Can. J. Math. Vol. 48 (5), 1996 pp. 997–1017

NONABELIAN FULLY-RAMIFIED SECTIONS

MARK L. LEWIS

ABSTRACT. Let G be a finite group and let K and L be normal subgroups of G such that $|K : L|$ and $|G : K|$ are relatively prime, and assume that $|K : L|$ is odd. Let H be a subgroup of G such that $G = HK$ and $H \cap K = L$. Let φ be an irreducible character of L that is invariant under the action of H and is fully ramified with respect to K/L. If $\chi \in \mathrm{Irr}(G)$ is a constituent of φ^G, then we prove that χ_H has a unique irreducible constituent having odd multiplicity.

1. **Introduction.** Given a finite group G, we write $\mathrm{Irr}(G)$ for the set of all irreducible characters of G. For a subgroup H of G, we would like to understand the relationship between the sets $\mathrm{Irr}(H)$ and $\mathrm{Irr}(G)$. In general, there is very little that can be said about how these two sets relate. If we know something about the structure of how H is embedded in G, then often we can apply knowledge of $\mathrm{Irr}(G)$ to gain information about $\mathrm{Irr}(H)$. In many cases, we obtain this data by constructing a bijection between the characters that lie in a subset $\mathcal{A}$ of $\mathrm{Irr}(G)$ and the characters that lie in a subset $\mathcal{B}$ of $\mathrm{Irr}(H)$. Such a bijection is called a *character correspondence*, and these correspondences have been studied in numerous papers. In this paper, we are going to take a closer look at one correspondence that has played an important role in the development of the character theory of finite solvable groups.

In order to present this result, we fix notation that encapsulates the knowledge of how the subgroup H is embedded in G. The notation that we use in this paper is broader than the notation found in the paper where this correspondence originally appeared, [4]. We extend this bijection to our more general setting, and so, we feel justified in using this broader notation to present Isaacs' result. We say that (G, K, L) is a *normal triple* if $L \subseteq K$ are normal subgroups of G. Such a normal triple is called *solvable*, nilpotent, or abelian if the quotient group K/L is solvable, nilpotent, or abelian. We call the subgroup H of G a complement for the normal triple (G, K, L) if $G = HK$ and $H \cap K = L$.

Given a normal subgroup N of G, there is an action of G on the elements of $\mathrm{Irr}(N)$ defined by $\varphi^g(n^g) = \varphi(n)$ where the character φ lies in $\mathrm{Irr}(N)$ and the elements g lies in G and n lies in N. Now, we can define $(G, K, L, \epsilon, \varphi)$ to be a *basic configuration* if (G, K, L) is a solvable normal triple, $\epsilon \in \mathrm{Irr}(K)$ is G-invariant, and $\varphi \in \mathrm{Irr}(L)$ is a constituent of ϵ_L. We will call this basic configuration abelian or nilpotent if the normal triple (G, K, L) is abelian or nilpotent. We say that H is a *stabilizing complement* for $(G, K, L, \epsilon, \varphi)$ if H

This appeared in somewhat different form in the author's thesis at the University of Wisconsin under the supervision of I. M. Isaacs and as such was supported by a DOE Fellowship.

Received by the editors April 20, 1995; revised December 15, 1995.

AMS subject classification: 20C15.

is a complement for (G,K,L) and φ is H-invariant, and from the definition of the basic configuration, we observe that ϵ is necessarily invariant under the action of H.

We also need to set notation for another situation that has been studied in many different papers. Given a subgroup H of G and an irreducible character $\varphi \in \mathrm{Irr}(H)$, we write $\mathrm{Irr}(G \mid \varphi)$ for the subset of $\mathrm{Irr}(G)$ consisting of irreducible constituents of φ^G. Suppose that L is a normal subgroup of G and that the irreducible character $\varphi \in \mathrm{Irr}(L)$ is G-invariant; we say that φ is *fully ramified* with respect to G/L if $\mathrm{Irr}(G \mid \varphi)$ contains exactly one character. Thus, a basic configuration (G,K,L,ϵ,φ) is called *fully ramified* if φ is fully ramified with respect to K/L.

In [4], Isaacs studied the case where (G,K,L,ϵ,φ) is an abelian fully-ramified configuration. In the notation of that paper, an abelian fully-ramified configuration is called a *character five*. One result that he proved in that paper is the following.

THEOREM (ISAACS). *Let (G,K,L,ϵ,φ) be an abelian fully-ramified configuration such that $|G:L|$ is odd. Then there exists a stabilizing complement H so that for each character $\theta \in \mathrm{Irr}(H \mid \varphi)$ there is a unique irreducible constituent χ of θ^G having odd multiplicity, and θ is the unique constituent of χ_H having odd multiplicity.*

Given an abelian fully-ramified configuration (G,K,L,ϵ,φ) such that $|G:L|$ is odd, one can use this theorem to define a correspondence between $\mathrm{Irr}(H \mid \varphi)$ and $\mathrm{Irr}(G \mid \epsilon)$ by associating the character $\theta \in \mathrm{Irr}(H \mid \varphi)$ with the unique irreducible constituent χ of θ^G having odd multiplicity. Furthermore, observe that this bijection is invariant under any map that fixes H, K, L, and φ. In order to generalize this result, we need to fix some more notation. We will call a normal triple (G,K,L) *coprime* if $(|G:K|,|K:L|) = 1$ and a configuration (G,K,L,ϵ,φ) is coprime if (G,K,L) is. We now present an analog to Isaacs' theorem for coprime fully-ramified configurations (*i.e.*, nonabelian configurations.)

THEOREM A. *Let (G,K,L,ϵ,φ) be a coprime fully-ramified configuration such that $|G:L|$ is odd. Then there exists a stabilizing complement H so that for each character $\theta \in \mathrm{Irr}(H \mid \varphi)$ there is a unique irreducible constituent χ of θ^G having odd multiplicity and θ is the unique irreducible constituent of χ_H having odd multiplicity.*

As in Isaacs' theorem, we can use Theorem A to define a correspondence between the sets $\mathrm{Irr}(H \mid \varphi)$ and $\mathrm{Irr}(G \mid \epsilon)$. In fact, this bijection is invariant under any isomorphism of G that fixes K, L, and φ. The result that we have presented as Isaacs' theorem is weaker than what appears in [4]. In particular, Isaacs is able to construct a correspondence with the assumption that either $|G:K|$ or $|K:L|$ is odd in place of the assumption that $|G:L|$ is odd that we used. In fact, even in the nonabelian case, it is possible to build on the techniques we develop here to construct such bijections under weaker conditions than we assume here. In [9], for instance, a bijection is constructed in the coprime situation without the assumption that $|G:L|$ is odd. Isaacs was able to deal with abelian configurations without assuming coprimeness, and in [9], we succeed in constructing bijections in certain nonabelian noncoprime configurations. Extending the results in that paper, we are able, in [10], to prove that this bijection can be constructed to be invariant under isomorphisms of G that fix K, L, and φ. We show, in [11], that the bijection that we construct in [10] is the same as the bijection determined by Theorem A. Furthermore, in [12], we build an example that shows that the coprime hypothesis is necessary to get a unique constituent with odd multiplicity.

Obviously, character correspondences play an important role in this paper. For an expository account regarding this subject, we suggest reading [6]. Perhaps, the best known of these is the Glauberman-Isaacs correspondence. The Glauberman "half" originally appeared in [2], and is described completely in Chapter 12 of [5]. The Isaacs "half" can be found in [4]. (The fact that both "halves" are the same when both are defined was proved by Wolf in [16].) There have been many papers published investigating the properties of this correspondence including (but not limited to) [17], [18], [8], [13], and [14]. Recently, Navarro has written a paper outlining many of the problems related to this correspondence that are still open (see [15]). While the Glauberman-Isaacs correspondence has received the most attention, it is not the only one that has been studied. A different correspondence is the subject of [7]. In this paper, the other correspondence that we will use frequently is a consequence of a theorem of Clifford's and can be found as Theorem 6.11(b) of [5].

To show the utility of Theorem A, we would like to apply it to prove a corollary in the area of coprime actions. Let the group S act on the group G so that $(|S|,|G|) = 1$. We will write Γ for the semi-direct product obtained by S acting on G. By Corollary 8.16 of [5], there is a unique extension χ^* of χ to Γ such that the determinantal order $o(\chi^*)$ of χ^* is relatively prime to $|S|$, and χ^* is called the *canonical* extension of χ.

THEOREM B. *Let the group S act on G such that $(|S|,|G|) = 1$ and $|S|\,|G|$ is odd. Suppose that L is a normal subgroup S-invariant subgroup of G and that the character $\varphi \in \mathrm{Irr}(L)$ is fully ramified with respect to G/L and S-invariant. Assume that the subgroup H of G contains L and is S-invariant and that φ is fully ramified with respect to H/L. If ϵ and γ are the unique irreducible constituents of φ^G and φ^H respectively, then ϵ and γ are S-invariant and χ^* is the unique irreducible constituent of $(\gamma^*)^{SG}$ having odd multiplicity.*

2. **Coprime and controlled triples.** Let (G,K,L) be a coprime triple, and so, by the Schur-Zassenhaus theorem (see Hauptsatz I.18.1 and I.18.2 of [3]), we know that (G,K,L) has a complement and that all the complements are conjugate. If H is a complement in this situation, then it follows that H/L acting on K/L is a coprime action. We will say that the action is fixed-point-free if $\mathbf{C}_{K/L}(H) = 1$, and since in this case all the complements are conjugate, it is easy to see that this property is independent of the choice of complement. Therefore, we may define (G,K,L) to be a *fixed-point-free coprime normal triple* in this case. Given a normal triple (G,K,L), assume that there exists a normal subgroup M of G containing K such that (M,K,L) is a fixed-point-free coprime normal triple. In this situation, we say that (G,K,L) is a controlled normal triple and that (G,K,L,ϵ,φ) is a *controlled configuration* if it is a basic configuration with a controlled normal triple.

The next lemma, which is a consequences of Glauberman's lemma, Theorem 13.8 of [5], will be used to look at the character theory of coprime and controlled triples.

