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THE NATURAL PARTIAL ORDER ON SOME TRANSFORMATION SEMIGROUPS

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Abstract

For a semigroup S, let S^1 be the semigroup obtained from S by adding a new symbol 1 as its identity if S has no identity; otherwise let $S^1 = S$. Mitsch defined the natural partial order $\leqslant$ on a semigroup S as follows: for $a, b \in S$, $a \leqslant b$ if and only if $a = xb = by$ and $a = ay$ for some $x, y \in S^1$. In this paper, we characterise the natural partial order on some transformation semigroups. In these partially ordered sets, we determine the compatibility of their elements, and find all minimal and maximal elements.

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1. Introduction and preliminaries

Let S be a semigroup. The semigroup S^1 is obtained from S by adding 1 as its identity if S has no identity; otherwise, let $S^1 = S$. If, for each $a \in S$, there exists an element $x \in S$ such that $a = axa$, then we call S a *regular semigroup*. An *inverse semigroup* S is a regular semigroup in which every element a has a unique element $a^{-1} \in S$, the *inverse of* a, such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. For $a \in S$, if $a = a^2$ then a is called an *idempotent*, and we denote by $E(S)$ the set of idempotents of S.

In the study of algebraic semigroups, an order relation on a semigroup may be defined via the multiplication of the semigroup. An important such order relation is the natural partial order. There are many studies in the literature about this order. The natural partial order arose in 1952 when Wagner [9] introduced the order, denoted by $\leqslant$, on inverse semigroups in the following way. Let S be an inverse semigroup and let a and b be elements in S; then we define

$$a \leqslant b \quad \text{if and only if} \quad a = aa^{-1}b.$$

Note that aa^{-1} and $a^{-1}a$ are idempotents in S. Later, in 1980, Hartwig [1] and Nambooripad [7] independently extended the notion of the natural partial order to regular semigroups. For a regular semigroup S and $a, b \in S$,

$$a \leqslant b \quad \text{if and only if} \quad a = eb = bf \text{ for some } e, f \in E(S). \tag{1.1}$$

Next, in 1986, Mitsch [6] showed that, given a semigroup S, the relation $\leqslant$ defined by

$$a \leqslant b \quad \text{if and only if} \quad a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1, \tag{1.2}$$

where $a, b \in S$, is a partial order. The partial orders (1.1) and (1.2) are actually the same on regular semigroups. Then (1.2) can be considered as the natural partial order on general semigroups. In 1994, Higgins stated in [2, Theorem 2.5] that if T is a regular subsemigroup of a semigroup S and $a, b \in T$, then $a \leqslant b$ on T if and only if $a \leqslant b$ on S. An element c in a semigroup S is *left* (respectively, *right*) *compatible* with the natural partial order $\leqslant$ if, for each $a, b \in S$, we have $ca \leqslant cb$ (respectively, $ac \leqslant bc$) on S whenever $a \leqslant b$ on S.

Much research on the natural partial order concerns semigroups of transformations. In this paper, we are interested in two semigroups of transformations, the semigroup of almost one-to-one transformations and the semigroup of almost onto transformations.

Let X be a nonempty set and let

$$P(X) = \{\alpha\colon A \to X : A \subseteq X\},$$
$$T(X) = \{\alpha \in P(X) : \operatorname{dom} \alpha = X\}.$$

It is clear that, under composition, $P(X)$ and $T(X)$ are semigroups. We call $P(X)$ the *partial transformation semigroup on* X, and $T(X)$ the *full transformation semigroup on* X. Note that $T(X)$ is a regular subsemigroup of $P(X)$.

In 1986, the natural partial order (1.1) was studied on $T(X)$ by Kowol and Mitsch [4]. Moreover, they described the minimal and maximal elements in $(T(X), \leqslant)$. Next, in 2003, Marques-Smith and Sullivan [5] studied the natural partial order (1.2) on $P(X)$, and then generalised the results of Kowol and Mitsch [4]. Furthermore, they characterised the left and the right compatible elements in $(P(X), \leqslant)$ and $(T(X), \leqslant)$ and determined the minimal and maximal elements in $(P(X), \leqslant)$.

