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# Derivation and Properties of the Angular Momentum Relative Amended Potential

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**Abstract.** The paper concerns the determination of the Angular Momentum Relative Amended Potential (AMR potential) in the framework of the Full n-Body problem and some of its basic properties are discussed. The AMR potential is derived using two different approaches : first using a Routh reduction of the system relative to rotation about the total angular momentum, second as a variation of the Sundman Inequality, using the Cauchy inequality.

**Keywords.** Amended Potential, Full Body Problem, Routh Reduction

## 1. Introduction

In this contribution the "Angular Momentum Relative Amended Potential" (AMR potential) is derived and some of its basic properties are discussed. This amended potential was specifically developed for the Full $n$-Body problem [Scheeres (2012)], which concerns the dynamics of $n$ rigid bodies as they attract each other gravitationally, and as they impact and rest on each other. In the current contribution this problem is stated using Jacobi coordinates as derived in [Scheeres & Brown (2023)], with a new discussion on the Lagrangian dynamics of this statement. The AMR potential is derived using two different approaches. First it is derived using a Routh reduction of the system relative to rotation about the total angular momentum, which shows that it is directly related to the forces acting within the system. Then it is derived as a variation of the Sundman Inequality, using the Cauchy inequality. It is compared with a few other amended potentials in common usage, derived as related versions of the Sundman Inequality or through geometric mechanics as the Smale Amended Potential [Simo et al. (1991)]. The results provide a rigorous and multi-faceted derivation of the angular momentum relative amended potential, establishing its use for evaluation of equilibria and their energetic stability.

## 2. Model

Consider the finite-density $n$-body problem where all of the component bodies are rigid and can thus rotate and rest on each other as well as orbit about each other, as derived and discussed in [Scheeres (2012, 2016)]. To pose the problem define $N+1$ finite density bodies $P_i$, each of which has a mass distribution $\mathcal{B}_i$, where $i = 0, 1, 2, ..., N$. Each body has a mass $m_i = \int_{\mathcal{B}_i} dm$, a rigid body inertia dyadic $\bar{\mathbf{I}}_i^R = -\int_{\mathcal{B}_i} \tilde{\rho} \cdot \tilde{\rho}\, dm$, a center of mass position $\mathbf{r}_i$, an orientation dyadic $\bar{\mathbf{T}}_i$, a velocity $\mathbf{v}_i$, and an angular velocity $\mathbf{\Omega}_i$, all generally specified with respect to an inertial frame, except that the orientation dyadic will orient the body frame in inertial space and the rigid body inertia dyadic is specified in a body-fixed frame. The attitude definition of each body can use Euler Angles, which allow a traditional Lagrangian approach to specifying the equations of motion. With this assumption, the orientation dyads and the angular velocities for each body are functions of its Euler Angles and their rates, $\boldsymbol{\Theta}_i = \left[\theta_i^1, \theta_i^2, \theta_i^3\right]$ and $\dot{\boldsymbol{\Theta}}_i = \left[\dot{\theta}_i^1, \dot{\theta}_i^2, \dot{\theta}_i^3\right]$, leading to the functional form $\bar{\mathbf{T}}(\boldsymbol{\Theta}_i)$ and $\boldsymbol{\Omega}(\boldsymbol{\Theta}_i, \dot{\boldsymbol{\Theta}}_i)$.

One important implication of the finite density rigid body assumption is that any two bodies have a constraint on how close their centers of mass can come to each other, denoted as $\|\mathbf{r}_{ij}\| \geqslant d(\mathbf{r}_{ij}, \mathbf{T}_{ij})$, where $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ and $\bar{\mathbf{T}}_{ij} = \bar{\mathbf{T}}_j^T \cdot \bar{\mathbf{T}}_i$ and equality occurs when the two bodies touch. This form of the constraint implicitly implies that the bodies are mutually convex, although this assumption can be relaxed. As discussed in [Scheeres (2019)] this formulation can accommodate the holonomic and non-holonomic constraints that arise when the components are in contact. As a final note, the mutual potential between any two bodies $i$ and $j$ is $\mathcal{U}_{ij}(\mathbf{r}_{ij}, \bar{\mathbf{T}}_{ij}) = -\mathcal{G} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} dm_i \; dm_j / \left|\mathbf{r}_{ij} + \rho_j - \rho_i\right|$ where $\mathcal{G}$ is the gravitational constant. This leads to the total gravitational potential $\mathcal{U} = \sum_{0 \leqslant i < j \leqslant N} \mathcal{U}_{ij}$.

