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REFINEMENTS OF THE FIRST AND SECOND POSITIVE CRANK MOMENTS

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Abstract

We revisit Berkovich and Garvan's two bijections: the first gives symmetry of cranks and the second relates partitions with crank $\leq k$ to those with k in the rank-set of partitions. Using these, we give a combinatorial proof for the relationship between the first positive crank moment and the sum of sizes of Durfee squares. We also study refinements of the first and second positive crank moments.

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1. Introduction

A partition λ is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$. We say that λ is a partition of n, denoted $\lambda \vdash n$, if $\sum_i \lambda_i = n$. The partition function $p(n)$ denotes the number of partitions of n. To explain Ramanujan's famous partition function congruences,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11},$$

the crank $c(\lambda)$ of a partition λ was defined by Andrews and Garvan [2] to be

$$c(\lambda) = \begin{cases} \lambda_1 & \text{if } o(\lambda) = 0, \\ \omega(\lambda) - o(\lambda) & \text{if } o(\lambda) \geq 1, \end{cases}$$

where $o(\lambda)$ is the number of 1's in λ and $\omega(\lambda)$ is the number of parts in λ that are strictly larger than $o(\lambda)$. Let $M(k, n)$ be the number of partitions of n with crank k.

Dyson [4] introduced the rank-set of λ, the infinite set of integers associated with λ, with the convention that $\lambda_{i+1} = 0$ for $i \geq s$, by

$$R(\lambda) = \{i - \lambda_{i+1} \mid i = 0, 1, 2, \ldots\}.$$

Let $q(k,n)$ be the number of partitions of n whose rank-sets contain k. Dyson [4] proved that

$$M(k,n) = q(k,n) - q(k-1,n), \quad n > 1,$$

by using the mappings between ordinary partitions and the vector partitions defined by Garvan [5]. Thirteen year later, Berkovich and Garvan [3] gave a direct mapping without any reference to vector partitions. In [3], they also gave a bijection for

$$M(k,n) = M(-k,n), \tag{1.1}$$

the symmetry of cranks.

Andrews *et al.* [1] introduced the jth positive crank moment,

$$M_j^{+}(n) = \sum_{k=1}^{\infty} k^j M(k,n),$$

and gave an interesting partition identity for the first positive moment of crank.

Theorem 1.1. *For all positive integers n,*

$$M_1^{+}(n) = \sum_{\lambda \vdash n} d(\lambda),$$

where $d(\lambda)$ is the size of the Durfee square of λ.

Dyson [4] obtained an interesting identity for the second positive moment of crank.

Theorem 1.2. *For all positive integers n,*

$$M_2^{+}(n) = np(n).$$

In this paper, we give a combinatorial proof of Theorem 1.1. To prove Theorem 1.1, we first review two bijections introduced by Berkovich and Garvan [3] and use them to give some refinements. We propose the modified positive moments

$$M_j^{+}(d,n) = \sum_{k=1}^{\infty} k^j M(d,k,n),$$

where $M(d,k,n)$ is the number of partitions of n with crank k having Durfee squares of size d.

Theorem 1.3. *For $d \geq 1$,*

$$M_1^{+}(d,n) = dp(d,n),$$

where $p(d,n)$ is the number of partitions of n having Durfee squares of size d.

Theorem 1.4. *For $d \geq 1$,*

$$M_2^{+}(d,n) = np(d,n).$$

This paper is organised as follows. In Section 2, we recall the two bijections in [3] and give a combinatorial proof of Theorem 1.1. In Section 3, we prove Theorems 1.3 and 1.4 using generating functions and we also give combinatorial proofs.

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2. Two bijections and the first positive moment

We recall the two bijections between partitions in [3]. First, we consider a mapping S between the partitions of n with negative cranks and those with positive cranks.

The set of partitions of $n > 1$ can be partitioned into three sets:

$$A = \{\lambda \mid o(\lambda) \geq \lambda_1\},$$
$$B = \{\lambda \mid o(\lambda) = 0\},$$
$$C = \{\lambda \mid 0 < o(\lambda) < \lambda_1\}.$$

From the definition of crank, we know that partitions in A have negative cranks and partitions in B have positive cranks. Define the mapping

$$S_1 : A \to B$$

as follows. For a partition λ in A, $S_1(\lambda)$ is defined by deleting $o(\lambda)$ 1's and then adding $o(\lambda)$ as the first part. Conversely, for a partition $\pi = S_1(\lambda)$, by deleting a part π_1 from π and then adding π_1 1's, we can recover λ again.