Throughout, all maps act on the right-hand side of the argument, and α^{-1} stands for the inverse relation of α, where $\alpha \in P(X)$. For a nonempty subset A of X, we let 0 and 1_A be the empty transformation and the identity map on A, respectively. Both are contained in $P(X)$ where 0 is the zero element and 1_X the identity element.

Let $\alpha \in T(X)$ and let $x \in X$. Then α is said to be *one-to-one at* x if $|(x\alpha)\alpha^{-1}| = 1$. From [8], if $\{x \in X : \alpha \text{ is not one-to-one at } x\}$ is finite, then α is called an *almost one-to-one transformation on* X. We denote by $AM(X)$ the set of almost one-to-one transformations on X, that is,

$$\mathrm{AM}(X) = \{\alpha \in T(X) : \{x \in X : \alpha \text{ is not one-to-one at } x\} \text{ is finite}\}.$$

For convenience, we let $K(\alpha) = \{x \in \operatorname{ran} \alpha : |x\alpha^{-1}| > 1\}$ and $\bigcup_{x\in K(\alpha)} x\alpha^{-1} = \{x \in X : \alpha \text{ is not one-to-one at } x\}$. Then

$$AM(X) = \Big\{\alpha \in T(X) : \bigcup_{x\in K(\alpha)} x\alpha^{-1} \text{ is finite}\Big\}.$$

Note that for each $\alpha \in T(X)$, $\alpha \in AM(X)$ if and only if $K(\alpha)$ and $x\alpha^{-1}$ are finite sets for every x in $K(\alpha)$. Let

$$AE(X) = \{\alpha \in T(X) : X \setminus \operatorname{ran} \alpha \text{ is finite}\}.$$

Each element of $AE(X)$ is called an *almost onto transformation on* X. From [8], $AM(X)$ and $AE(X)$ are subsemigroups of $T(X)$ with the identity 1_X. It is easy to see that if X is finite, then $AM(X) = T(X) = AE(X)$. As the natural partial order on $T(X)$ has been thoroughly studied, in this paper we suppose X is an infinite set if we do not specify otherwise. In this case, it was shown in [3] that $AM(X)$ and $AE(X)$ are not regular subsemigroups of $T(X)$. Therefore, by [2, Theorem 2.5], the natural partial order (1.2) on these semigroups cannot be reproduced from $T(X)$. Throughout this paper, $\leqslant$ denotes the natural partial order (1.2).

For convenience, let A_i and B_j be disjoint nonempty subsets of X and $x_i, y_j \in X$ for all $i \in I$, $j \in J$, where I and J are index sets. Define

$$\alpha = \begin{pmatrix} A_i & B_j \\ x_i & y_j \end{pmatrix}_{i \in I,\ j \in J},$$

which means $\alpha \in P(X)$ such that

$$x\alpha = \begin{cases} x_i & \text{if } x \in A_i, \\ y_j & \text{if } x \in B_j. \end{cases}$$

In particular, we let

$$\begin{pmatrix} A_i & b \\ x_i & y \end{pmatrix}_{i \in I} = \begin{pmatrix} A_i & \{b\} \\ x_i & y \end{pmatrix}_{i \in I},$$

where $b \in X \setminus \bigcup_{i \in I} A_i$ and $y \in X$. Without loss of generality, more mapping symbols in this article can be easily applied.

The following remark and some known results on the natural partial order will be used.

Remark 1.1. Let S be a semigroup.

(i) If S contains the zero 0, then 0 is the minimum.

(ii) If S contains the identity 1 and $s \in S$, then:

 (a) $s \leqslant 1$ if and only if s is an idempotent;

 (b) 1 is a maximal element.

(iii) For any subsemigroup T of S and $a, b \in T$, $a \leqslant b$ on T implies $a \leqslant b$ on S.

Theorem 1.2 [5]. *For $\alpha, \beta \in T(X)$, $\alpha \leqslant \beta$ on $T(X)$ if and only if the following conditions hold:*

(i) $\operatorname{ran} \alpha \subseteq \operatorname{ran} \beta$;

(ii) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$;

(iii) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$.