### 2.1. *Jacobi Coordinates*

Given the standard form of the problem, transform the positions and velocities of the bodies into Jacobi Coordinates, which makes them all relative to each other in a recursive way. Note that the rigid body components are not affected by this transformation, which only operates on the centers of mass of each body.

The Jacobi formulation defines a sequence of transformations where the position and velocity of each body $P_i$ is measured from the collective center of mass of the bodies $P_j$, $j = 0, 1, \ldots, i-1$ [Scheeres & Brown (2023)]. To start the sequence define: $M_0 = m_0, \mathbf{R}_0 = \mathbf{V}_0 = \mathbf{0}, \mathbf{R}_0^C = \mathbf{r}_0$, and $\mathbf{V}_0^C = \mathbf{v}_0$. Then the remaining Jacobi coordinates are defined as

$$\mathbf{R}_i = \mathbf{r}_i - \mathbf{R}_{i-1}^C \tag{2.1}$$

$$\mathbf{V}_i = \mathbf{v}_i - \mathbf{V}_{i-1}^C \tag{2.2}$$

$$M_i = M_{i-1} + m_i \tag{2.3}$$

$$\mathbf{R}_i^C = \frac{1}{M_i}\left[M_{i-1}\mathbf{R}_{i-1}^C + m_i \mathbf{r}_i\right] \tag{2.4}$$

$$\mathbf{V}_i^C = \frac{1}{M_i}\left[M_{i-1}\mathbf{V}_{i-1}^C + m_i \mathbf{v}_i\right] \tag{2.5}$$

all for $i = 1, 2, \ldots, N$. The center of mass position and velocity vectors $\mathbf{R}_i^C$ and $\mathbf{V}_i^C$ are computed accounting for all bodies with that index and lower. The relative position and velocity vectors $\mathbf{R}_i$ and $\mathbf{V}_i$ are of body $i$ relative to the center of mass of all bodies with index $i-1$ and lower. The vector $\mathbf{R}_N^C$ is the total center of mass of the entire system.

Finally, note that the difference of two body positions $\mathbf{r}_j$ and $\mathbf{r}_i$ where $j > i$ can be stated as

$$\mathbf{r}_j - \mathbf{r}_i = \mathbf{R}_j - \mathbf{R}_i + \mathbf{R}_{j-1}^C - \mathbf{R}_{i-1}^C \;=\; \mathbf{R}_j - \mathbf{R}_i + \sum_{k=i}^{j-1} \frac{m_k}{M_k}\mathbf{R}_k \tag{2.6}$$

Thus, when considering the potential energy, the mutual potential between bodies $j$ and $i$, $\mathbf{r}_{ij}$, is only a function of $\mathbf{R}_k$ from $k = i, i+1, \ldots, j$.

### 2.2. *Angular Momentum, Energy and Moments of Inertia*

A key advantage of the Jacobi coordinates is that they explicitly decouple the quantities of angular momentum, moment of inertia and kinetic energy at a given order from each other and are only a function of the lower order indices. Define the angular momentum,

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kinetic energy and inertia tensor at each order, respectively,

$$\mathbf{H}_i = \frac{M_{i-1}m_i}{M_i}\mathbf{R}_i \times \mathbf{V}_i + \bar{\mathbf{T}}_i \cdot \bar{\mathbf{I}}_i^R \cdot \boldsymbol{\Omega}_i \tag{2.7}$$

$$T_i = \frac{1}{2}\frac{M_{i-1}m_i}{M_i}\left(\mathbf{V}_i \cdot \mathbf{V}_i\right) + \frac{1}{2}\boldsymbol{\Omega}_i \cdot \bar{\mathbf{I}}_i^R \cdot \boldsymbol{\Omega}_i \tag{2.8}$$

$$\bar{\mathbf{I}}_i = \frac{M_{i-1}m_i}{M_i}\left[R_i^2\bar{\mathbf{U}} - \mathbf{R}_i\mathbf{R}_i\right] + \bar{\mathbf{T}}_i \cdot \bar{\mathbf{I}}_i^R \cdot \bar{\mathbf{T}}_i^T \tag{2.9}$$

