\mathbb{P}}_{D_1}$ is equivalent to the product $\mathbb{P}_{D_2} \times \mathbb{P}_{D_1}$. Moreover, $M_{C_1^*,C_2^*}[C_1, C_2] = L[C_1, C_2]$, and the latter is then the first-order part of a hyperclass-forcing extension of the Kelley–Morse model $\langle V_{\mathrm{Ord}}^M, \in, \mathcal{C}\rangle$ (where $\mathcal{C}$ consists of the subsets of V_{Ord}^M in M).*

PROOF. Let f_1 be the function on the elements α of $\hat{C}_1^*$ such that $f_1(\alpha)$ is the ω-sequence of elements of C_1 limiting up to α. Let f_2 be the function on the elements α of $\hat{C}_2^*$ such that $f_2(\alpha)$ is the ω-sequence of elements of C_2 limiting up to α. The arguments in [29] already verify that f_2 satisfies the Mathias criterion for $\mathbb{P}_{D_2}$ and f_1 satisfies the Mathias criterion for $\mathbb{P}_{D_1}$. Indeed, we will now argue that $\mathbb{P}_{D_1}$ densely embeds into $\bar{\mathbb{P}}_{D_1} = (\dot{\mathbb{P}}_{D_1})_{f_2}$. Since class forcing notions with set-sized antichains which densely embed produce the same forcing extensions (see [14]), we will be able to assume without loss that we are actually forcing with $\mathbb{P}_{D_1}$.

It suffices to argue that for every function F on D_1 in $M[f_2]$ such that $F(\alpha) \in \mu_\alpha$, there is a function $F^* \in M$ such that $F^*(\alpha) \in \mu_\alpha$ and $F^*(\alpha) \subseteq F(\alpha)$ for every $\alpha \in D_1$. Let $\dot{F}$ be a $\mathbb{P}_{D_2}$-name for F such that $\mathbb{1}_{\mathbb{P}_{D_2}}$ forces that $\dot{F}(\alpha) \in \check{\mu}_\alpha$ for every measurable α (this requires mixing names). We will argue in M, by induction on $\beta \leq \mathrm{Ord}$, that we can define cohering functions f_β on $D_1 \restriction \beta$ such that $f_\beta(\alpha) \in \mu_\alpha$ and $\mathbb{1}_{\mathbb{P}_{D_2}} \Vdash \check{f}_\beta(\alpha) \subseteq \dot{F}(\alpha)$ for every $\alpha \in D_1 \restriction \beta$. Suppose inductively that we can construct f_γ for $\gamma < \beta$ as required. Let's argue that we can construct f_β. If β is a limit ordinal, then f_β is just the union of the f_γ. So suppose that $\beta = \beta^* + 1$ and assume that F is defined at β^* because otherwise there is nothing to prove. Observe that $F \restriction \beta$ must be added by $\mathbb{P}_{D_{2<\beta}}$ by Proposition 7.2. Since $\beta^* \in C_1$ cannot be a limit of elements of C_2, $\mathbb{P}_{D_{2<\beta}}$ must have size $\lambda < \beta^*$. Let $\dot{f}$ be a $\mathbb{P}_{D_{2<\beta}}$-name for $F \restriction \beta$ such that $\mathbb{1}_{\mathbb{P}_{D_2}} \Vdash \dot{f} = \dot{F} \restriction \beta$. For every condition $p \in \mathbb{P}_{D_{2<\beta}}$, if p forces that some $A \in \mu_{\beta^*}$ is contained in $\dot{f}(\beta^*)$, then choose some such A_p. Since there are at most λ-many such sets $A_p \in \mu_{\beta^*}$ and $\lambda < \beta^*$, we can intersect them all to obtain a set $A \in \mu_{\beta^*}$ such that $\mathbb{1}_{\mathbb{P}_{D_{2<\beta}}} \Vdash \check{A} \subseteq \dot{f}(\beta^*)$. It follows that f_β defined to extend f_{β^*} with $f_\beta(\beta^*) = A$ satisfies our requirements.