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Theorem 1.3 [5]. *Suppose $\gamma \in T(X)$ and $|X| \geq 3$. Then:*

(i) *γ is left compatible on $(T(X), \leqslant)$ if and only if γ is onto;*
(ii) *γ is right compatible on $(T(X), \leqslant)$ if and only if γ is one-to-one or constant.*

Theorem 1.4 [4]. *Let $\alpha \in T(X)$. Then:*

(i) *α is a minimal element in $(T(X), \leqslant)$ if and only if α is a constant map;*
(ii) *α is a maximal element in $(T(X), \leqslant)$ if and only if α is a one-to-one or onto map.*

The next proposition was extracted from the proof of [5, Theorem 2].

Proposition 1.5. *Let $\alpha, \beta \in T(X)$ be such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$. Then the following assertions hold.*

(i) *$\alpha = \binom{A_i}{x_i}_{i\in I}$ and $\beta = \begin{pmatrix} B_i & C_j \\ x_i & y_j \end{pmatrix}_{i\in I,\ j\in J}$ for some $A_i, B_i, C_j \subseteq X$ and distinct $x_i, y_j \in X$, where $i \in I$ and $j \in J$.*
(ii) *If $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$, then $B_i \subseteq A_i$ for all $i \in I$.*
(iii) *If $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$, then $A_i = B_i \cup \bigcup_{j\in J_i} C_j$, where $J_i = \{j \in J : A_i \cap C_j \neq \emptyset\}$ and $i \in I$.*

In this paper, we give necessary and sufficient conditions for two elements in $AM(X)$ and $AE(X)$ to be related under the natural partial order (1.2). The left and right compatible elements and the minimal and maximal elements in these posets are characterised.

2. The natural partial order

For any $\alpha, \beta \in AM(X)$, it is clear that $\alpha \leqslant \beta$ on $AM(X)$ implies $\alpha \leqslant \beta$ on $T(X)$. In this section we show that the converse is also true.

Example 2.1. Since X is an infinite set, there are disjoint sets $A, B \subsetneq X$ such that $|X| = |A| = |B|$ and $X = A \cup B$. Choose distinct elements $a, b \in B$. It is clear that $|X \setminus \{a, b\}| = |A|$. Let ϕ be a bijection from $X \setminus \{a, b\}$ onto A and define $\alpha, \beta \in AM(X)$ by

$$\alpha = \begin{pmatrix} \{a, b\} & x \\ a & x\phi \end{pmatrix}_{x\in X\setminus\{a,b\}} \quad \text{and} \quad \beta = \begin{pmatrix} a & b & x \\ b & a & x\phi \end{pmatrix}_{x\in X\setminus\{a,b\}}.$$

Clearly, $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$. Then we will show that $\alpha \leqslant \beta$ on $T(X)$. We have that

$$\begin{aligned} \alpha\beta^{-1} &= \{(a, b)\} \cup 1_{X\setminus\{a\}}, \\ \alpha\alpha^{-1} &= \{(a, b), (b, a)\} \cup 1_X, \\ \beta\beta^{-1} &= 1_X. \end{aligned}$$

So $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$. By Theorem 1.2, we have $\alpha \leqslant \beta$ on $T(X)$. Following the proof of Theorem 1.2, one will get $\lambda \in T(X)$ and $\mu \in T(X) \setminus AM(X)$ such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. This does not allow us to conclude that $\alpha \leqslant \beta$ on $AM(X)$.

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Due to this example, it is worth studying the natural partial order on $AM(X)$. In Theorem 2.3 we provide necessary and sufficient conditions for two elements in $AM(X)$ to be related. We start with a lemma.