The term $\bar{\mathbf{U}}$ is the unity dyadic, the product $\mathbf{RR}$ is a dyad, and the rigid body angular momentum and rotational inertia are mapped from a body-fixed frame into the inertial frame with the orientation dyadic $\bar{\mathbf{T}}_i$. Then the system total Angular Momentum, Kinetic Energy and Inertia Dyadic are, respectively,

$$\mathbf{H} = \sum_{i=0}^{N}\mathbf{H}_i + M_N\mathbf{R}_N^C \times \mathbf{V}_N^C \tag{2.10}$$

$$T = \sum_{i=0}^{N} T_i + \frac{1}{2}M_N\mathbf{V}_N^C \cdot \mathbf{V}_N^C \tag{2.11}$$

$$\bar{\mathbf{I}} = \sum_{i=0}^{N}\bar{\mathbf{I}}_i + M_N\left[\mathbf{R}_N^C \cdot \mathbf{R}_N^C\ \bar{\mathbf{U}} - \mathbf{R}_N^C\mathbf{R}_N^C\right] \tag{2.12}$$

and the total system energy can be defined as

$$E = \sum_{i=0}^{N} E_i + \frac{1}{2}M_N\mathbf{V}_N^C \cdot \mathbf{V}_N^C \tag{2.13}$$

$$E_i = T_i + \mathcal{U}_i \tag{2.14}$$

where $\mathcal{U}_i = \sum_{j=0}^{i-1}\mathcal{U}_{ji}$. The self-potential of a given body could be included, but is not in this formulation.

## 2.3. *Lagrangian and Equations of Motion*

The coordinates of the system can be gathered into a standard coordinate and velocity vector

$$\mathbf{Q} = \left[\boldsymbol{\Theta}_0, \ldots, \mathbf{R}_i, \boldsymbol{\Theta}_i, \ldots, \mathbf{R}_N, \boldsymbol{\Theta}_N, \mathbf{R}_N^C\right] \tag{2.15}$$

$$\dot{\mathbf{Q}} = \left[\dot{\boldsymbol{\Theta}}_0, \ldots, \mathbf{V}_i, \dot{\boldsymbol{\Theta}}_i, \ldots, \mathbf{V}_N, \dot{\boldsymbol{\Theta}}_N, \mathbf{V}_N^C\right] \tag{2.16}$$

With this notation, the Lagrangian is defined as

$$\mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) = T(\mathbf{Q}, \dot{\mathbf{Q}}) - \mathcal{U}(\mathbf{Q}) \tag{2.17}$$

The equations of motion for the $i$th Jacobi coordinate are

$$\frac{M_{i-1}m_i}{M_i}\dot{\mathbf{V}}_i = -\frac{\partial\mathcal{U}}{\partial\mathbf{R}_i} \tag{2.18}$$

The rotational equations of motion for the $i$th rigid body are

$$\frac{d}{dt}\left(\boldsymbol{\Omega}_i \cdot \bar{\mathbf{I}}_i^R \cdot \frac{\partial\boldsymbol{\Omega}_i}{\partial\dot{\boldsymbol{\Theta}}_i}\right) = \boldsymbol{\Omega}_i \cdot \bar{\mathbf{I}}_i^R \cdot \frac{\partial\boldsymbol{\Omega}_i}{\partial\boldsymbol{\Theta}_i} - \frac{\partial\mathcal{U}}{\partial\boldsymbol{\Theta}_i} \tag{2.19}$$

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## 2.4. *Routh Reduction of the Linear Momentum*

The first step is to remove the linear momentum, which can be formally done via a Routh reduction [Greenwood (1977)]. The Lagrangian is independent of the final center of mass term $\mathbf{R}_N^C$, thus the linear momentum integral is found from $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{V}_N^C} = 0$, leading to

$$M_N \mathbf{V}_N^C = \mathbf{P} \ = \ \text{Constant} \tag{2.20}$$

Then the center of mass velocity can be solved for as a function of the constant: $\mathbf{V}_N^C = \frac{1}{M_N}\mathbf{P}$

This momentum can be removed from the Lagrangian by first forming the Routhian function

$$\begin{aligned}\mathcal{L}_R &= \mathcal{L} - \mathbf{V}_N^C \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{V}_N^C} \\ &= T - \mathcal{U} - \frac{1}{2M_N}\mathbf{P}\cdot\mathbf{P}\end{aligned} \tag{2.21}$$

The Routhian can be used to recover the center of mass motion after the rest of the solution is found [Greenwood (1977)]

$$\mathbf{V}_N^C = -\frac{\partial \mathcal{L}_R}{\partial \mathbf{P}} \ = \ \frac{1}{M_N}\mathbf{P}$$

which can be solved by quadrature.