LEMMA 2.1. *Let (G,K,L) be a coprime normal triple. Assume that $\varphi \in \mathrm{Irr}(L)$ is G-invariant and that φ extends to K. Then some irreducible character extending φ in K is G-invariant. Furthermore, if (G,K,L) is a fixed-point-free coprime normal triple, then this G-invariant extension is unique.*

PROOF. We write $\delta \in \mathrm{Irr}(K)$ for an extension of φ, and we know by Gallagher's theorem (Corollary 6.17 of [5]) that the set of irreducible characters in K lying over φ is $\{\delta\gamma \mid \gamma \in \mathrm{Irr}(K/L)\}$. Thus, if we set $\Omega = \{\delta\gamma \mid \gamma \in \mathrm{Irr}(K/L), \gamma(1) = 1\}$, then it is easy to see that Ω is the set of extension of φ to $\mathrm{Irr}(K)$. Since φ is G-invariant, it follows that G permutes the irreducible constituents of φ^K. Because this action of G fixes character degrees, we know that G must permute the elements of Ω. Furthermore, since K acts trivially on Ω under this action, we may view this action as a G/K-action. We denote the set of linear characters of K/L by $A = \mathrm{lin}(K/L)$, and we remark that A is an abelian group with $|A| = |(K/L) : (K/L)'|$. This implies that $|A|$ divides $|K : L|$. Since G/K fixes character degrees in its action on $\mathrm{Irr}(K/L)$, it follows that G/K acts by conjugation on A in a coprime manner. Also, it is easy to see that A acts transitively on Ω by right multiplication. Since $(\alpha\beta)^g = \alpha^g\beta^g$ for all $\alpha \in \Omega$, $\beta \in A$, and $g \in G/K$, we are now in the situation of Glauberman's lemma (Theorem 13.8 of [5]). By that theorem, we know that there exists some extension $\alpha \in \Omega$ that is G-invariant.

We know by the Schur-Zassenhaus theorem (see Hauptsatz I.18.1 of [3]) that there exists a complement H for (G,K,L). Assume that the action of H/L on K/L is fixed-point-free. Since H/L is isomorphic to G/K, we can view the action in the last paragraph as being an H/L-action. Since this is a coprime action, we know that $K/L = ([K,H]L/L)\mathbf{C}_{K/L}(H)$. Because the action is fixed-point-free, we have that $\mathbf{C}_{K/L}(H) = 1$. Together, these two facts imply that $K = [K,H]L$. By a corollary to Glauberman's lemma (Corollary 3.9 of [5]), we know that $\mathbf{C}_A(H)$ acts transitively on the H/L-fixed points of Ω. Thus, if we can show that $\mathbf{C}_A(H) = 1$, then we are done. Suppose that $\lambda \in \mathbf{C}_A(H)$. This implies that λ is a linear character of K/L and that $\lambda^h = \lambda$ for all $h \in H$. We see that $\lambda^h(k) = \lambda(k)$ for all $k \in K$. Using the fact that λ is a homomorphism (*i.e.* λ is linear), it is easy to see that $\lambda([h,k]) = 1$ for all $h \in H$ and $k \in K$. This implies that $[K,H] \subseteq \ker(\lambda)$. Since we also know that $L \subseteq \ker(\lambda)$, it follows that $[K,H]L \subseteq \ker(\lambda)$. Recalling that $[K,H]L = K$, we now see that $\ker(\lambda) = K$ and that $\lambda = 1_K$. ■

We now get an immediate corollary for controlled triples.

COROLLARY 2.2. *Let (G,K,L) be a controlled normal triple. Suppose that $\varphi \in \mathrm{Irr}(L)$ is G-invariant and that φ extends to K. Then φ has a unique G-invariant extension to K.*

PROOF. We have a normal subgroup M containing K such that (M,K,L) is a fixed-point-free coprime normal triple. Thus, by Lemma 2.1, we see that φ has a unique M-invariant extension to K. On the other hand, it is easy to see that G permutes the M-invariant extensions of φ to K. The uniqueness of the M-invariant extension implies that it must be G-invariant. ■

3. **The "bilinear" form and good elements.** First, we present a tool that Isaacs used in [4] to study fully-ramified sections and that we use to study the nonabelian case.

Given a normal subgroup L of G and a G-invariant character $\varphi \in \mathrm{Irr}(L)$, a uniquely defined complex number $\langle\langle x,y\rangle\rangle_\varphi$ was constructed in [4] for all $x,y \in G$ such that xL and yL commute. In particular, if $[x,y] \in L$, then define $\langle\langle x,y\rangle\rangle_\varphi$ in the following manner. Since $\langle L,y\rangle/L$ is cyclic and φ is invariant, it follows that there exists some character $\psi \in \mathrm{Irr}(\langle L,y\rangle)$ extending φ, and since x normalizes $\langle L,y\rangle$, we see that ψ^x is another extension of φ. Thus, by Gallagher's theorem (6.17 of [5]), there exists some linear character $\lambda \in \mathrm{Irr}(\langle L,y\rangle/L)$ such that $\psi^x = \lambda\psi$, and hence, we may define $\langle\langle x,y\rangle\rangle_\varphi$ to be $\lambda(y)$. In [4], Isaacs proved that this is well-defined and when φ is linear that $\langle\langle x,y\rangle\rangle_\varphi = \varphi([x,y])$, and he also showed there that $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is constant on L-cosets in G. Thus, we view $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ as a form that is defined on commuting pairs in G/L. Given a subgroup X such that X/L is central in G/L, we define the *perpendicular space* for X in G with respect to $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ as

$$X^\perp = \{k \in K \mid \langle\langle k,X\rangle\rangle = 1\}.$$

We now assume that (G,K,L) is an abelian normal triple, and so, we may view $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ as a bilinear form defined on $K/L\times K/L$. The form is called *nondegenerate* on K/L when the only elements $a \in K$ that have the property $\langle\langle a,K\rangle\rangle_\varphi = 1$ lie in L. In particular, $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is nondegenerate on K/L if and only if $K\cap K^\perp = L$. In Lemma 2.1 of [4], Isaacs proved that φ is fully ramified with respect to K/L if and only if $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is nondegenerate on K/L. The next result that we will require is the following lemma which can be found as Lemma 2.4 in [4].

LEMMA 3.1. *Let (G,L,θ) be a character triple with $L \subseteq H \subseteq G$. Fix a character $\chi \in \mathrm{Irr}(H \mid \theta)$ and elements $g \in G$ and $h \in H$ such that $[g,h] \in L$. Then:*

(a) $\chi(ghg^{-1}) = \langle\langle g,h\rangle\rangle_\varphi\chi(h)$.

(b) If $g \in H$ and $\langle\langle g,h\rangle\rangle_\varphi \neq 1$, then $\chi(g) = 0 = \chi(h)$. ∎

In fact, the function $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is an example of the following well-known general phenomenon. We say that $(\cdot,\cdot)$ is a *pairing* between the groups A and B if $(\cdot,\cdot)$ maps $A\times B$ into the multiplicative group of the complex numbers and $(\cdot,\cdot)$ is a homomorphism in each coordinate. We define $A_0 = \{a \in A \mid (a,B) = 1\}$ and $B_0 = \{b \in B \mid (A,b) = 1\}$.

Let (G,K,L) be a normal triple and φ a G-invariant character in $\mathrm{Irr}(L)$. In Lemma 2.1 of [4], Isaacs proved that if X is a subgroup such that X/L is in the center of K/L, then $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is a G-invariant pairing between K and X, and it is easy to see that $K_0 = X^\perp$. The following two lemmas are well-known.

LEMMA 3.2. *Let $(\cdot,\cdot)$ be a pairing between A and B. Then A/A_0 and B/B_0 are isomorphic abelian groups.*

PROOF. Note that if $b \in B$, then the map $(\cdot,b)$ is a linear character of A with A_0 in its kernel. It is easy to see that the map $b \longmapsto (\cdot,b)$ defines a homomorphism from B into the linear characters of A/A_0, written $\mathrm{lin}(A/A_0)$. Furthermore, the kernel of this map is exactly B_0. Thus, we have that $|B : B_0| \leq |\,\mathrm{lin}(A/A_0)| \leq |A : A_0|$. From symmetry, we get the reverse inequality, and so, we have $|A : A_0| = |B : B_0|$. This implies that B/B_0 is isomorphic to $\mathrm{lin}(A/A_0)$ and thus, to A/A_0. Since $\mathrm{lin}(A/A_0)$ is abelian, the result now follows. ∎

COROLLARY 3.3. *Let (G, L, φ) be a character triple. Assume that U is a subgroup of G containing L such that $U/L \subseteq \mathbf{Z}(G/L)$. Then $|G : U^{\perp}| \leq |U : L|$. Furthermore, if φ is fully-ramified with respect to G/L, then $|G : U^{\perp}| = |U : L|$.*

PROOF. Recall that $\langle\langle \cdot, \cdot \rangle\rangle_{\varphi}$ is pairing between U and G and that $G_0 = U^{\perp}$. It is easy to see that $L \subseteq U_0$, and from Lemma 3.2, we have that $|G : U^{\perp}| \leq |U : L|$. Now assume that φ is fully ramified with respect to G/L, and we work to prove that $U_0 = L$. Suppose that $u \in U_0$, and since φ is fully ramified with respect to G/L, we know that G acts transitively on $\mathrm{Irr}(\langle L, u \rangle \mid \varphi)$. On the other hand, from Lemma 3.1, we see that all the elements of $\mathrm{Irr}(\langle L, u \rangle \mid \varphi)$ are G-invariant. The only way both of these facts can occur simultaneously is if $|\mathrm{Irr}(\langle L, u \rangle \mid \varphi)| = 1$. Since φ is u-invariant, we know that φ extends to $\langle L, u \rangle$ and so by Gallagher's theorem (6.17 of [5]), $|\mathrm{Irr}(\langle L, u \rangle \mid \varphi)| = |\langle L, u \rangle : L|$. Thus, we conclude that $u \in L$. Therefore, $U_0 = L$ and the result follows from Lemma 3.2. ∎

Next, we include any easy consequence of Corollary 3.3.