Lemma 2.2. *Let $\alpha, \beta \in AM(X)$ be defined by*

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}_{i\in I} \quad \text{and} \quad \beta = \begin{pmatrix} B_i & C_j \\ x_i & y_j \end{pmatrix}_{i\in I,\ j\in J}$$

for some $A_i, B_i, C_j \subseteq X$ and distinct $x_i, y_j \in X$, where $i \in I$ and $j \in J$. Assume that $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$, $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ and let

$$\mu = \begin{pmatrix} A_i\beta & a \\ x_i & a \end{pmatrix}_{i\in I,\ a\in A},$$

where $A = X \setminus \bigcup_{i\in I} A_i\beta$. Then the following hold:

(i) *for each $i \in I$, $x_i \notin A$;*
(ii) *for each $i \in I$, $x_i\mu^{-1} \cap A = \emptyset$;*
(iii) *$\mu \in T(X)$;*
(iv) *$K(\mu) = \{x_i : |x_i\mu^{-1}| > 1\}$ is a finite set;*
(v) *for each $x \in K(\mu)$, $x\mu^{-1}$ is a finite set.*

Moreover, $\mu \in AM(X)$.

Proof. (i) By Proposition 1.5(ii), $x_i \in B_i\beta \subseteq A_i\beta$, so $x_i \notin A$ for all $i \in I$.
(ii) If there is $i \in I$ and $b \in x_i\mu^{-1} \cap A$, then $x_i = b\mu = b \in A$, which contradicts (i).
(iii) This follows from (i), (ii) and Proposition 1.5(iii).
(iv) By (ii),

$$\begin{aligned}\{i \in I : |x_i\mu^{-1}| > 1\} &= \{i \in I : |A_i\beta| > 1\} \\ &\subseteq \{i \in I : |A_i| > 1\},\end{aligned}$$

which is a finite set, since $\alpha \in AM(X)$. Also, $K(\mu)$ is finite.
(v) Let $x \in K(\mu)$. Then $x\mu^{-1} = A_i\beta$ for some $i \in I$. Thus $|x\mu^{-1}| = |A_i\beta| \leqslant |A_i| = |x_i\alpha^{-1}|$ is finite, since $\alpha \in AM(X)$. □

Now we are ready to give the first main result.

Theorem 2.3. *For $\alpha, \beta \in AM(X)$, $\alpha \leqslant \beta$ on $AM(X)$ if and only if the following conditions hold:*

(i) $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$;
(ii) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$;
(iii) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$.

Proof. Let $\alpha, \beta \in AM(X)$. The necessity follows from Remark 1.1(iii) and Theorem 1.2.

To see the sufficiency, we suppose that the three conditions (i)–(iii) hold. Since $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$,

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}_{i\in I} \quad \text{and} \quad \beta = \begin{pmatrix} B_i & C_j \\ x_i & y_j \end{pmatrix}_{i\in I,\ j\in J}$$

for some $A_i, B_i, C_j \subseteq X$ and distinct elements $x_i, y_j \in X$, with $i \in I$ and $j \in J$. For each $i \in I$, choose $b_i \in B_i$ and let

$$\lambda = \begin{pmatrix} A_i \\ b_i \end{pmatrix}_{i\in I}.$$

Recall that $K(\alpha) = \{x_i : |x_i\alpha^{-1}| > 1\}$ and $K(\lambda) = \{b_i : |b_i\lambda^{-1}| > 1\}$. Since $\alpha \in AM(X)$ and $\bigcup_{b_i\in K(\lambda)} b_i\lambda^{-1} = \bigcup_{x_i\in K(\alpha)} x_i\alpha^{-1}$, we have $\lambda \in AM(X)$. Consequently,

$$\alpha = \lambda\beta.$$

By Proposition 1.5(ii), we have $\emptyset \neq B_i\beta \subseteq A_i\beta$ for all $i \in I$. Let $A = X \setminus \bigcup_{i\in I} A_i\beta$ and define

$$\mu = \begin{pmatrix} A_i\beta & a \\ x_i & a \end{pmatrix}_{i\in I,\ a\in A}.$$