This reduction is trivial, but sets the stage for the more consequential reduction of the angular momentum. One can arbitrarily set $\mathbf{P} = \mathbf{0}$, which resets the Lagrangian to the original form, $\mathcal{L}_R = \mathcal{L}$.

## 2.5. *Routh Reduction of the Angular Momentum Magnitude*

Instead of removing the entire angular momentum, as in [Simo et al. (1991)], one can focus on the simpler reduction of the angular momentum magnitude, which is key to the AMR amended potential definition.

The system has a constant angular momentum which can be stated as $\mathbf{H} = H\hat{\mathbf{H}}$. A simple rotation of the system about the angular momentum direction, which is fixed in inertial space, yields a well defined angular coordinate and its time derivative, $\theta$ and $\dot{\theta}$. Note the simple relationship between the system moment of inertia, spin rate and angular momentum magnitude

$$\dot{\theta} = \frac{H}{I_H} \tag{2.22}$$

$$I_H = \hat{\mathbf{H}} \cdot \bar{\mathbf{I}} \cdot \hat{\mathbf{H}} \tag{2.23}$$

The $I_H$ term is only a function of the Jacobi coordinates and Euler angles and is the instantaneous system moment of inertia about the angular momentum direction. It is important to note that the angular rate $\dot{\theta}$ is not constant and is a function of the coordinates.

The Lagrangian system can be rewritten in a rotating coordinate frame defined by the angular velocity $\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{H}}$. This is easily done in by rewriting the Jacobi velocities using the transport rule:

$$\mathbf{V}_i = \mathbf{R}_i' + \boldsymbol{\Omega} \times \mathbf{R}_i \tag{2.24}$$

where the $()'$ signifies a time derivative in the rotating frame. Similarly, the angular velocities of the rigid bodies must be redefined relative to this overall rotation, yielding

$$\boldsymbol{\Omega}_i = \omega_i + \boldsymbol{\Omega} \tag{2.25}$$

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where $\omega_i$ represents the new rotation of these bodies relative to the rotating frame. The Euler angles and their time derivatives are also redefined relative to this rotating frame.

Rewriting the kinetic energy of the system one finds

$$T = \sum_{i=1}^{N} \frac{M_{i-1} m_i}{2\ M_i} \left[ \mathbf{R}_i' \cdot \mathbf{R}_i' + 2\boldsymbol{\Omega} \cdot \mathbf{R}_i \times \mathbf{R}_i' - \boldsymbol{\Omega} \cdot \tilde{\mathbf{R}}_i \cdot \tilde{\mathbf{R}}_i \cdot \boldsymbol{\Omega} \right] + \frac{1}{2} \sum_{i=0}^{N} \left[ \omega_i \cdot \bar{\mathbf{I}}_i^R \cdot \omega_i + 2\boldsymbol{\Omega} \cdot \bar{\mathbf{I}}_i^R \cdot \omega_i + \boldsymbol{\Omega} \cdot \bar{\mathbf{I}}_i^R \cdot \boldsymbol{\Omega} \right] \tag{2.26}$$

which can be rewritten as

$$T = T_R + \boldsymbol{\Omega} \cdot \mathbf{H}_R + \frac{1}{2} \boldsymbol{\Omega} \cdot \bar{\mathbf{I}} \cdot \boldsymbol{\Omega} \tag{2.27}$$

$$\mathbf{H}_R = \sum_{i=1}^{N} \frac{M_{i-1} m_i}{2\ M_i} \mathbf{R}_i \times \mathbf{R}_i' + \sum_{i=0}^{N} \bar{\mathbf{I}}_i^R \cdot \omega_i \tag{2.28}$$

The quantity $\mathbf{H}_R$ is the total angular momentum of the system relative to the rotating frame, which must equal zero and is a constant. However, it cannot be eliminated from the kinetic energy as the partials of this term are still required to properly state the equations of motion. Also note that the kinetic energy term $T_R$ is the system kinetic energy relative to the rotating frame, and the velocities are the time derivatives of the coordinates relative to the rotating frame, discussed in more detail in [Scheeres (2019)].