The third set C can be partitioned into three sets: C_- with negative cranks, C_0 with 0 crank and C_+ with positive cranks. Consider the mapping

$$S_2 : C_- \to C_+.$$

For a partition $\lambda \in C_-$, $S_2(\lambda)$ is obtained from λ in the following way.

Case 1. If $\lambda_1 = \lambda_2$:

(i) delete $o(\lambda)$ 1's and then add $o(\lambda)$ as a part;

(ii) conjugate the partition obtained from (i);

(iii) the $\lambda_{\omega(\lambda)}$th part of the conjugation is $\omega(\lambda)$. Delete this part and add $\omega(\lambda)$ 1's.

Case 2. If $\lambda_1 > \lambda_2$, $\omega(\lambda) > 1$:

(i) subtract $\lambda_1 - \lambda_2$ from the largest part λ_1;

(ii) now the first two parts of the partition of $n - \lambda_1 + \lambda_2$ obtained from (i) are the same and we apply the mapping in Case 1;

(iii) add $\lambda_1 - \lambda_2$ to the largest part.

Case 3. If $\omega(\lambda) = 1$:

(i) subtract $\lambda_1 - o(\lambda) - 1$ from the largest part λ_1;

(ii) delete $o(\lambda)$ 1's, then add $o(\lambda)$ as a part and then conjugate the partition;

(iii) add $\lambda_1 - o(\lambda) - 1$ to the largest part.

The inverse mapping is just the same as Cases 1 and 2 in S_2.

For a partition λ with negative crank $c(\lambda)$, $S(\lambda)$ is defined to be

$$S(\lambda) = \begin{cases} S_1(\lambda) & \text{if } o(\lambda) = 0, \\ S_2(\lambda) & \text{if } o(\lambda) \geq 1. \end{cases}$$

Then the crank of $S(\lambda)$ is $-c(\lambda)$. This bijection gives a combinatorial proof of (1.1).

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Now we consider the second bijection, which relates partitions of n with crank $\leq k$ to those with k in the rank-set of partitions. Given any pair $[\lambda, m]$ with a partition λ of n and a negative integer $m = i - \lambda_{i+1}$ in $R(\lambda)$, a mapping is defined by

$$[\lambda, m] \to L[\lambda, m] = \langle \pi, m \rangle.$$

The partition π of n is defined by deleting a part λ_{i+1} and then adding λ_{i+1} 1's. Since $w(\pi) \leq i < \lambda_{i+1} \leq o(\pi)$, $c(\pi) = w(\pi) - o(\pi) \leq m < 0$. Conversely, given any partition π of n with negative crank and any negative integer $m \geq c(\pi)$, we can find a nonnegative integer $i < d(\pi)$ such that

$$i - 1 - \pi_i < m \leq i - \pi_{i+1}$$

with the convention that $\pi_0 = \infty$. Delete $i - m$ 1's and then add a part $i - m$ between π_i and π_{i+1}. This gives a partition λ and a negative integer $m = i - \lambda_{i+1} = i - (i - m)$ in $R(\lambda)$.

We will use these bijections to prove our results. For any partition λ, define $R_-(\lambda)$ to be the set of all negative integers in the rank-set $R(\lambda)$.

Lemma 2.1. *For a partition λ,*

$$d(\lambda) = |R_-(\lambda)|,$$

where $d(\lambda)$ is the size of the Durfee square of λ.

Proof. If $d(\lambda) = d$, then $d \leq \lambda_d$ and $\lambda_{d+1} \leq d$. So, $(d-1) - \lambda_d < 0$ and $d - \lambda_{d+1} \geq 0$. Since $i - \lambda_{i+1} < j - \lambda_{j+1}$ for $i < j$, $R_-(\lambda) = \{i - \lambda_{i+1} \mid i = 0, 1, 2, \ldots, d-1\}$. □

Proof of Theorem 1.1. The bijection L gives the following relations:

$$\sum_{\lambda \vdash n} |R_-(\lambda)| = |\{[\lambda, m] : m \in R_-(\lambda), \lambda \vdash n\}| = |\{\langle \pi, m \rangle : c(\pi) \leq m < 0, \pi \vdash n\}|.$$

By the symmetry (1.1) and Lemma 2.1,

$$M_1{}^+(n) = |\{\langle \pi, m \rangle \mid c(\pi) \leq m < 0, \pi \vdash n\}| = \sum_{\lambda \vdash n} d(\lambda).$$ □

3. Refinements of the first and second positive crank moments

The modified positive crank generating function is given by

$$C_k(d, q) = \sum_{n=1}^{\infty} M_k{}^+(d, n) q^n.$$

In this paper, we will focus on the first and second modified positive moments of crank. The following simple observation will play an important role in finding the generating function for $M(d, k, n)$.