LEMMA 3.4. *Let (K, L, φ) be a fully-ramified triple. Suppose that U is a subgroup of K such that $U/L \subseteq \mathbf{Z}(K/L)$, and consider a character $\delta \in \mathrm{Irr}(U \mid \varphi)$. Write V for the stabilizer of δ in K. Then (V, U, δ) is a fully-ramified triple. Furthermore, if δ is an extension of φ, then $V = U^{\perp}$.*

PROOF. Because δ is invariant in V, it suffices to show that δ is fully ramified with respect to V/U. Since U is normal in K and V is the stabilizer of δ in K, it follows, by Clifford's theorem (Theorem 6.11 of [5]), that $|\mathrm{Irr}(K \mid \delta)| = |\mathrm{Irr}(V \mid \delta)|$. On the other hand, it is easy to see that $\mathrm{Irr}(K \mid \delta) \subseteq \mathrm{Irr}(K \mid \varphi) = \{\epsilon\}$. Thus, $|\mathrm{Irr}(V \mid \delta)| = 1$, and because δ is V-invariant, we conclude that δ is fully ramified with respect to V/U.

Assume that φ extends to U, and so, we know by Gallagher's theorem (6.17 of [5]) that

$$|\mathrm{Irr}(U \mid \varphi)| = |\mathrm{Irr}(U/L)| = |U : L|.$$

Since φ is fully ramified with respect to K/L, we have that K acts transitively on the set $\mathrm{Irr}(U \mid \varphi)$. Hence, we know that $|K : V| = |U : L|$. By Lemma 3.1, we know that $U^{\perp}$ fixes δ, and so $U^{\perp} \subseteq V$. From Corollary 3.3, we have that $|K : U^{\perp}| = |U : L|$. Thus, we see that $|K : U^{\perp}| = |K : V|$. It is now immediate that $U^{\perp} = V$. ∎

In order to prove Theorem A, we need to generalize one of the tools used in [4] to the case where the configuration is a nilpotent fully-ramified configuration. The tool that Isaacs used to study fully-ramified abelian sections is the idea of "good" elements. We remark that in [4] good elements were defined only when K/L was abelian, and so, we now extend the definition of good elements to the case when K/L need not be abelian.

DEFINITION. Let (G, K, L) be a normal triple, $\varphi \in \mathrm{Irr}(L)$ be a G-invariant character, and $C/L = \mathbf{C}_{K/L}(g)$ for some element $g \in G$. We say that g is *good with respect to* *the character triple* (K,L,φ) if φ has a g-invariant extension to $\langle L,c\rangle$ for every element $c\in C$.

Note in this definition that if one extension of φ to $\langle L,c\rangle$ is g-invariant, then it follows that all the extensions of φ to $\langle L,c\rangle$ are g-invariant. We remark that if g is good with respect to the character triple (K,L,φ), then it follows that every member of gL is good with respect to this same character triple. Thus, we will often say that an L-coset is good with respect to the character triple (K,L,φ) if the members of the L-coset are good with respect to this character triple. We will show that this definition is equivalent to the definition of good elements presented in [4].

This next lemma specifies the relationship between good elements and the form discussed above. In particular, the elements of G that are good with respect to the triple (K,L,φ) are precisely those elements g such that $\langle\langle g,c\rangle\rangle_\varphi=1$ for every element $c\in C$, where $C/L=\mathbf{C}_{K/L}(g)$. Given an element $g\in G$ and a character $\chi\in\mathrm{Irr}(G\mid\varphi)$, suppose that $\chi(g)\neq 0$. Observe that Lemma 3.1 in combination with the next lemma states that g must be good with respect to triple (K,L,φ).

LEMMA 3.5. *Let (G,K,L) be a normal triple, $\varphi\in\mathrm{Irr}(L)$ be a G-invariant character, and $C/L=\mathbf{C}_{K/L}(g)$ for some element $g\in G$. Then g is good with respect to the triple (K,L,φ) if and only if $\langle\langle g,c\rangle\rangle_\varphi=1$ for every element $c\in C$. Furthermore, if φ is linear, then g is good with respect to the triple (K,L,φ) if and only if $[g,C]\subseteq\ker(\varphi)$.*

PROOF. We know that g is good with respect to the triple (K,L,φ) if and only if φ has a g-invariant extension to $\langle L,c\rangle$ for every element $c\in C$. On the other hand, it is easy to see from Lemma 3.1 for each element $c\in C$ that φ has a g-invariant extension to $\langle L,c\rangle$ if and only if $\langle\langle g,c\rangle\rangle_\varphi=1$. Therefore, it follows that g is good with respect to the triple (K,L,φ) if and only if $\langle\langle g,c\rangle\rangle_\varphi=1$ for every element $c\in C$.

Now, we assume that φ is linear, and as we stated earlier, we know that we may write $\langle\langle g,c\rangle\rangle_\varphi=\varphi([g,c])$ for all $c\in C$. Thus, by applying the previous paragraph we see that g is good with respect to the triple (K,L,φ) if and only if $[g,C]\subseteq\ker(\varphi)$. ∎

Suppose again we have a character triple (G,L,φ) where φ is linear and faithful. If $\langle\langle x,y\rangle\rangle_\varphi=1$ for elements $x,y\in G$ such that $[x,y]\in L$, then we have that $[x,y]=1$. Furthermore, suppose there exists a normal subgroup K containing L such that φ is fully ramified with respect to K/L. If g is good with respect to the triple (K,L,φ), then g centralizes $\mathbf{C}_{K/L}(x)$.

Assume now that we have an abelian normal triple (G,K,L) and a G-invariant character $\varphi\in\mathrm{Irr}(L)$. When there exists some element $g\in G$ such that the order mod L of g is relatively prime to $|K:L|$, Isaacs showed in [4] that g must be good with respect to the triple (K,L,φ). The following easy lemma shows that we do not need to assume that K/L be abelian.

LEMMA 3.6. *Let (G,K,L) be a normal triple and $\varphi\in\mathrm{Irr}(L)$ be a G-invariant character. If the order mod L of some element $g\in G$ is relatively prime to $|K:L|$, then g is good with respect to the triple (K,L,φ).*

PROOF. Notice that we may assume that $G = \langle K, g\rangle$. Let $C/L = \mathbf{C}_{K/L}(g)$, and consider $c \in C$. We will write $B = \langle L, c\rangle$. Since B/L is cyclic where φ is B-invariant, we have that φ extends to B. Because B is a normal subgroup in $\langle B, g\rangle$, we may apply Lemma 2.1 to $\langle B, g\rangle$ to conclude that φ has a g-invariant extension to B. Therefore, g is good with respect to the triple (K, L, φ). ∎

4. **Conjugacy of good cosets.** In the main result of this section, we prove an analog of Theorem 3.2 of [4]. In particular, we assume that we have normal subgroups K and L of G such that K/L is nilpotent and that we have a G-invariant character $\varphi \in \mathrm{Irr}(L)$ that is fully ramified with respect to K/L. In this situation, we prove that all the L-cosets containing elements that are good with respect to the triple (K, L, φ) are conjugate by elements of K. Before we can prove the main result of this section, we need to prove the following technical lemma.

LEMMA 4.1. *Let (A, U, L) be a normal triple such that L is central in A, that A/L is a p-group for some prime p, that U/L is an elementary abelian p-group, and that U/L is central in A/L. Assume a cyclic group G acts on A such that $[L, G] = 1$ and $[A, G] \subseteq U$. Write $C/L = \mathbf{C}_{A/L}(G)$, and suppose that G centralizes C. Let $\varphi \in \mathrm{Irr}(L)$ be a character such that φ has an A-invariant extension to U. Then there exists a G-invariant extension $\delta \in \mathrm{Irr}(U)$ of φ such that $[A, G] \subseteq \ker(\delta)$.*

PROOF. We assume that (A, U, L) forms a counterexample normal triple with $|A| + |U : L|$ as small as possible.

Write $\nu \in \mathrm{Irr}(U)$ for the A-invariant extension of φ mentioned in the hypothesis. By Gallagher's theorem (6.17 of [5]), we know that every irreducible constituent of φ^U is of the form $\nu\lambda$ for some character $\lambda \in \mathrm{Irr}(U/L)$, and because U/L is central in A/L, it follows that every member of $\mathrm{Irr}(U \mid \varphi)$ is an A-invariant extension of φ.

Suppose first that there exists a G-invariant subgroup V such that $L < V < U$. Since U/L is central in A/L, it follows that V is normal in A, and we set $B/V = \mathbf{C}_{A/V}(G)$. We will show that (B, V, L) is a normal triple satisfying the hypotheses of the lemma. From the definition of B, we see that $[B, G] \subseteq V$. We recall that ν is an A-invariant extension of φ, and so, it follows that $\nu_V \in \mathrm{Irr}(V)$ is an B-invariant extension of φ. In particular, φ has a B-invariant extension to V. Since $|V : L| < |U : L|$ and $|B| \leq |A|$, we have $|B| + |V : L| < |A| + |U : L|$, and thus, we know that (B, V, L) is a normal triple satisfying the conclusion of the lemma. Therefore, there exists a G-invariant character $\gamma \in \mathrm{Irr}(V)$ extending φ such that $[B, G] \subseteq \ker(\gamma)$.

Because every irreducible constituent of γ^U is an irreducible constituent of φ^U, we know that every character in $\mathrm{Irr}(U \mid \gamma)$ is an A-invariant extension of φ. In particular, γ has an A-invariant extension to U, and so, γ is itself an A-invariant extension of φ. Thus, it follows that $\ker(\gamma)$ is a normal subgroup of A, and we write $\bar{}$ for the map from A to $A/\ker(\gamma)$. We now show that $(\bar{A}, \bar{U}, \bar{V})$ is a normal triple satisfying the hypotheses of the lemma. Since γ is also G-invariant, it follows that G normalizes $\ker(\gamma)$ and that it acts on $\bar{A}$. Because γ is linear, faithful, and $\bar{A}$-invariant in $\bar{V}$, it is easy to see that $\bar{V}$ is

https://doi.org/10.4153/CJM-1996-052-8 Published online by Cambridge University Press

central in $\bar{A}$. Recall that $B/V = \mathbf{C}_{A/V}(G)$ and that $[B, G] \subseteq \ker(\gamma)$, and so, this implies that G centralizes $\bar{B}$. Since $|\bar{U} : \bar{V}| = |U : V| < |U : L|$ and $|\bar{A}| \leq |A|$, we observe that $|\bar{A}| + |\bar{U} : \bar{V}| < |A| + |U : L|$, and by the choice of the normal triple (A, U, L) as a minimal counterexample, we see that there exists a G-invariant extension $\delta \in \mathrm{Irr}(\bar{U})$ of γ such that $[A, G] \subseteq \ker(\delta)$. Because γ extends φ, we conclude that δ is a G-invariant extension of φ in contradiction to the choice of normal triple (A, U, L), and we are done in this case.