Make the explicit substitution $\boldsymbol{\Omega} = \dot{\theta} \hat{\mathbf{H}}$ yielding the transformed Lagrangian

$$\mathcal{L} = T_R + \dot{\theta} \hat{\mathbf{H}} \cdot \mathbf{H}_R + \frac{1}{2} \dot{\theta}^2 \hat{\mathbf{H}} \cdot \bar{\mathbf{I}} \cdot \hat{\mathbf{H}} - \mathcal{U} \tag{2.29}$$

With the velocity $\dot{\theta}$ exposed as such, and the overall orientation angle about the angular momentum vector $\theta$ well defined but absent from the Lagrangian, the angular momentum integral can be identified as $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$, leading to

$$\dot{\theta} \hat{\mathbf{H}} \cdot \bar{\mathbf{I}} \cdot \hat{\mathbf{H}} + \hat{\mathbf{H}} \cdot \mathbf{H}_R = H = \text{constant} \tag{2.30}$$

Solving for the angular rate results in

$$\dot{\theta} = \frac{H}{I_H} \left[ 1 - \frac{\hat{\mathbf{H}} \cdot \mathbf{H}_R}{H} \right] \tag{2.31}$$

where $I_H$ is, as before, the total moment of inertia of the system about the angular momentum vector. Again note that $\mathbf{H}_R = \mathbf{0}$, but this equality is not enforced until after the equations of motion and related quantities are defined.

Applying the Routh reduction to the Lagrangian yields the Routhian

$$\begin{aligned} \mathcal{L}_R &= \mathcal{L} - \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} \\ &= \mathcal{L} - \dot{\theta} \left[ \dot{\theta} I_H + \hat{\mathbf{H}} \cdot \mathbf{H}_R \right] \\ &= T_R - \mathcal{U} - \frac{1}{2} \dot{\theta}^2 I_H \end{aligned} \tag{2.32}$$

To complete the transformation substitute $\dot{\theta}$ into the expression to find

$$\mathcal{L}_R = T_R - \mathcal{U} - \frac{H^2}{2 I_H} \left[ 1 - \frac{\hat{\mathbf{H}} \cdot \mathbf{H}_R}{H} \right]^2 \tag{2.33}$$

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Note that the amended potential has the additional term $\left[1 - \frac{\hat{\mathbf{H}}\cdot\mathbf{H}_R}{H}\right]^2$ which contributes to the equations of motion as the partials of this term are not generally equal to zero.

Once the system is solved, the rotation term can be reconstructed following the Routhian approach

$$\begin{aligned}\dot{\theta} &= -\frac{\partial \mathcal{L}_R}{\partial H} \\ &= \frac{H}{I_H}\left[1 - \frac{\hat{\mathbf{H}}\cdot\mathbf{H}_R}{H}\right]^2 + \frac{H^2}{2I_H}2\left[1 - \frac{\hat{\mathbf{H}}\cdot\mathbf{H}_R}{H}\right]\frac{\hat{\mathbf{H}}\cdot\mathbf{H}_R}{H^2} \\ &= \frac{H}{I_H}\end{aligned} \tag{2.34}$$

where, in the last step $\mathbf{H}_R = \mathbf{0}$.

Finally, as the system is time invariant, the Jacobi integral can be found, equal to $\dot{\mathbf{Q}} \cdot \frac{\partial \mathcal{L}_R}{\partial \dot{\mathbf{Q}}} - \mathcal{L}_R$, which yields the classical energy

$$E = T_R + \mathcal{E} \tag{2.35}$$

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U} \tag{2.36}$$

Note that in this derivation, $\dot{\mathbf{Q}} \cdot \partial\mathbf{H}_R/\partial\dot{\mathbf{Q}} = \mathbf{H}_R = 0$ as the angular momentum is linear in the velocities. From this last statement, along with the fact that $T_R \geqslant 0$, the useful inequality is found

$$\mathcal{E} \leqslant E \tag{2.37}$$

This inequality is sharp, meaning that equality can occur when the internal kinetic energy is zero. This happens, in particular, when the system is at an equilibrium. It can also be verified that when the system is at an equilibrium the condition $\delta\mathcal{E} = \partial\mathcal{E}/\partial\mathbf{Q}\cdot\delta\mathbf{Q} \geqslant 0$ holds for all feasible $\delta\mathbf{Q}$. If the bodies are not in contact, then the variation in each coordinate is zero, whereas if any bodies are in contact then the allowable variations will only increase energy [Scheeres (2019)].