By Lemma 2.2, we get $\mu \in AM(X)$. Let $x \in X$. Then $x \in A_i$ for some $i \in I$, so $x\alpha = x_i = x\beta\mu$ and $x\alpha = x_i = x_i\mu = x\alpha\mu$, since $x_i \in B_i\beta \subseteq A_i\beta$. Therefore,

$$\alpha = \beta\mu \quad \text{and} \quad \alpha = \alpha\mu,$$

which implies that $\alpha \leqslant \beta$ on $AM(X)$. □

Recall $\alpha, \beta \in AM(X)$ in Example 2.1, where

$$\alpha = \begin{pmatrix} \{a, b\} & x \\ a & x\phi \end{pmatrix}_{x\in X\setminus\{a,b\}} \quad \text{and} \quad \beta = \begin{pmatrix} a & b & x \\ b & a & x\phi \end{pmatrix}_{x\in X\setminus\{a,b\}}.$$

Since $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$, by Theorem 2.3, $\alpha \leqslant \beta$ on $AM(X)$.

The following corollary is obtained from Theorems 1.2 and 2.3. This result confirms that the converse of [2, Theorem 2.5] is not true.

Corollary 2.4. *Let $\alpha, \beta \in AM(X)$. Then the following are equivalent:*

(i) *$\alpha \leqslant \beta$ on $AM(X)$;*
(ii) *$\alpha \leqslant \beta$ on $T(X)$;*
(iii) *$\alpha \leqslant \beta$ on $P(X)$.*

As we have seen above, $AM(X)$ inherits the natural partial order from $T(X)$. However, this is not the case for $AE(X)$. Furthermore, in the main theorem we give necessary and sufficient conditions for two elements in $AE(X)$ to be related under the natural partial order.

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Example 2.5. Since X is an infinite set, there are disjoint subsets $A, B \subsetneq X$ such that $|X| = |A| = |B|$ and $X = A \cup B$. Choose $d \in A$ and $e \in B$. It is clear that $|A| = |X \setminus \{d, e\}|$. Let ϕ be a bijection from A onto $X \setminus \{d, e\}$, and define $\alpha, \beta \in AE(X)$ by

$$\alpha = \begin{pmatrix} a & B \\ a\phi & d \end{pmatrix}_{a \in A} \quad \text{and} \quad \beta = \begin{pmatrix} a & e & B \setminus \{e\} \\ a\phi & d & e \end{pmatrix}_{a \in A}.$$

Clearly, $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$. Then we will show that $\alpha \leqslant \beta$ on $T(X)$. Since

$$\begin{aligned} \alpha\beta^{-1} &= 1_A \cup (B \times \{e\}), \\ \alpha\alpha^{-1} &= 1_A \cup (B \times B), \\ \beta\beta^{-1} &= 1_{A \cup \{e\}} \cup (B \setminus \{e\} \times B \setminus \{e\}), \end{aligned}$$

we get that $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$. By Theorem 1.2, we have $\alpha \leqslant \beta$ on $T(X)$. We show that $\alpha \nleqslant \beta$ on $AE(X)$. Suppose on the contrary that $\alpha \leqslant \beta$ on $AE(X)$. Then there exists $\lambda \in AE(X)$ such that $\alpha = \lambda\beta$. Let $x \in X$.

Case 1: $x \in A$. Then $x\phi = x\alpha = x\lambda\beta$. Therefore, $x\lambda = x$.

Case 2: $x \in B$. Then $d = x\alpha = x\lambda\beta$. Therefore, $x\lambda = e$.

Hence we can write

$$\alpha = \begin{pmatrix} a & B \\ a & e \end{pmatrix}_{a \in A}.$$

Then $X \setminus \operatorname{ran}\lambda = B \setminus \{e\}$, which is an infinite set and this contradicts the fact that λ is in $AE(X)$.