We may assume that there exist no proper nontrivial G-invariant subgroups of U/L, and we let $D/L = \mathbf{C}_{U/L}(G)$. Because D is G-invariant, we have that either $D = U$ or $D = L$, and so, we first assume that $D = U$. We now know that $U \subseteq C$ and so, by hypothesis, G centralizes U. In particular, all the extensions of φ to U are G-invariant, and hence, to get a contradiction in this case it suffices to find some extension of φ to U with $[A, G]$ in its kernel. Write $\mathrm{lin}(U)$ for the abelian group of linear characters of U, and note that $\mathrm{Irr}(U/L)$ and $\mathrm{Irr}(U \mid \varphi)$ are subsets of $\mathbf{C}_{\mathrm{lin}(U)}(A)$.

Fix an element $g \in G$ so that $G = \langle g \rangle$. We define a map $f : \mathbf{C}_{\mathrm{lin}(U)}(A) \longrightarrow \mathrm{lin}(A/U)$ by $f(\mu)(a) = \mu([a, g])$ for all $a \in A$ and $\mu \in \mathbf{C}_{\mathrm{lin}(U)}(A)$. To see that this map makes sense, we show that if $\mu \in \mathbf{C}_{\mathrm{lin}(U)}(A)$, then $f(\mu)$ is a homomorphism with U in the kernel. To see that $f(\mu)$ is a homomorphism, we consider the elements $a, b \in A$. Thus, we have that

$$f(\mu)(ab) = \mu([ab, g]) = \mu([a, g]^b[b, g]) = \mu([a, g])\mu([b, g]) = \big(f(\mu)(a)\big)\big(f(\mu)(b)\big)$$

where the third equality follows from the fact that μ is an A-invariant linear character. To see that U is in the kernel of $f(\mu)$, we recall that $[U, G] = 1$. This implies that $f(\mu)(u) = \mu([g, u]) = \mu(1) = 1$ for every element $u \in U$. We now claim that f itself is a homomorphism. Fix $\mu, \nu \in \mathbf{C}_{\mathrm{lin}(U)}(A)$. We see that

$$f(\mu\nu)(a) = \mu\nu([a, g]) = \mu([a, g])\nu([a, g]) = f(\mu)(a)f(\nu)(a)$$

for every element $a \in A$. Thus, we conclude that $f(\mu\nu) = f(\mu)f(\nu)$ and that f is a homomorphism.

Suppose $\mu \in \ker(f)$; so that for every element $a \in A$, we have that $\mu([a, g]) = 1$. This implies that $[A, G] \subseteq \ker(\mu)$. Hence, if $f\big(\mathrm{Irr}(U/L)\big) = 1$, then we have that

$$[A, G] \subseteq \bigcap_{\mu \in \mathrm{Irr}(U/L)} \ker(\mu) = L$$

where the last equality comes from Lemma 2.21 of [5]. This implies that $A \subseteq C$, and it follows that G centralizes A. Therefore, we conclude that $[A, G] = 1 \subseteq \ker(\delta)$ for all extensions $\delta \in \mathrm{Irr}(U)$ of φ. Therefore, any extension of φ to U satisfies the lemma, and we have a contradiction to the choice of (A, U, L) in the case when $f\big(\mathrm{Irr}(U/L)\big) = 1$.

We assume that $f\big(\mathrm{Irr}(U/L)\big) > 1$. We fix a character $\lambda \in f\big(\mathrm{Irr}(U/L)\big)$ with $\lambda \neq 1$ and choose a character $\mu \in \mathrm{Irr}(U/L)$ so that $f(\mu) = \lambda$. Since U/L is an elementary p-group, we know that λ has order p. Thus, if $B = \ker(\lambda)$, then $|A : B| = p$. Note that (B, U, L) is a normal triple satisfying the hypotheses of the lemma with $|B| + |U : L| < |A| + |U : L|$. By the choice of normal triple (A, U, L), we know that there exists a character $\gamma \in \mathrm{Irr}(U \mid \varphi)$ such that $[B, G] \subseteq \ker(\gamma)$. It follows from this fact that $f(\gamma) \in \mathrm{Irr}(A/B) = \langle \lambda \rangle$ and so $f(\gamma) = \lambda^n$ for some integer n such that $0 \leq n < p$. This implies that $f(\gamma) = f(\mu)^n = f(\mu^n)$ and so $f(\gamma\mu^{p-n}) = 1$. If we fix $\delta = \gamma\mu^{p-n}$, then since $\mu \in \mathrm{Irr}(U/L)$ we conclude that δ extends φ. By the definition of f, we observe that $[A, G] \subseteq \ker(\delta)$. This, however, is a contradiction to the choice of (A, U, L). Therefore, we are done in the case when $D = U$.

Finally, we may assume that $D = L$. Let P be a Sylow p-subgroup of G and Q be a p-complement for G. Since G is cyclic, we know that P and Q are normal in G. We see that the p-group P acts on the p-group U/L, and so there exist nontrivial fixed points. In particular, we see that $\mathbf{C}_{U/L}(P) > 1$. Since $\mathbf{C}_{U/L}(P)$ is G-invariant, we note that because U/L has no proper nontrivial G-invariant subgroups, this implies that $\mathbf{C}_{U/L}(P) = U/L$. Since $D = L$, it follows that $\mathbf{C}_{U/L}(Q) = 1$. Hence, (GU, U, L) is a controlled triple. Thus, by Corollary 2.2 there exists a unique G-invariant extension $\delta \in \mathrm{Irr}(U)$ of φ. If we can show that $[A, G] \subseteq \ker(\delta)$, then we would have a contradiction to the choice of (A, U, L).

Write $E/L = \mathbf{C}_{A/L}(Q)$, and note that E is G-invariant and that $U \cap E = L$. Since the action of Q on A/L is coprime, we see that $A/L = ([A, Q]L/L)(E/L)$, and so, $A = [A, Q]E$. Because $[A, Q] \subseteq [A, G] \subseteq U$, this implies that $A = UE$. We observe that $[E, G] \subseteq [A, G] \subseteq U$. Since we also have that $[E, G] \subseteq E$, it follows that $[E, G] \subseteq E \cap U = L$ and so $E \subseteq C$. By hypothesis, we have that G centralizes E. Since δ is linear and G-invariant, it is easy to see that $[U, G] \subseteq \ker(\delta)$. Therefore, we conclude that $[A, G] = [UE, G] = [U, G] \subseteq \ker(\delta)$ where the middle equality is true since G centralizes E. This completes the proof of the lemma. ∎

Since our definition of good elements is equivalent to the definition stated in [4] when K/L is abelian, we can present the following result, which appears as Theorem 3.2 of that paper. Recall that the basic configuration $(G, K, L, \epsilon, \varphi)$ is fully ramified if φ is fully ramified with respect to K/L.

THEOREM 4.2. *Let $(G, K, L, \epsilon, \varphi)$ be an abelian fully-ramified configuration. Fix an element $g \in G$. Then all the L-cosets in gK that are good with respect to the character triple (K, L, φ) are conjugate under K.* ∎

We would like to be able to prove this theorem in a more general situation, and so, we present the following result which proves the same conclusion when K/L is nilpotent. This theorem will be used in later papers in order to further generalize the results of [4].

THEOREM 4.3. *Let $(G, K, L, \epsilon, \varphi)$ be a nilpotent fully-ramified configuration. Fix an element $g \in G$. Then all the L-cosets in gK that are good with respect to the character triple (K, L, φ) are conjugate under K.*

PROOF. We assume that the theorem is not true. It is easy to see from the definition of good cosets that determining whether an L-coset is good with respect to the character triple (K, L, φ) can be done entirely in the character triple (G, L, φ). In particular, the good L-cosets with respect to the triple (K, L, φ) are invariant under character triple isomorphisms. If (G, L, φ) is a counter example, then every character triple isomorphic to

(G, L, φ) is one also, and thus, by Theorem 11.28 of [5], there exists one where φ is linear. We assume that G, K, L, and φ provide a counter example with φ linear and $|K|$ as small as possible. We begin by noting that $\langle K, g\rangle$ satisfies the hypotheses of the theorem and that determining whether an L-coset of gK is good with respect to the triple (K, L, φ) may be done in $\langle K, g\rangle$. Thus, it suffices to assume that $G = \langle K, g\rangle$.

Because G, K, L and φ form a counterexample, we see that Theorem 4.2 implies that K/L is not abelian. Also, since we have a counterexample with φ linear and $|K|$ as small as possible, it is easy to see that φ is faithful and thus, that L is central.

STEP 1. *Suppose that $K = UV$ where U and V are normal subgroups in G such that $L \subseteq U$, that $L \subseteq V$, and that $[U, V] = 1$. Then either $U = K$ or $V = K$.*

PROOF 1. Assume that there exist subgroups U and V as above with $U < K$ and $V < K$. Since φ is fully ramified with respect to K/L, we know that K acts transitively on the set $\mathrm{Irr}(U \mid \varphi)$. On the other hand, we observe that V fixes all the members of $\mathrm{Irr}(U \mid \varphi)$ and so every irreducible constituent of φ^U is invariant in K. The only way that all the irreducible constituents of φ^U can simultaneously be transitively permuted by K and invariant in K is if $\mathrm{Irr}(U \mid \varphi)$ has exactly one member. Since φ is invariant in U, this implies that φ is fully ramified with respect to U/L. By the symmetry of the hypotheses, we observe that φ is fully ramified with respect to V/L.