## 3. Alternate Derivation

The AMR potential can also be derived using a different approach related to Cauchy's inequality and resulting in a modified version of the Sundman inequality. The total angular momentum of the system can be defined as an integral over the entire mass distribution, which contains all of the rigid bodies [Scheeres (2019)].

$$\mathbf{H} = \int_{\mathcal{B}} \mathbf{r}\times\mathbf{v}\ dm \tag{3.1}$$

The magnitude is then $H$ and the unit vector is $\hat{\mathbf{H}} = \mathbf{H}/H$.

Also recall the similarly general definitions for kinetic energy, polar moment of inertia and inertia tensor.

$$T = \frac{1}{2}\int_{\mathcal{B}} \mathbf{v}\cdot\mathbf{v}\ dm \tag{3.2}$$

$$I_P = \int_{\mathcal{B}} \mathbf{r}\cdot\mathbf{r}\ dm \tag{3.3}$$

$$\bar{\mathbf{I}} = -\int_{\mathcal{B}} \tilde{\mathbf{r}}\cdot\tilde{\mathbf{r}}\ dm \tag{3.4}$$

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The usual Sundman Inequality derivation is made using the polar moment of inertia, as found in [Pollard (1966)]. Starting with the angular momentum definition, $H^2 = \left|\int_{\mathcal{B}} \mathbf{r}\times\mathbf{v}\ dm\right|^2 \leqslant \left(\int_{\mathcal{B}} |\mathbf{r}\times\mathbf{v}|\ \ dm\right)^2$. Then the Cauchy-Schwartz inequality can be applied to find $\left(\int_{\mathcal{B}} |\mathbf{r}\times\mathbf{v}|\ \ dm\right)^2 \leqslant \left(\int_{\mathcal{B}} v^2\ dm\right)\left(\int_{\mathcal{B}} r^2\ dm\right)$. From above these two integrals are the kinetic energy and polar moment of inertia, leading to $H^2 \leqslant 2T\ I_P$. Using the relationship $E = T + \mathcal{U}$, then $H^2 \leqslant 2(E-\mathcal{U})I_P$, the polar amended potential and related Sundman inequality is found as

$$\mathcal{E}_P = \frac{H^2}{2I_P} + \mathcal{U} \leqslant E \tag{3.5}$$

Here $\mathcal{E}_P$ is the polar moment of inertia version of the amended potential.

A sharper alternate approach to Sundman's inequality can be followed that utilizes the system moment of inertia about $\hat{\mathbf{H}}$, and results directly in the AMR amended potential. At the start, the change is trivial and just recognizes that the angular momentum vector is oriented along the $\hat{\mathbf{H}}$ vector. Thus, $H = \hat{\mathbf{H}}\cdot\mathbf{H} = \int_{\mathcal{B}} \hat{\mathbf{H}}\cdot\mathbf{r}\times\mathbf{v}\ dm$, where since the angular momentum direction is a constant it can be brought inside the integral. Note that $\hat{\mathbf{H}}\cdot\mathbf{r}\times\mathbf{v} = \mathbf{v}\cdot(\hat{\mathbf{H}}\times\mathbf{r})$ and apply the Cauchy-Schwartz inequality to $H^2$ to find

$$H^2 = \left(\int_{\mathcal{B}} \mathbf{v}\cdot\hat{\mathbf{H}}\times\mathbf{r}\ dm\right)^2 \leqslant \left(\int_{\mathcal{B}} v^2\ dm\right)\left(-\hat{\mathbf{H}}\cdot\int_{\mathcal{B}} \tilde{\mathbf{r}}\cdot\tilde{\mathbf{r}}\ dm\cdot\hat{\mathbf{H}}\right)$$

which immediately leads to $H^2 \leqslant 2T\ I_H$. Again using the relationship $E = T + \mathcal{U}$, then $H^2 \leqslant 2(E-\mathcal{U})I_H$ which leads to

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U} \leqslant E \tag{3.6}$$

Note that the defined lower quantity is the AMR amended potential $\mathcal{E}$. Substituting the Jacobi integral from the rotating, reduced dynamical system, $E = T_R + \mathcal{E}$, it is clear that the AMR amended potential equals the energy when the system is at an equilibrium.

## 4. Comparisons with Other Amended Potentials

There are several versions of the amended potential that can be used to define different variations of the Sundman inequality, relating angular momentum, moment of inertia and energy. It is instructive to compare these to gain insight into the relations between them. These are succinctly summarized with a few theorems and proofs.