Lemma 3.1. *For a partition λ of n,*

$$c(\lambda) > 0 \quad \Leftrightarrow \quad o(\lambda) < d(\lambda).$$

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Proof. If $o(\lambda) \geq d(\lambda)$, then there are at most $d(\lambda)$ parts greater than $o(\lambda)$. Therefore, $c(\lambda) \leq 0$. Conversely, suppose that $o(\lambda) < d(\lambda)$. If $o(\lambda) = 0$, then $c(\lambda) = \lambda_1 > 0$. If $o(\lambda) \neq 0$, then there are at least $d(\lambda)$ parts greater than $o(\lambda)$. In both cases, $c(\lambda) > 0$. □

We will use the standard q-series notation: $(a;q)_0 := 1$ and

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

For a fixed integer $d \geq 1$, let $C(d,z,q)$ be the generating function of $M(d,k,n)$ with $k > 0$.

Theorem 3.2. *For $d \geq 1$,*

$$C(d,z,q) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} M(d,k,n)z^k q^n = \frac{q^{d^2}(1-q)}{(q;q)_d}\sum_{j=0}^{d-1}\frac{z^{d-j}q^j}{(q;q)_j(zq^{j+1};q)_{d-j}}.$$

Proof. For $1 \leq j \leq d-1$,

$$\frac{z^{d-j}q^{d^2+j}}{(q;q)_d(q^2;q)_{j-1}(zq^{j+1};q)_{d-j}}$$

generates partitions with $o(\lambda) = j$ having Durfee square of size d. The exponent of z is clearly $w(\lambda) - o(\lambda)$, which is $c(\lambda)$. By Lemma 3.1, $c(\lambda) > 0$. The generating function for partitions having Durfee squares of size d without 1's is

$$\frac{z^d q^{d^2}}{(q^2;q)_{d-1}(zq;q)_d}.$$

Hence,

$$\frac{q^{d^2}(1-q)}{(q;q)_d}\sum_{j=0}^{d-1}\frac{z^{d-j}q^j}{(q;q)_j(zq^{j+1};q)_{d-j}}$$

is the generating function of $M(d,k,n)$ with $k > 0$. □

Using $C(d,z,q)$, we can find the generating functions for the first and second modified positive moments in Theorems 3.3 and 3.4.

Theorem 3.3. *For $d \geq 1$,*

$$C_1(d,q) = \frac{dq^{d^2}}{(q;q)_d^2}.$$

Proof. Apply the differential operator $z(d/dz)$ on both sides in Theorem 3.2:

$$\begin{aligned} z\frac{d}{dz}C(d,z,q) &= \frac{zq^{d^2}(1-q)}{(q;q)_d}\frac{d}{dz}\sum_{j=0}^{d-1}\frac{z^{d-j}q^j}{(q;q)_j(zq^{j+1};q)_{d-j}} \\ &= \frac{q^{d^2}(1-q)}{(q;q)_d}\sum_{j=0}^{d-1}\bigg\{\frac{z^{d-j}q^j}{(q;q)_j(zq^{j+1};q)_{d-j}}\bigg(\sum_{l=j+1}^{d}\frac{1}{1-zq^l}\bigg)\bigg\}. \end{aligned}$$

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Setting $z = 1$ yields

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} kM(d,k,n)q^n = \frac{q^{d^2}(1-q)}{(q;q)_d}\sum_{j=0}^{d-1}\Big\{\frac{q^j}{(q;q)_d}\Big(\sum_{l=j+1}^{d}\frac{1}{1-q^l}\Big)\Big\}$$
$$= \frac{q^{d^2}(1-q)}{(q;q)_d^2}\sum_{j=0}^{d-1}\sum_{l=j+1}^{d}\frac{q^j}{1-q^l} = \frac{q^{d^2}(1-q)}{(q;q)_d^2}\sum_{l=1}^{d}\sum_{j=0}^{l-1}\frac{q^j}{1-q^l}$$
$$= \frac{q^{d^2}(1-q)}{(q;q)_d^2}\sum_{l=1}^{d}\frac{1}{1-q} = \frac{dq^{d^2}}{(q;q)_d^2}. \qquad \square$$

The above theorem immediately implies Theorem 1.3: $M_1{}^{+}(d,n) = dp(d,n)$, a refinement of Theorem 1.1.