We have a contradiction if we can show that gL is conjugate to gkL in K for every choice of element $k \in K$ such that gkL is good with respect to the triple (K, L, φ). Fix some element $k \in K$ and assume that gkL is a good L-coset with respect to the triple (K, L, φ). We know that there exist elements $u \in U$ and $v \in V$ such that $k = vu$. It is obvious that gL is good with respect to the triple (V, L, φ). Let $C/L = \mathbf{C}_{V/L}(gk)$. Since U centralizes V, it follows that $C/L = \mathbf{C}_{V/L}(gv)$. Because gk is good in K with respect to the triple (K, L, φ) and $C/L \subseteq \mathbf{C}_{K/L}(gk)$, it follows that gk centralizes C by Lemma 3.5. Since $u \in U$ centralizes V, it follows that gv centralizes C. By Lemma 3.5, we observe that gvL is good with respect to the triple (V, L, φ), and we note that G, V, L, and φ satisfy the hypotheses of the theorem with $|V| < |K|$. Hence, all the L-cosets in gV that are good with respect to the triple (V, L, φ) are conjugate by elements of V, and thus, gL and gvL are conjugate by an element of V.

Since gvL is conjugate to an L-coset that is good with respect to the triple (U, L, φ), it follows that gvL is itself good with respect to the triple (U, L, φ). Notice that G, U, L, and φ satisfy the hypotheses of the theorem with $|U| < |K|$, and so, all the L-cosets in gvU that are good with respect to the triple (U, L, φ) are conjugate by elements of U. Since gkL is good with respect to the triple (K, L, φ), we note that $gkL = gvuL$ is good with respect to the triple (U, L, φ), and we now conclude that gvL and $gvuL$ are conjugate by an element of U. Therefore, gL and gkL are conjugate by an element of K which is a contradiction. ■

STEP 2. *K/L is a p-group for some prime p.*

PROOF 2. Since L is central and K/L is nilpotent, it follows that K is nilpotent. Let p be a prime divisor of $|K : L|$, and write $P \in \mathrm{Syl}_p(K)$. We pick $Q \in \mathrm{Hall}_{p'}(K)$ and note that $QL < K$. We know that $K = PQ$ and that PL and QL are both normal in G. Since K is nilpotent, it follows that $[P,Q] = 1$, and because L is central, this implies that $[PL,QL] = 1$. Thus, we now have that PL and QL are subgroups as in Step 1. Since $QL < K$, we can conclude that $K = PL$, and the result follows. ∎

Let U/L be a G chief factor such that $U/L \subseteq \mathbf{Z}(K/L)$, and since K/L is not abelian, we know that $U < K$. Let $V = \mathbf{C}_K(U)$, and because K and U are both normal in G, we note that V is normal in G. Since L is central, it follows that $L \subseteq V$ and by Lemma 3.5 since φ is linear and faithful, that $V = U^\perp$.

STEP 3. *φ extends to U.*

PROOF 3. By the Going-Up theorem (Problem 6.12 of [5]), φ being invariant in G implies either that φ extends to U or that φ is fully ramified with respect to U/L, and so, we assume that φ is fully ramified with respect to U/L. From Section 3, we know that $\langle\langle\cdot,\cdot\rangle\rangle_\varphi$ is nondegenerate on U/L and so $V \cap U = L$. From Corollary 3.3, we see that $|K:V| = |U:L| = |VU:V|$ and so $K = VU$. Since V centralizes U, it follows that U and V are subgroups as in Step 1. Because $U < K$ and $V < K$, we get a contradiction. Therefore, we see that φ extends to U. ∎

We observed after Step 2 that $V = U^\perp$. Since U/L is central in K/L and φ extends to V, we may use Lemma 3.4 to see that V is the stabilizer of δ in K.

STEP 4. *If xL and yL are L-cosets in gK that are good with respect to the triple (K,L,φ), then xU and yU are conjugate by an element of K.*

PROOF 4. Let $A/U = \mathbf{C}_{V/U}(x)$. We notice that (A,U,L) is a normal triple, that L is central in A, that U/L is central in A/L, that A/L is a p-group (by Step 2), that U/L is an elementary abelian p-group, that $[A,x] \subseteq U$, and that $[L,x] = 1$. Furthermore, since g is good with respect to the triple (K,L,φ) and since φ is linear and faithful, we may use Lemma 3.5 to observe that if $C/L = \mathbf{C}_{A/L}(x)$, then x centralizes C. Finally, since φ has a V-invariant extension to U (by Step 3 and Lemma 3.4), we see that φ has an A-invariant extension to U. Therefore, we may apply Lemma 4.1, to see that there exists a x-invariant extension $\delta \in \mathrm{Irr}(U)$ of φ such that $[A,x] \subseteq \ker(\delta)$.

Since δ is linear, this implies that xL is good with respect to the triple (V,U,δ). We write T to be stabilizer in G of δ, and by Lemma 3.4, we see that $T \cap K = V$. By Clifford's theorem (6.11 of [5]), we know that $|\mathrm{Irr}(V \mid \delta)| = |\mathrm{Irr}(K \mid \delta)|$. Since φ is fully ramified with respect to K/L, we recall that $|\mathrm{Irr}(K \mid \varphi)| = 1$. Therefore, it follows that $|\mathrm{Irr}(V \mid \delta)| = 1$, and since δ is invariant in V, we see that δ is fully ramified with respect to V/U. Similarly, for yL there exists an extension δ' of φ to U such that δ' is y-invariant and that yL is good with respect to the triple (V,U,δ').

We know, since φ is fully ramified with respect to K/L, that there exists an element $k \in K$ such that $(\delta')^k = \delta$ and thus the stabilizer of δ' in G is conjugate to T via k. Since $\varphi^k = \varphi$, we have that conjugation by k takes L-cosets that are good with respect to the triple (V,U,δ') to L-cosets that are good with respect to the triple (V,U,δ). Since yL is good with respect to the triple (V,U,δ'), we have that $(yL)^k \in T$ is good with respect to the triple (V, U, δ). Thus, we may assume, up to conjugacy, that $y \in T$. Since $x \in T$, we then have that $xy^{-1} \in T \cap K = V$ and so x and y lie in the same V-coset of T. Note that T, V, U, and δ satisfy the hypotheses of the theorem with δ linear and $|V| < |K|$. Therefore, since x and y lie in the same V-coset of T, we see that xU and yU are conjugate by an element of K. ∎

Step 4 implies that we may assume that all the L-cosets in gK that are with respect to the triple (K, L, φ) lie in gU. This is true since any L-coset in gK that is good with respect to the triple (K, L, φ) must be conjugate to some L-coset in gU. Thus, we assume that it is in gU. It follows that gL is not the only L-coset in gU that is good with respect to the triple (K, L, φ). Also, we write $C/L = \mathbf{C}_{K/L}(g)$.

STEP 5. *$|U : L| = p$ and $U \subseteq C$.*

PROOF 5. Observe that g normalizes $C \cap U$ and since U/L is central in K/L, we see that K normalizes $C \cap U$. Because $G = \langle K, g \rangle$, it follows that $C \cap U$ is normal in G. Since U/L is a G-chief factor, we know that either $C \cap U = U$ or $C \cap U = L$. It is easy to see that $|U : U \cap C|$ is equal to the number of L-cosets in gU that are conjugate to gL by elements of U. If $C \cap U = L$, then we have that there are $|U : L|$ conjugates of gL in gU. We also note that the number of L-cosets in gU is $|U : L|$ and so gL is conjugate by an element of U to every L-coset of gU. By Step 4, this implies that gL is conjugate to every L-coset in gK that is good with respect to the triple (K, L, φ), and so, this is a contradiction to the choice of G, K, L, and φ. Thus, we see that $C \cap U = U$ or equivalently that $U \subseteq C$. This implies that g centralizes U/L. Since $U/L \subseteq \mathbf{Z}(K/L)$, we remark that $U/L \subseteq \mathbf{Z}(G/L)$. Because U/L is a chief factor for G, we conclude that $|U : L| = p$. ∎

STEP 6. *$C \subseteq V$.*

PROOF 6. By Step 5, we know that $|U : L| = p$, and by Step 4, there exists an element u lying in U but not in L such that gu is good with respect to the triple (K, L, φ). Thus, we observe that $U = \langle L, u \rangle$. Recall that $V = \mathbf{C}_K(U)$, and since L is central, it follows that $V = \mathbf{C}_K(u)$. Because $uL \in \mathbf{Z}(K/L)$, it follows that $C/L = \mathbf{C}_{K/L}(gu)$, and since φ is linear and faithful and g and gu are good with respect to the triple (K, L, φ), we may use Lemma 3.5 to see that g and gu centralize C. This implies that C centralizes u. The result is now clear. ∎

STEP 7. *$C/U = \mathbf{C}_{K/U}(g)$.*

PROOF 7. Let $B/U = \mathbf{C}_{K/U}(g)$, and note that $C \subseteq B$. By Step 2 we know that $|B : C|$ is a p-power. Observe that $|B : C|$ is the number of conjugates of gL in gU. By Step 5, we have that $|U : L| = p$, and so, it follows that gL is either conjugate by elements of K only to itself in gU or to all the L-cosets in gU. Thus, if $B > C$, then we know that gL is conjugate to all the L-cosets of gU. As a consequence of Step 4, we see that this contradicts the choice of G, K, L, and φ, and so, we conclude that $B = C$. ∎

https://doi.org/10.4153/CJM-1996-052-8 Published online by Cambridge University Press

STEP 8. *Let $\delta \in \mathrm{Irr}(U)$ be any extension of φ. Then g is good with respect to the triple (V, U, δ).*

PROOF 8. Since g is good with respect to the triple (K, L, φ), we have, by Lemma 3.5, that $[C, g] \subseteq \ker(\varphi) = 1$. From Step 7, we have that $C/U = \mathbf{C}_{K/U}(g)$ On the other hand, by Step 6, we see that $C/U = (C \cap V)/U = \mathbf{C}_{V/U}(g)$. This implies that $[C, g] = 1 \subseteq \ker(\delta)$, and so by Lemma 3.4, we may conclude that g is good with respect to the triple (V, U, δ). ∎

STEP 9. *Final contradiction.*

PROOF 9. Let $\delta \in \mathrm{Irr}(U)$ be any extension of φ. (By Step 3, we know that φ has an extension to U.) From Lemma 3.4, we know that V is the stabilizer in K of δ. Since V centralizes in U, it follows that all the members of $\mathrm{Irr}(U \mid \varphi)$ are invariant in V. By Step 4, we know that $U \subseteq C$. Because g is good with respect to the triple (K, L, φ) and φ is linear and faithful, we may apply Lemma 3.5 to see that g centralizes U. Because g acts trivially by conjugation on U, it follows that $[K, g]$ centralizes U, and so, we have that $[K, g] \subseteq V$. Thus, we have that $(gV)^k = gV$ for all $k \in K$.