Theorem 2.6. *For $\alpha, \beta \in AE(X)$, $\alpha \leqslant \beta$ on $AE(X)$ if and only if the following conditions hold:*

(i) $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$;
(ii) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$;
(iii) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$;
(iv) $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ *is a finite set.*

Proof. Let $\alpha, \beta \in AE(X)$. Assume that $\alpha \leqslant \beta$ on $AE(X)$. By Remark 1.1(iii) and Theorem 1.2, we get (i)–(iii). Since $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$,

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}_{i \in I} \quad \text{and} \quad \beta = \begin{pmatrix} B_i & C_j \\ x_i & y_j \end{pmatrix}_{i \in I,\ j \in J} \tag{2.1}$$

for some $A_i, B_i, C_j \subseteq X$ and distinct elements $x_i, y_j \in X$, where $i \in I$ and $j \in J$. It is easy to see that $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1} = \bigcup_{j \in J} C_j$. We claim that $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite by showing that J and C_j are finite sets for all $j \in J$. Since $\{y_j : j \in J\} = \operatorname{ran}\beta \setminus \operatorname{ran}\alpha \subseteq X \setminus \operatorname{ran}\alpha$ and $\alpha \in AE(X)$, J is finite. By assumption, $\alpha = \lambda\beta$ for some $\lambda \in AE(X)$. Let $j \in J$. If there is $c \in C_j \cap \operatorname{ran}\lambda$, then $x\lambda = c$ for some $x \in X$, which

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implies $x\alpha = x\lambda\beta = c\beta = y_j$, which is a contradiction. Then $C_j \subseteq X \setminus \operatorname{ran}\lambda$, and therefore C_j is finite.

Conversely, (i) implies that α and β satisfy (2.1), and therefore, by Proposition 1.5(iii), $A_i = B_i \cup \bigcup_{j\in J_i} C_j$, with $J_i = \{j \in J : A_i \cap C_j \neq \emptyset\}$ and $i \in I$. For each $i \in I$, we choose $b_i \in B_i$ and define $\lambda \in T(X)$ by

$$\lambda = \begin{pmatrix} x & \bigcup_{j\in J_i} C_j \\ x & b_i \end{pmatrix}_{x\in B_i,\ i\in I}.$$

Since $\operatorname{ran}\lambda = \bigcup_{i\in I} B_i$, $X \setminus \operatorname{ran}\lambda = \bigcup_{j\in J} C_j = (\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite. Thus $\lambda \in AE(X)$. Let $x \in X$. Then $x \in A_i$ for some $i \in I$.

Case 1: $x \in B_i$. Then $x\alpha = x_i = x\beta = x\lambda\beta$.

Case 2: $x \in C_j$ for some $j \in J_i$. Then $x\alpha = x_i = b_i\beta = x\lambda\beta$.

Thus

$$\alpha = \lambda\beta.$$

Next, let $A = X \setminus \bigcup_{i\in I} A_i\beta$ and let $d \in \operatorname{ran}\alpha$. Define

$$\mu = \begin{pmatrix} A_i\beta & A \\ x_i & d \end{pmatrix}_{i\in I}.$$

By Proposition 1.5(iii), μ is well defined. Since $\alpha \in AE(X)$ and $\operatorname{ran}\alpha = \operatorname{ran}\mu$, we have $\mu \in AE(X)$. Let $x \in X$. Then $x \in A_i$ for some $i \in I$, so $x\alpha = x_i = x\beta\mu$ and $x\alpha = x_i = x_i\mu = x\alpha\mu$, since $x_i \in B_i\beta \subseteq A_i\beta$. Therefore,

$$\alpha = \beta\mu \quad \text{and} \quad \alpha = \alpha\mu.$$

Hence $\alpha \leqslant \beta$ on $AE(X)$. □

By Theorems 1.2 and 2.6, we have the following corollary.

Corollary 2.7. *Let $\alpha, \beta \in AE(X)$. Then the following assertions are equivalent.*

(i) *$\alpha \leqslant \beta$ on $AE(X)$.*
(ii) *$\alpha \leqslant \beta$ on $T(X)$ and $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite.*
(iii) *$\alpha \leqslant \beta$ on $P(X)$ and $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite.*

As a consequence, we have the following proposition.