### 4.1. *Comparison with $\mathcal{E}_P$*

THEOREM 4.1.

$$\mathcal{E}_P \leqslant \mathcal{E}$$

If there are any three-dimensional distributions of material then $\mathcal{E}_P < \mathcal{E}$, while if the entire system is planar, including the rigid bodies, then $\mathcal{E}_P = \mathcal{E}$.

*Proof.* To prove this it is sufficient to show that $I_H \leqslant I_P$ in general, and is either strictly less than or equal for the different cases considered. Note that $I_P = \frac{1}{2}\text{Trace}(\bar{\mathbf{I}}) = \frac{1}{2}(I_1 + I_2 + I_3)$ where $I_i$ are the principal moments of inertia of the system, assumed ordered as $I_1 \leqslant I_2 \leqslant I_3$. Then $I_H \leqslant I_3$ and the most extreme case to consider is

$$I_3 \leqslant \frac{1}{2}(I_1 + I_2 + I_3), \tag{4.1}$$

however this can be reduced to $I_3 \leqslant I_1 + I_2$ which is a fundamental result. It can easily be shown that equality only holds for fully planar mass distributions. Thus, if the system has

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3-dimensional rigid bodies the result becomes a strict inequality, $I_3 < \frac{1}{2}(I_1 + I_2 + I_3)$. If the systems are fully planar then $I_H = I_3 = I_P$. □

Thus the polar moment of inertia compresses three dimensional mass distributions, including the moments of inertias for rigid bodies, into a lumped value that does not provide a sharp inequality. This implies that at equilibrium, the polar amended potential is not equal to the total energy for systems with three dimensional mass distributions.

### 4.2. *Smale Amended Potential*

A version of the amended potential that is used in geometric mechanics analysis of these problems is the Smale Amended Potential [Simo et al. (1991)]:

$$\mathcal{E}_{GM} = \frac{1}{2}\mathbf{H}\cdot\bar{\mathbf{I}}^{-1}\cdot\mathbf{H} + \mathcal{U} \tag{4.2}$$

This potential bounds the AMR potential.

Theorem 4.2.

$$\mathcal{E} \leqslant \mathcal{E}_{GM} \leqslant E$$

The two potentials are equal when the system is in a relative equilibrium.

*Proof.* One only needs to prove $H^2/I_H^2 \leqslant \mathbf{H}\cdot\bar{\mathbf{I}}^{-1}\cdot\mathbf{H}$. Normalize by $H^2$, choose a diagonal coordinate representation for $\bar{\mathbf{I}}$ denoted as $\hat{\mathbf{e}}_i, i = 1, 2, 3$, and note that $\hat{\mathbf{H}} = \sum_{i=1}^{3} h_i \hat{\mathbf{e}}_i$. Then the inequality can be reduced to

$$\begin{aligned}
1 &\leqslant \left(h_1^2 I_1 + h_2^2 I_2 + h_3^2 I_3\right)\left(h_1^2\frac{1}{I_1} + h_2^2\frac{1}{I_2} + h_3^2\frac{1}{I_3}\right) \\
&= h_1^4 + h_2^4 + h_3^4 + h_1^2 h_2^2\left(\frac{I_2}{I_1} + \frac{I_1}{I_2}\right) + h_2^2 h_3^2\left(\frac{I_3}{I_2} + \frac{I_2}{I_3}\right) + h_3^2 h_1^2\left(\frac{I_1}{I_3} + \frac{I_3}{I_1}\right) \\
&\quad + 2\left(h_1^2 h_2^2 + h_2^2 h_3^2 + h_3^2 h_1^2\right) - 2\left(h_1^2 h_2^2 + h_2^2 h_3^2 + h_3^2 h_1^2\right) \\
&= \left(h_1^2 + h_2^2 + h_3^2\right)^2 + h_1^2 h_2^2\frac{(I_1 - I_2)^2}{I_1 I_2} + h_2^2 h_3^2\frac{(I_2 - I_3)^2}{I_2 I_3} + h_3^2 h_1^2\frac{(I_3 - I_1)^2}{I_3 I_1} \\
&= 1 + h_1^2 h_2^2\frac{(I_1 - I_2)^2}{I_1 I_2} + h_2^2 h_3^2\frac{(I_2 - I_3)^2}{I_2 I_3} + h_3^2 h_1^2\frac{(I_3 - I_1)^2}{I_3 I_1}
\end{aligned}$$