Since K fixes gV, we observe that K permutes the U-cosets in gV. Let $T = \mathrm{I}_G(\delta)$, and note that T^k, V, U, and δ^k satisfy the hypotheses of the theorem with δ^k linear and $|V| < |K|$ for all the elements of $k \in K$. Thus, we have that they satisfy the conclusion of the theorem that all the U-cosets in gV that are good with respect to the triple (V, U, δ^k) are conjugate by elements of V for every element $k \in K$.

Observe that conjugation by an element $k \in K$ takes a U-coset that is good with respect to the triple (V, U, δ) to a U-coset that is good with respect to the triple (V, U, δ^k). Since gU is good with respect to the triple (V, U, δ), we have that $g^k U$ is good with respect to the triple (V, U, δ^k). On the other hand, by Step 8, we also know that gU is good with respect to the triple (V, U, δ^k). Hence, there exists $v \in V$ such that $g^v U = g^k U$. Therefore, we know that gU is V-conjugate to every U-coset that is K-conjugate to gU. Since C is the stabilizer of gU in K, we may now use a Frattini Argument to see that $K = VC$. By Step 6, we have that $C \subseteq V$, and so $K = V$. This, however, contradicts Corollary 3.3 which says that $|K : V| = |U : L| > 1$. ∎

The abelian analog of the following corollary is not explicitly in [4] but is clearly an immediate consequence of the results in Section 3 there.

COROLLARY 4.4. *Let $(G, K, L, \epsilon, \varphi)$ be a nilpotent fully-ramified configuration with stabilizing complement H, and let $\chi \in \mathrm{Irr}(G \mid \varphi)$. Assume that all the elements of H are good with respect to the triple (K, L, φ). Then we have that $\chi(g) = 0$ if the element $g \in G$ is not conjugate to an element of H.*

PROOF. Let $g \in G$, and assume that $\chi(g) \neq 0$. We must show that g is conjugate to some element of H. Since $\chi(g) \neq 0$, we know by Lemma 3.1 that g is good with respect to the triple (K, L, φ) and by Theorem 4.3 that all the L-cosets of gK that are good with respect to the triple (K, L, φ) are conjugate by elements of K. We know that $gK \cap H$ is not the empty set, and thus, there exists $h \in gK \cap H$ such that $hL = gK \cap H$. Since $h \in H$, we know that hL is good with respect to the triple (K, L, φ), and so, hL is conjugate to gL by an element of K. Therefore, we may conclude that g is conjugate to an element of H, as desired. ∎

We close this section with a question that the author has not been able to resolve. Throughout this paper the standard hypothesis is that K/L is nilpotent. While we were able to use this hypothesis to prove Theorem A, it would be more straightforward if we instead had assumed only that K/L is solvable. Thus, the question is: given a G-invariant character $\varphi \in \mathrm{Irr}(L)$ which is fully ramified with respect to K/L, are all of the good L-cosets with respect to the triple (K, L, φ) conjugate by an element of K when K/L is solvable (*i.e.*, not nilpotent).

5. **Character correspondences.** Before we can prove Theorem A, we need to present two character correspondences. The first of these is Lemma 10.5 of [4].

EXTENSION CORRESPONDENCE (EC). Let $(G, K, L, \epsilon, \varphi)$ be a basic configuration, and assume that $\epsilon_L = \varphi$. If H is a stabilizing complement, then restriction defines a bijection $\mathrm{Irr}(G \mid \epsilon) \longrightarrow \mathrm{Irr}(H \mid \varphi)$. ∎

When we use the Extension Correspondence, we will normally mean the inverse of the bijection found in EC.

LEMMA 5.1. *Let $(G, U, L, \delta, \varphi)$ be an abelian configuration with stabilizing complement H, and assume that $\delta_L = \varphi$. Then there exists a bijection $\hat{}\colon \mathrm{Irr}(G \mid \varphi) \longrightarrow \mathrm{Irr}(G \mid \varphi)$ satisfying for every character $\alpha \in \mathrm{Irr}(G \mid \varphi)$:*

(a) $\alpha_H = \hat{\alpha}_H$,
(b) $\hat{\hat{\alpha}} = \alpha$, and
(c) if $|G : L|$ is odd, then $\alpha = \hat{\alpha}$ if and only if $\alpha \in \mathrm{Irr}(G \mid \delta)$.

PROOF. By Gallagher's theorem (6.17 of [5]) we know that $\mathrm{Irr}(U \mid \varphi) = \{\delta\lambda \mid \lambda \in \mathrm{Irr}(U/L)\}$. Since δ is G-invariant, it is easy to see for every character $\lambda \in \mathrm{Irr}(U/L)$ that the stabilizer of $\delta\lambda$ in G is the stabilizer of λ in G. We will use the notation $\bar{(\,)}$ to denote the complex conjugate. For every element $h \in H$ and every character $\lambda \in \mathrm{Irr}(U/L)$, we know that $(\bar{\lambda})^h = \overline{\lambda^h}$, and thus, it follows that the stabilizer of λ in G equals the stabilizer of $\bar{\lambda}$ in G. We conclude for every such λ that the stabilizer of $\delta\lambda$ in G equals the stabilizer of $\delta\bar{\lambda}$, and since U/L is abelian, it follows that $\delta\lambda$ and $\delta\bar{\lambda}$ are both extensions of φ.

Given a character $\alpha \in \mathrm{Irr}(G \mid \varphi)$, we need to produce a character $\hat{\alpha} \in \mathrm{Irr}(G \mid \varphi)$. We will construct the $\hat{}$-bijection in several steps.

Fix a character $\lambda \in \mathrm{Irr}(U/L)$, and let T be the stabilizer of $\delta\lambda$ in G. Since φ, $\delta\lambda$, and $\delta\bar{\lambda}$ are all T-invariant, we may use EC to see that restriction defines bijections between $\mathrm{Irr}(T \mid \delta\lambda)$ and $\mathrm{Irr}(T \cap H \mid \varphi)$ and between $\mathrm{Irr}(T \mid \delta\bar{\lambda})$ and $\mathrm{Irr}(T \cap H \mid \varphi)$. We define the map $g_\lambda\colon \mathrm{Irr}(T \mid \delta\lambda) \longrightarrow \mathrm{Irr}(T \mid \delta\bar{\lambda})$ in the following manner. For any character $\beta \in \mathrm{Irr}(T \mid \delta\lambda)$, write $\gamma = \beta_{T\cap H}$, and using EC, we observe that $\gamma \in \mathrm{Irr}(T \cap H \mid \varphi)$. Thus, by EC again, there exists a unique character extending γ lying in $\mathrm{Irr}(T \mid \delta\bar{\lambda})$, and we will define $g_\lambda(\beta)$ to be this character so that $g_\lambda(\beta)_{T\cap H} = \gamma$.

We construct the map $f_\lambda\colon \mathrm{Irr}(G \mid \delta\lambda) \longrightarrow \mathrm{Irr}(G \mid \delta\bar{\lambda})$ as follows. First, consider a character $\alpha \in \mathrm{Irr}(G \mid \delta\lambda)$. By Clifford's theorem (6.11 of [5]), we know that there exists a unique character such that $\beta^G = \alpha$. We now define the character $f_\lambda(\alpha) = \big(g_\lambda(\beta)\big)^G$, and using Clifford's theorem again, we conclude that $f_\lambda(\alpha) \in \mathrm{Irr}(G \mid \delta\bar{\lambda})$. Observe that f_λ is a well-defined map.

We want to define $\hat{\alpha}$ as $f_\lambda(\alpha)$, where the character $\lambda \in \mathrm{Irr}(U/L)$ is chosen so that $\delta\lambda$ is a constituent of α_U. For this to be well-defined, we must show that $f_\lambda(\alpha)$ is independent of the choice of λ. If we have $\mu \in \mathrm{Irr}(U/L)$ so that $\delta\mu$ is also a constituent of α_U, then $\delta\lambda$ and $\delta\mu$ lie in the same H-orbit. We know that any character in the same H-orbit as $\delta\lambda$ is of the form $(\delta\lambda)^h = \delta\lambda^h$ for some element $h \in H$, and so, by Gallagher's theorem (6.17 of [5]), we have that $\mu = \lambda^h$. Hence, we must show that $f_\lambda(\alpha) = f_{\lambda^h}(\alpha)$ for every element $h \in H$.

As before, we write T for the stabilizer of $\delta\lambda$ in G and β for the Clifford correspondent (see Theorem 6.11 of [5]) for α with respect to $\delta\lambda$. Fix an element of $h \in H$, and observe that T^h is the stabilizer of $(\delta\lambda)^h$ in G and that β^h is the Clifford correspondent for α with respect to $(\delta\lambda)^h$. We know that conjugation by h is an isomorphism from T to T^h carrying $T \cap H$ to $T^h \cap H$, and so, it follows that $g_{\lambda^h}(\beta^h) = g_\lambda(\beta)^h$. Thus, we have the following equalities:

$$f_{\lambda^h}(\alpha) = \big(g_{\lambda^h}(\beta^h)\big)^G = \big(g_\lambda(\beta)^h\big)^G = \big(g_\lambda(\beta)\big)^G = f_\lambda(\alpha).$$

It is easy to see that $f_{\bar{\lambda}}\big(f_\lambda(\alpha)\big) = \alpha$, and so, we see that $\hat{\hat{\alpha}} = \alpha$. Hence, the map $\alpha \longmapsto \hat{\alpha}$ is a bijection from $\mathrm{Irr}(G \mid \varphi)$ to $\mathrm{Irr}(G \mid \varphi)$.

By Problem 5.2 of [5], we have that

$$\alpha_H = (\beta^G)_H = (\beta_{T\cap H})^H = \gamma^H = \big(g_\lambda(\beta)_{T\cap H}\big)^H = \big(g_\lambda(\beta)^G\big)_H = \big(f_\lambda(\alpha)\big)_H = \hat{\alpha}_H,$$

where λ, β, and γ are the same characters as before.