Theorem 3.4. *For $d \geq 1$,*

$$C_2(d,q) = \frac{2q^{d^2}}{(q;q)_d^2}\Big(\sum_{l=1}^{d}\frac{l}{1-q^l}\Big) - C_1(d,q). \tag{3.1}$$

Proof. Apply the differential operator $z(d/dz)$ twice on both sides in Theorem 3.2:

$$z\frac{d}{dz}\Big(z\frac{d}{dz}C(d,z,q)\Big) = z\frac{d}{dz}\Big(\frac{q^{d^2}(1-q)}{(q;q)_d}\sum_{j=0}^{d-1}\Big\{\frac{z^{d-j}q^j}{(q;q)_j(zq^{j+1};q)_{d-j}}\Big(\sum_{l=j+1}^{d}\frac{1}{1-zq^l}\Big)\Big\}\Big)$$
$$= \sum_{j=0}^{d-1}\frac{z^{d-j}q^{d^2+j}(1-q)}{(q;q)_d(q;q)_j(zq^{j+1};q)_{d-j}}\Big\{\Big(\sum_{l=j+1}^{d}\frac{1}{1-zq^l}\Big)^2 + \sum_{l=j+1}^{d}\frac{zq^l}{(1-zq^l)^2}\Big\}.$$

Setting $z = 1$ yields

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k^2M(d,k,n)q^n = \frac{q^{d^2}(1-q)}{(q;q)_d^2}\sum_{j=0}^{d-1}q^j\Big\{\Big(\sum_{l=j+1}^{d}\frac{1}{1-q^l}\Big)^2 + \sum_{l=j+1}^{d}\frac{q^l}{(1-q^l)^2}\Big\}$$
$$= \frac{q^{d^2}(1-q)}{(q;q)_d^2}\sum_{j=0}^{d-1}\Big\{\sum_{l=j+1}^{d}\frac{q^j(1+q^l)}{(1-q^l)^2} + \sum_{j+1\leq l_1<l_2\leq d}\frac{2q^j}{(1-q^{l_1})(1-q^{l_2})}\Big\}.$$

Since

$$\sum_{j=0}^{d-1}\sum_{l=j+1}^{d}\frac{q^j(1+q^l)}{(1-q^l)^2} = \sum_{l=1}^{d}\sum_{j=0}^{l-1}\frac{q^j(1+q^l)}{(1-q^l)^2}$$
$$= \sum_{l=1}^{d}\frac{(1+q+\cdots+q^{l-1})(1+q^l)}{(1-q^l)^2} = \frac{1}{1-q}\sum_{l=1}^{d}\frac{1+q^l}{1-q^l}$$
$$= \frac{1}{1-q}\sum_{l=1}^{d}\Big(\frac{2}{1-q^l} - 1\Big)$$

and

$$\sum_{j=0}^{d-1}\sum_{j+1\le l_1<l_2\le d}\frac{2q^j}{(1-q^{l_1})(1-q^{l_2})}=\sum_{l_1=1}^{d}\sum_{l_2=l_1+1}^{d}\frac{2(1+q+\cdots+q^{l_1-1})}{(1-q^{l_1})(1-q^{l_2})}$$
$$=\frac{1}{1-q}\sum_{l_1=1}^{d}\sum_{l_2=l_1+1}^{d}\frac{2}{1-q^{l_2}}=\frac{1}{1-q}\sum_{l_2=2}^{d}\sum_{l_1=1}^{l_2-1}\frac{2}{1-q^{l_2}}$$
$$=\frac{1}{1-q}\sum_{l_2=2}^{d}\frac{2(l_2-1)}{1-q^{l_2}}=\frac{1}{1-q}\sum_{l=1}^{d}\frac{2l-2}{1-q^{l}},$$

we arrive at

$$C_2(d,q)=\frac{2q^{d^2}}{(q;q)_d^2}\sum_{l=1}^{d}\frac{l}{1-q^l}-C_1(d,q).\qquad\square$$