or

$$0 \leqslant h_1^2 h_2^2\frac{(I_1 - I_2)^2}{I_1 I_2} + h_2^2 h_3^2\frac{(I_2 - I_3)^2}{I_2 I_3} + h_3^2 h_1^2\frac{(I_3 - I_1)^2}{I_3 I_1} \tag{4.3}$$

which is true by inspection. When the system is at an equilibrium, it must rotate uniformly about only one of the principle moments of inertia of the system, meaning $h_i h_j = 0$ for $i \neq j$. In this case the bound becomes an equality, meaning that the two potentials are equal when evaluated at an equilibrium. □

Despite the fact that the Smale potential bounds the AMR potential, they both yield conditions for equilibria and by extension for energetic stability of an equilibria.

Theorem 4.3. *If the system is in a non-contact equilibrium configuration, then*

$$\delta\mathcal{E}_{GM} = \delta\mathcal{E} = 0$$

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*Proof.* Note that, based on the Lagrangian derivation, when the system is at such an equilibrium then $\delta\mathcal{E}=0$. To prove the other equality first express the first variation of both quantities:

$$\delta\mathcal{E}_{GM}=-\frac{1}{2}\mathbf{H}\cdot\bar{\mathbf{I}}^{-1}\cdot\delta\bar{\mathbf{I}}\cdot\bar{\mathbf{I}}^{-1}\cdot\mathbf{H}+\delta\mathcal{U} \tag{4.4}$$

$$\delta\mathcal{E}=-\frac{1}{2}\left(\frac{H}{I_H}\right)^2\delta I_H+\delta\mathcal{U} \tag{4.5}$$

Cancelling like terms reduces the proof to showing the following result

$$\left(\frac{H}{I_H}\right)^2\delta I_H=\mathbf{H}\cdot\bar{\mathbf{I}}^{-1}\cdot\delta\bar{\mathbf{I}}\cdot\bar{\mathbf{I}}^{-1}\cdot\mathbf{H} \tag{4.6}$$

When in an equilibrium $\mathbf{H}\cdot\bar{\mathbf{I}}^{-1}=\bar{\mathbf{I}}^{-1}\cdot\mathbf{H}=\boldsymbol{\Omega}=\dot{\theta}\hat{\mathbf{H}}$ by definition, as it will be rotating about a principal moment of inertia. Thus, substituting for the definition of $I_H$ and noting that $H/I_H=\dot{\theta}$, equality is shown to be trivially true. □

This is an interesting result, given that the one potential in general bounds the other, and implies that either potential can be used to evaluate equilibria and stability.

### 4.3. *Disaggregated AMR Amended Potential*

A different potential version that bounds the AMR amended potential can be defined. This form disaggregates the potential into a series of component parts. This approach is specialized to the Jacobi coordinates, and was derived and proven in [Scheeres & Brown (2023)]. Define a disaggregated version of the potential as

$$\mathcal{E}_i=\frac{H_i^2}{2I_{H_i}}+\mathcal{U}_i \tag{4.7}$$

where $H_i$ is the angular momentum arising from the $i$th Jacobi coordinate, $I_{H_i}=\hat{\mathbf{H}}\cdot\bar{\mathbf{I}}_i\cdot\hat{\mathbf{H}}$, and $\mathcal{U}_i=\sum_{j=0}^{i}\mathcal{U}_{ij}$. Note that $\sum_{i=1}^{N}H_i=H$.

Then in [Scheeres & Brown (2023)] it was proven that:

$$\mathcal{E}\leqslant\sum_{i=1}^{N}\mathcal{E}_i\leqslant E \tag{4.8}$$

and that at an equilibrium the two potentials are equal. The individual $\mathcal{E}_i$ components do not provide information on the dynamics. One should note, specifically, that the angular momentum terms $H_i$ are not necessarily conserved, and thus can individually vary when taking variations. Only when they are summed is that quantity conserved.

## 5. Conclusions

In this contribution the Angular Momentum Relative amended potential is derived and compared with other amended potential versions. When the dynamical system is composed of rigid bodies, it is shown that the AMR potential dominates the potential defined using the polar moment of inertia. It is also shown that the Smale amended potential bounds the AMR potential, but that they are equal at an equilibrium and that they can both yield conditions for evaluating such equilibria and their stability.

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