We now assume that $|G : L|$ is odd, and prove that (c) holds. If $\alpha \in \mathrm{Irr}(H \mid \delta)$, then we know that the character λ in the above construction equals 1_U. If we let β be the Clifford correspondent (see 6.11 of [5]) for α with respect to φ, then we have that $g_{1_U}(\beta) = \beta$, and so,

$$\hat{\alpha} = f_{1_U}(\beta) = \big(g_{1_U}(\beta)\big)^G = \beta^G = \alpha.$$

On the other hand, when $\alpha \notin \mathrm{Irr}(H \mid \delta)$, it follows that the character λ in the above construction does not equal 1_U, and since $|U : L|$ is odd, we know that $\lambda \neq \bar{\lambda}$. Furthermore, we will prove that λ and $\bar{\lambda}$ lie in different orbits under the action of H. Suppose that λ and $\bar{\lambda}$ lie in the same orbit under the action of H and hence, that there is some element $h \in H$ such that $\lambda^h = \bar{\lambda}$. It is easy to see that $(\bar{\lambda})^h = \bar{\bar{\lambda}} = \lambda$, and because L fixes the characters of $\mathrm{Irr}(U/L)$, this implies that 2 divides $|H : L|$. Since this contradicts the assumption that $|H : L|$ is odd, we observe that λ and $\bar{\lambda}$ lie in different orbits of H, and because δ is also H-invariant, it follows that $\delta\lambda$ and $\delta\bar{\lambda}$ lie in different orbits under the action of H. Since α lies in $\mathrm{Irr}(G \mid \delta\lambda)$ and $\hat{\alpha}$ lies in $\mathrm{Irr}(G \mid \delta\bar{\lambda})$, we conclude that $\alpha \neq \hat{\alpha}$. ∎

6. **Theorem A.** We are now ready to prove Theorem A of the Introduction. The following result is the induction step and contains most of the work for proving Theorem A.

THEOREM 6.1. *Let $(G, K, L, \epsilon, \varphi)$ be a coprime fully-ramified configuration such that $|G : L|$ is odd, and let H be a stabilizing complement. Assume that U is a normal H-invariant subgroup of G that lies in K and contains L so that U/L is an abelian group. Fix the character $\theta \in \mathrm{Irr}(H \mid \varphi)$.*

(i) Assume that φ is fully-ramified with respect to U/L, and let δ be the unique irreducible constituent of φ^U. Then there exists a character γ that is the unique irreducible constituent of θ^{HU} having odd multiplicity. Furthermore, assume that χ is the unique irreducible constituent of γ^G having odd multiplicity and that γ is the unique irreducible constituent of χ_{HU} having odd multiplicity.

(ii) Assume that $\delta \in \mathrm{Irr}(U)$ is an H-invariant extension of φ. Write V for the stabilizer of δ in K and σ for the Clifford correspondent for ϵ with respect to φ. Then there is a character $\gamma \in \mathrm{Irr}(HU \mid \delta)$ such that $\gamma_H = \theta$. Assume that η is the unique irreducible constituent of γ^{HV} having odd multiplicity and that γ is the unique irreducible constituent of η_{HU} having odd multiplicity. Then $\chi = \eta^G$ is irreducible.

Then χ is the unique irreducible constituent of θ^G such that $[\chi_H, \theta]$ is odd, and furthermore, θ is the unique irreducible constituent of χ_H such that $[\chi_H, \theta]$ is odd.

PROOF. Assume first that we are in case (i). By Isaacs' theorem in the Introduction, there is a unique character $\gamma \in \mathrm{Irr}(HU \mid \delta)$ having odd multiplicity as constituent of θ^{HU}. Thus, we can write $\gamma^G = \chi + 2\Gamma$ and $\theta^{HU} = \gamma + 2\Theta$ for some possibly reducible or zero characters $\Gamma \in \mathrm{Char}(G)$ and $\Theta \in \mathrm{Char}(HU)$. Putting these together, we conclude that $\theta^G = \gamma^G + 2\Gamma^G = \chi + 2(\Theta + \Gamma^G)$, and therefore, χ is the unique irreducible constituent of θ^G having odd multiplicity. In a similar manner, we can write $\gamma_H = \theta + 2\Delta$ and $\chi_{HU} = \gamma + 2\Lambda$ for some possibly reducible or zero characters $\Delta \in \mathrm{Char}(H)$ and $\Lambda \in \mathrm{Char}(HU)$. Combining these, we have $\chi_H = (\gamma + 2\Lambda)_H = \theta + 2(\Delta + \Lambda_H)$. Therefore, θ is the unique irreducible constituent of χ_H having odd multiplicity.

Suppose now that we are in case (ii). Write T for the stabilizer of δ in G, and so, $V = T \cap K$ and $T = HV$. By EC, there exists a character $\gamma \in \mathrm{Irr}(HU \mid \delta)$ such that $\gamma_H = \theta$, and we can write $\gamma^T = \eta + 2\Gamma$ for some possibly reducible or zero character $\Gamma \in \mathrm{Char}(T)$. Similarly, we may write $\eta_{HU} = \gamma + 2\Xi$ for some possibly reducible or zero character $\Xi \in \mathrm{Char}(T)$. By Clifford's theorem (6.11 of [5]), we know that $\chi = \eta^G$ and that $\chi_T = \eta + \Delta$, where the (possibly reducible or zero) character $\Delta \in \mathrm{Char}(T)$ consists of all the irreducible constituents of χ_T that are not constituents of δ^T.

Assume that U/L is a p-group for some prime p. By Lemma 5.1, we have a bijection $\hat{}: \mathrm{Irr}(HU \mid \varphi) \longrightarrow \mathrm{Irr}(HU \mid \varphi)$ such that every character $\alpha \in \mathrm{Irr}(HU \mid \varphi)$ satisfies $\alpha_H = \hat{\alpha}_H$. Thus, we have that $[\theta, \alpha_H] = [\theta, \hat{\alpha}_H]$ for every character $\alpha \in \mathrm{Irr}(HU \mid \varphi)$, and so, we see that α and $\hat{\alpha}$ have equal multiplicities in θ^{HU}. By EC, we know that γ is the unique irreducible constituent of θ^{HU} lying in $\mathrm{Irr}(HU \mid \delta)$. The remaining constituents of θ^{HU} come in pairs α and $\hat{\alpha}$ for characters $\alpha \in \mathrm{Irr}(HU \mid \delta\lambda)$ where the character $\lambda \in \mathrm{Irr}(U/L)$ satisfies $\lambda \neq 1_U$. Thus, we pick the characters $\alpha_i \in \mathrm{Irr}(HU \mid \varphi)$, such that we have one character from each pair, and since $|G : L|$ is odd, we apply Lemma 5.1(c) to see that $\alpha_i \neq \hat{\alpha}_i$ for each index i. Therefore, we may write $\theta^{HU} = \gamma + \sum_i a_i(\alpha_i + \hat{\alpha}_i)$ where the coefficients are nonnegative integers.

Since $(|H : L|, |K : L|) = 1$, we have by Glauberman's lemma (13.8 of [5]) that H normalizes some Sylow p-subgroup $P/L \in \mathrm{Syl}_p(K/L)$. Since U is a normal subgroup of K, we see that $U \subseteq P$, and by Theorem 2 of [1], it follows that φ is fully ramified with respect to P/L. Let $\kappa \in \mathrm{Irr}(P)$ be the unique irreducible constituent of φ^P, and note that HP is a group. Since $(|H : L|, |P : L|) = 1$, we know by Lemma 3.6 that all the elements of H are good with respect to the triple (P, L, φ). Because $(HP, P, L, \kappa, \varphi)$ is a nilpotent fully-ramified configuration, we may use Corollary 4.4 to observe that if $\beta \in \mathrm{Irr}(HP \mid \varphi)$, then $\beta(g) = 0$ if g is not conjugate to an element of H. Consider some character $\alpha \in \mathrm{Irr}(HU \mid \varphi)$. Since α^{HP} and $\hat{\alpha}^{HP}$ are sums of elements of $\mathrm{Irr}(HP \mid \varphi)$, we have that α^{HP} and $\hat{\alpha}^{HP}$ vanish on elements of HP that are not conjugate to elements of H. Thus, if $(\alpha^{HP})_H = (\hat{\alpha}^{HP})_H$, then it follows that $\alpha^{HP} = \hat{\alpha}^{HP}$. We are ready to prove the following equalities for each element $h \in H$:

$$\alpha^{HP}(h) = \frac{1}{|HU|} \sum_y \alpha(h^y) = \frac{1}{|HU|} \sum_y \hat{\alpha}(h^y) = \hat{\alpha}^{HP}(h),$$

where each of the sums are over those elements $y \in HP$ such that $h^y \in HU$. Except for the middle equality, all of these are clear by the definition of induced characters, and so, we work to prove the middle equality. We consider an element $h^y \in HU$. We know that $o(h^yL) = o(hL)$ and that $H/L \in \mathrm{Hall}_{p'}(HU/L)$. We see that $h^yL \in (H/L)^u$ for some element $u \in U$, and so, $h^y \in H^u$. Thus, there exists $h_1 \in H$ such that $h^y = h_1^u$. Since $\alpha, \hat{\alpha} \in \mathrm{Irr}(HU)$ and since $h^y = h_1^u$, we observe that $\alpha(h^y) = \alpha(h_1^u)$ and $\hat{\alpha}(h^y) = \hat{\alpha}(h_1^u)$. Because α and $\hat{\alpha}$ are class functions, we have that $\alpha(h_1^u) = \alpha(h_1)$ and $\hat{\alpha}(h_1^u) = \hat{\alpha}(h_1)$. Since $\alpha_H = \hat{\alpha}_H$, we have that $\alpha(h_1) = \hat{\alpha}(h_1)$. Thus, we observe that $\alpha(h^y) = \hat{\alpha}(h^y)$ for all elements $h \in H$ and $y \in HP$ such that $h^y \in HU$. We may now conclude that $\alpha^{HP} = \hat{\alpha}^{HP}$, and this implies that $\alpha^G = \hat{\alpha}^G$. In particular, we know that χ has the same multiplicity in α^G as in $\hat{\alpha}^G$.

We now have that

$$\theta^G = (\gamma^T)^G + \sum_i a_i(\alpha_i + \hat{\alpha_1})^G = (\eta + 2\Gamma)^G + 2\sum_i a_i(\alpha_i)^G = \chi + 2\Big(\Gamma^G + \sum_i a_i(\alpha_i)^G\Big).$$

Therefore, χ is the unique irreducible constituent of θ^G such that $[\chi_H, \theta]$ is odd.