The right-hand side of (3.1) is the generating function for $np(d,n)$. We know that

$$\sum_{n=0}^{\infty}p(d,n)q^n=\frac{q^{d^2}}{(q;q)_d^2}.$$

Applying the differential operator $q(d/dq)$ on both sides,

$$\sum_{n=1}^{\infty}np(d,n)q^n=q\frac{d}{dq}\Big(\frac{q^{d^2}}{(q;q)_d^2}\Big)=\frac{d^2q^{d^2}}{(q;q)_d^2}+\frac{2q^{d^2+1}}{(q;q)_d^2}\Big(\sum_{l=1}^{d}\frac{lq^{l-1}}{1-q^l}\Big)$$
$$=\frac{d^2q^{d^2}}{(q;q)_d^2}+\frac{q^{d^2}}{(q;q)_d^2}\Big(\sum_{l=1}^{d}\frac{2lq^l}{1-q^l}\Big)$$
$$=\frac{d^2q^{d^2}}{(q;q)_d^2}+\frac{q^{d^2}}{(q;q)_d^2}\Big\{\sum_{l=1}^{d}\frac{2l}{1-q^l}-2(1+2+\cdots+d)\Big\}$$
$$=\frac{q^{d^2}}{(q;q)_d^2}\Big(-d+\sum_{l=1}^{d}\frac{2l}{1-q^l}\Big).$$

From these relations, we have Theorem 1.4: ${M_2}^{+}(d,n)=np(d,n)$, a refinement of Theorem 1.2.

Now we focus on the combinatorial explanation for the two refinements and again consider the two bijections from Section 2. Let λ be a partition of n with $d(\lambda)=d$ and let m be an integer in $R_{-}(\lambda)$. Let $L[\lambda,m]=\langle\pi,m\rangle$. Since $\pi_d=\lambda_{d+1}$, the size of the Durfee square of π is

$$d(\pi)=\begin{cases}d & \text{if } \lambda_{d+1}=d,\\ d-1 & \text{if } \lambda_{d+1}<d.\end{cases}$$

Let π be a partition of n with $d(\pi)=d$. For all cases in the bijection S, the size of the Durfee square of $S(\pi)$ is

$$d(S(\pi))=\begin{cases}d & \text{if } \pi_d=d,\\ d+1 & \text{if } \pi_d>d.\end{cases}$$

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Therefore, the two bijections satisfy the relations

$$[\lambda, m] \;\leftrightarrow\; \langle \pi, m\rangle \;\leftrightarrow\; \langle S(\pi), -m\rangle,$$

and $d(\lambda) = d(S(\pi))$. Hence, we have Theorem 1.3 again.

Finally, we consider a weighted sum

$$\sum_{k=1}^{n} (1 + 2 + \cdots + k) M(d, k, n).$$

The bijection

$$[\lambda, -m] \leftrightarrow \langle S(\pi), m\rangle,$$

where $L[\lambda, -m] = \langle \pi, -m\rangle$ with $-m = i - \lambda_{i+1}$ $(i = 0, 1, \ldots, d-1)$, gives

$$\sum_{k=1}^{n} (1 + 2 + \cdots + k) M(d, k, n) = \sum_{d(S(\pi))=d} \sum_{m=1}^{c(S(\pi))} m = \sum_{d(\lambda)=d} \sum_{i=0}^{d-1} (\lambda_{i+1} - i).$$

Note that for a partition λ of n, the size of the Durfee square of λ and the conjugation λ' of λ are the same. If $d(\lambda) = d$, then

$$\sum_{i=0}^{d-1} (\lambda_{i+1} - i) + \sum_{i=0}^{d-1} (\lambda'_{i+1} - i) = d + \sum_{i=1}^{\infty} \lambda_i = d + n.$$

Hence,

$$\sum_{k=1}^{n} \frac{k^2 + k}{2} M(d, k, n) = \sum_{d(\lambda)=d} \sum_{i=0}^{d-1} (\lambda_{i+1} - i) = \sum_{d(\lambda)=d} \frac{d+n}{2} = \frac{d+n}{2} p(d, n).$$

We know that $\sum_{k=1}^{n} kM(d, k, n) = dp(d, n)$, so

$$\sum_{k=1}^{n} k^2 M(d, k, n) = np(d, n),$$

giving Theorem 1.4, as desired.

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