This completes the argument that $\mathbb{P}_{D_1}$ densely embeds into $\bar{\mathbb{P}}_{D_1}$. The argument also shows that f_1 meets the Mathias criterion for $\bar{\mathbb{P}}_{D_1}$ because it met the Mathias criterion for $\mathbb{P}_{D_1}$ over M and every sequence of measure one sets from $M[f_2]$ can be thinned out on each coordinate to a sequence of measure one sets which exists in M. Thus, f_1 is $M[f_2]$-generic for $\bar{\mathbb{P}}_{D_1}$.

Finally, by Proposition 7.3, forcing with $\bar{\mathbb{P}}_{D_1}$ preserves the inaccessibility of Ord in $M[f_2]$, so that we can form the hyperclass forcing extension of the Kelley–Morse model whose first-order part is $M_{C_1^*,C_2^*}$ and the first-order part of the forcing extension is then the model $M_{C_1^*,C_2^*}[f_2][f_1] = M_{C_1^*,C_2^*}[C_1, C_2]$. Using Theorem 6.5, it is clear that $M_{C_1^*,C_2^*}[C_1, C_2] = L[C_1, C_2]$. ⊣

The characterization easily generalizes to n-many clubs $C_1, \dots, C_n$. Suppose that $m_n^\#$ exists and $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$ are clubs of uncountable cardinals such that C_i is Σ_1-stable relative to (the class canonically coding) $C_1, \dots, C_{i-1}$ for all $1 < i \leq n$. Let M be the (untruncated) iterate of $m_n^\#$ constructed as usual for the clubs $C_1^*, \dots, C_n^*$ consisting of the limit points of the clubs $C_1, \dots, C_n$ respectively.

Theorem 7.10. *The model $M[C_1, \dots, C_n]$ is a forcing extension of M by the class forcing iteration $\mathbb{P}_{D_n} * \cdots * \dot{\mathbb{P}}_{D_1}$, where D_i, for $1 \leq i \leq n$, is the class of strictly i-measurable cardinals. The iteration $\mathbb{P}_{D_n} * \cdots * \dot{\mathbb{P}}_{D_1}$ is equivalent to the product $\mathbb{P}_{D_n} \times \cdots \times \mathbb{P}_{D_1}$. Moreover, $M_{C_1^*,\dots,C_n^*}[C_1, \dots, C_n] = L[C_1, \dots, C_n]$, and the latter is then the first-order part of a hyperclass-forcing extension of the Kelley–Morse model $\langle V_{\mathrm{Ord}}^M, \in, \mathcal{C}\rangle$ (where $\mathcal{C}$ consists of the subsets of V_{Ord}^M in M).*

Theorem 7.7 also generalizes to show that such an extension cannot be obtained by a set forcing over M (or equivalently a class forcing over $M_{C_1^*,\dots,C_n^*}$).

§8. Open questions. The article did not answer several difficult questions about the structure of the stable core. In Sections 3 and 4, we showed how to code information into the stable core over small canonical inner models using the fact

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that these models must be contained in the stable core so that we can use them for decoding. In a very recent work Friedman showed that there is a large cardinal notion below a Woodin cardinal such that the stability predicate is definable over an iterate of a mouse with such a large cardinal. This immediately implies that we cannot code any set into the stable core and that there is a bound (below a Woodin cardinal) on the large cardinals that can exist in inner models of the stable core [6]. This leaves the following open questions regarding the structure of the stable core.

We still don't have a precise upper bound on the large cardinals that can exist in the stable core.

QUESTION 1. *Can the stable core have a measurable limit of measurable cardinals*?

For HOD, we know that the HOD of HOD can be smaller than HOD and that any universe V is the HOD of a class forcing extension of itself.

QUESTION 2. *Can the stable core of the stable core be smaller than the stable core*?

QUESTION 3. *When is V the stable core of an outer model*? *More precisely, is there a tame* (ZFC-*preserving*) *class forcing notion $\mathbb{P}$ such that for some V-generic filter $G \subseteq \mathbb{P}$ we have $V = L[S^{V[G]}]$*?

Finally, with regard to Section 7, we can ask whether the results there generalize to ω-many clubs.

QUESTION 4. *Is there a version of Theorem 7.10 for ω-many clubs*?

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