We now show that θ is the unique irreducible constituent of χ_H having odd multiplicity. Since δ is invariant in HV, it follows that the irreducible constituents of Δ_{HU} are not constituents of δ^{HU}. Consider an irreducible character β of HU that is not a constituent of δ^{HU}. Using Frobenius Reciprocity (Lemma 5.2 of [5]) and the fact that η is the unique irreducible constituent of χ_T that is a constituent of δ^T, we have that

$$[\Delta_{HU}, \beta] = [\Delta, \beta^T] = [\chi_T, \beta^T] = [\chi, \beta^G] = [\chi, \hat{\beta}^G] = [\chi_T, \hat{\beta}^T] = [\Delta, \hat{\beta}^T] = [\Delta_{HU}, \hat{\beta}].$$

Thus, the constituents of Δ_{HU} come in pairs β and $\hat{\beta}$ for characters $\beta \in \mathrm{Irr}(HU \mid \delta\lambda)$ where the character $\lambda \in \mathrm{Irr}(U/L)$ satisfies $\lambda \neq 1_U$. Thus, we pick the characters $\beta_i \in$

$\mathrm{Irr}(HU \mid \varphi)$, such that we have one character from each pair, and since $|G : L|$ is odd, we apply Lemma 5.1(c) to see that $\beta_i \neq \hat{\beta}_i$ for each index i. Therefore, we may write $\Delta_{HU} = \sum_i b_i(\beta_i + \hat{\beta}_i)$ where the coefficients are nonnegative integers. We have

$$\chi_H = (\eta + \Delta)_H = \Big(\gamma + 2\Xi + \sum_i b_i(\beta_i + \hat{\beta}_i)\Big)_H = \theta + 2\Big(\Xi_H + \sum_i b_i(\beta_i)_H\Big).$$

We conclude that θ is the unique irreducible constituent of χ_H having odd multiplicity.

Assume that we are in the general case of (ii), and let p be a prime divisor of $|U : L|$. If U/L is a p-group, then we are done by the previous paragraphs, and so, we assume that U/L is not a p-group. Let P/L be the Sylow p-subgroup of U/L, and observe that P is H-invariant. We write $\mu = \delta_P$, and it is easy to see that μ is irreducible and H-invariant. We write Q for the stabilizer of μ in K, and $\nu \in \mathrm{Irr}(Q \mid \mu)$ for the Clifford correspondent of ϵ with respect to μ. By Lemma 3.4, we know that μ is fully ramified with respect to Q/P, and so, (HQ, Q, P, ν, μ) is a coprime fully-ramified configuration with stabilizing complement HP. Observe that this configuration satisfies the same hypotheses as the original configuration with $|Q : P| < |K : L|$. We write $\omega = \gamma_{HP}$ and $\psi = \eta^{HQ}$. Thus, we may apply induction on $|K : L|$ to see that ψ is the unique irreducible constituent of ω^{HQ} having odd multiplicity and that ω is the unique irreducible constituent of ψ_{HP} having odd multiplicity. Observe that we may now apply the p-group case to get the result. ■

The following corollary is Theorem A of the Introduction.

COROLLARY 6.2. *Let $(G, K, L, \epsilon, \varphi)$ be a coprime fully-ramified configuration such that $|G : L|$ is odd. Then there exists a stabilizing complement H so that for each character $\theta \in \mathrm{Irr}(H \mid \varphi)$ there is a unique irreducible constituent χ of θ^G having odd multiplicity and θ is the unique irreducible constituent of χ_H having odd multiplicity.*

PROOF. Since (G, K, L) is a coprime triple, we know by the Schur-Zassenhaus theorem that there exists a complement H, and since φ is G-invariant it follows that H is a stabilizing complement.

We work by induction on $|K : L|$. If $L = K$, then it is clear that we are done, and so, we may assume that $L < K$. Thus, there is a subgroup U of K such that U/L is a chief factor for G. Since K/L is solvable, it follows that U/L is an abelian p-group for some prime p. Since φ is G-invariant, we know by the Going-Up theorem (Problem 6.12 of [5]) either that φ extends to U or that φ is fully ramified with respect to U/L. If φ extends to U, then by Lemma 2.1 we know that there exists an H-invariant extension δ in U, and let γ correspond to θ via EC. If φ is fully ramified with respect to U/L, then we let $\delta \in \mathrm{Irr}(U)$ be the unique constituent of φ^U, and we write γ for the unique irreducible constituent of θ^{HU} having odd multiplicity.

Let V be the stabilizer for δ in K, and let σ be the Clifford correspondent for ϵ with respect to δ. Observe that if φ is fully-ramified with respect to U/L, then $V = K$ and $\sigma = \epsilon$. Since φ is fully-ramified with respect to K/L, we know by Clifford's theorem (6.11 of [5]) that σ is the unique irreducible constituent of $(\delta)^V$. Because δ is invariant in V, it follows that δ is fully ramified with respect to V/U. In any case, we know that

$(HV, V, U, \sigma, \delta)$ is a coprime fully- ramified configuration with $|HV : U|$ odd. We may then apply the inductive hypothesis to see that γ^{HV} has a unique irreducible constituent of odd multiplicity η and that γ is the unique irreducible constituent of η_{HU} having odd multiplicity. Therefore, we may apply Theorem 6.1 to conclude that θ^G has a unique irreducible constituent χ with odd multiplicity and that θ is the unique irreducible constituent of χ_H with odd multiplicity. ∎

PROOF OF THEOREM B. Since φ is S-invariant and γ and ϵ are the unique irreducible constituents of φ^H and φ^G, it is obvious that γ and ϵ are S-invariant.

We note that $(SG, G, L, \epsilon, \varphi)$ and $(SH, H, L, \gamma, \varphi)$ are fully-ramified coprime configurations. By the Schur-Zassenhaus theorem, we know that all complements for (SG, G, L) are conjugate. Similarly, we know that all complements for (SH, H, L) are conjugate. Thus, we may use Corollary 6.2 to see that $(\varphi^*)^{SH}$ and $(\varphi^*)^{SG}$ have unique irreducible constituents $\psi \in \mathrm{Irr}(SH)$ and $\chi \in \mathrm{Irr}(SG)$ having odd multiplicity. In fact, there are reducible or zero characters Ψ and Φ so that $(\varphi^*)^{SH} = \psi + 2\Psi$ and $(\varphi^*)^{SG} = \chi + 2\Phi$. This implies that $\psi^{SG} + 2\Psi^{SG} = \chi + 2\Phi$. Therefore, we observe that χ is the unique constituent of ψ^{SG} having odd multiplicity. To prove the result, it now suffices to show that $\psi = \gamma^*$ and $\chi = \epsilon^*$.

Define E to be the extension of the rational numbers containing all the roots of unity dividing $|GS|$. Write F for the subfield of E containing roots of unity that have order dividing $|G|$, and observe that the Galois group for E over the rational numbers is the direct sum of the Galois group for E over F with the Galois group of F over the rational numbers. It follows that there is a Galois automorphism τ of E over the rational numbers that fixes the roots of unity having order that divide $|G|$ and acts like complex conjugation on roots of unity that divide $|S|$. It easy to see that φ^*, γ^*, and ϵ^* are the unique irreducible constituents of φ^{LS}, γ^{HS}, and ϵ^{GS} fixed by τ (see the discussion following Lemma 13.3 of [5]). Combining the observation that ψ and ψ^τ have the same multiplicity in $(\varphi^*)^{HS}$ and χ and χ^τ have the same multiplicity in $(\varphi^*)^{GS}$ with the fact that ψ and χ are the unique irreducible constituents of $(\varphi^*)^{HS}$ and $(\varphi^*)^{GS}$ having odd multiplicity, it follows that $\psi = \psi^\tau$ and $\chi = \chi^\tau$. Hence, we conclude that $\psi = \gamma^*$ and $\chi = \epsilon^*$. ∎

REFERENCES

1. F. DeMeyer and G. Janusz, *Finite Groups with an Irreducible Representation of Large Degree*, Math Z. **108**(1969), 145–153.
2. G. Glauberman, *Correspondences of Characters for Relatively Prime Operator Groups*, Canad. J. Math. **20**(1968), 1465–1488.
3. B. Huppert, *Endlichen Gruppen I*, Springer-Verlag, Berlin, 1967.
4. I. M. Isaacs, *Characters of Solvable and Symplectic Groups*, Amer. J. Math. **95**(1973), 594–635.
5. ______, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
6. ______, *Characters of Solvable Groups*, Proc. of Symp. in Pure Math. **37**(1980), 377–384.
7. ______, *Character Correspondences in Solvable Groups*, Adv. in Math. **43**(1982), 284–306.
8. I. M. Isaacs and G. Navarro, *Character Correspondences and Irreducible Induction and Restriction*, J. Algebra **140**(1991), 131–140.
9. M. L. Lewis, *Character Correspondences and Nilpotent Fully-Ramified Sections*, Trans. Amer. Math. Soc., submitted.

10. ———, *A New Canonical Character Correspondence in Solvable Groups*, J. Algebra, submitted.
11. ———, *The Canonical Correspondent and Constituents of Odd Multiplicity*, preprint.
12. ———, *A Family of Groups Containing a NonAbelian Fully-Ramified Section*, J. Algebra, to appear.
13. G. Navarro, *Fong Characters and Correspondences in π-Separable Groups*, Canad. J. Math. **43**(1991), 405–412.
14. ———, *Correspondences and Irreducible Products of Characters*, J. Algebra **158**(1993), 492–498.
15. ———, *Some Open Problems on Coprime Action and Character Correspondences*, Bull. London Math. Soc. **26**(1994), 513–522.
16. T. R. Wolf,*Character Correspondences in Solvable Groups*, Illnois J. Math. **22**(1978), 327–340.
17. ———, *Character Correspondences Induced by Subgroups of Operator Groups*, J. Algebra **57**(1979), 502–521.
18. ———, *Character Correspondences and π-Special Characters in π-Separable Groups*, Canad. J. Math. **39**(1987), 920–937.

Mathematics Department
400 Carver Hall
Iowa State University
Ames, Iowa 50011
U.S.A.
e-mail: mllewis@iastate.edu

https://doi.org/10.4153/CJM-1996-052-8 Published online by Cambridge University Press