Proposition 2.8. *The following assertions are equivalent.*

(i) *X is finite.*
(ii) *For $\alpha, \beta \in AE(X)$, $\alpha \leqslant \beta$ on $T(X)$ if and only if $\alpha \leqslant \beta$ on $AE(X)$.*
(iii) *For $\alpha, \beta \in AE(X)$, $\alpha \leqslant \beta$ on $P(X)$ if and only if $\alpha \leqslant \beta$ on $AE(X)$.*

Proof. See Corollary 2.7 and Example 2.5. □

Let $S(X)$ be either $AM(X)$ or $AE(X)$. Then we show that the natural partial order on $S(X)$ is not the identity relation on $S(X)$.

EXAMPLE 2.9. Let a and b be distinct elements in X. Define $\alpha, \beta \in AM(X) \cap AE(X)$ by

$$\alpha = \begin{pmatrix} \{a, b\} & x \\ a & x \end{pmatrix}_{x \in X \setminus \{a,b\}} \quad \text{and} \quad \beta = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \setminus \{a,b\}}.$$

Clearly, $\operatorname{ran} \alpha \subseteq \operatorname{ran} \beta$. Also,

$$\alpha\beta^{-1} = \{(a, b)\} \cup 1_{X \setminus \{a\}},$$
$$\alpha\alpha^{-1} = \{(a, b), (b, a)\} \cup 1_X,$$
$$\beta\beta^{-1} = 1_X.$$

So $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$. Then, by Theorem 2.3, we have $\alpha \leqslant \beta$ on $AM(X)$. Also, we have that $(\operatorname{ran} \beta \setminus \operatorname{ran} \alpha)\beta^{-1} = \{a\}$ is finite. By Theorem 2.6, $\alpha \leqslant \beta$ on $AE(X)$.

Hence it is valid to study the compatible elements, the minimal and maximal elements with respect to the natural partial order on $AM(X)$ and $AE(X)$.

3. Elements compatible with the natural partial order

Here we characterise the left compatible and right compatible elements in $(AM(X), \leqslant)$ and $(AE(X), \leqslant)$ when X is an infinite set.

THEOREM 3.1. *Let $\gamma \in AM(X)$. Then:*

(i) *γ is left compatible on $(AM(X), \leqslant)$ if and only if γ is onto;*
(ii) *γ is right compatible on $(AM(X), \leqslant)$ if and only if γ is one-to-one.*

PROOF. (i) Assume that γ is left compatible with $\leqslant$ on $AM(X)$. Choose $y \in \operatorname{ran} \gamma$. Let $x \in X$ and define

$$\alpha_x = \begin{pmatrix} y & a \\ x & a \end{pmatrix}_{a \in X \setminus \{y\}}.$$

Then α_x is an idempotent in $AM(X)$. By Remark 1.1(ii), we have $\alpha_x \leqslant 1_X$ on $AM(X)$. Since γ is left compatible, $\gamma\alpha_x \leqslant \gamma$ on $AM(X)$. By Theorem 2.3(i), we have $x \in \operatorname{ran} \gamma\alpha_x \subseteq \operatorname{ran} \gamma$. Since x is arbitrary, γ is onto.

The converse follows from Theorem 1.3(i) and Corollary 2.4.

(ii) Suppose that γ is right compatible with $\leqslant$ on $AM(X)$ but not one-to-one. Then there are distinct elements $a, b \in X$ such that $a\gamma = b\gamma$. Let $c \in X \setminus \{a, b\}$. We define $\alpha \in AM(X)$ by letting

$$\alpha = \begin{pmatrix} a & x \\ c & x \end{pmatrix}_{x \in X \setminus \{a\}}.$$

Clearly, α is an idempotent, and by Remark 1.1(ii) we have $\alpha \leqslant 1_X$ on $AM(X)$. Since γ is right compatible, $\alpha\gamma \leqslant \gamma$ on $AM(X)$. By Theorem 2.3(iii), $\gamma\gamma^{-1} \subseteq \alpha\gamma(\alpha\gamma)^{-1}$. We also have $(a, b) \in \gamma\gamma^{-1} \subseteq \alpha\gamma(\alpha\gamma)^{-1}$, as $a\gamma = b\gamma$. Then there exists $z \in X$ such that $(a, z) \in \alpha\gamma$

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and $(z, b) \in (\alpha\gamma)^{-1}$. So $c\gamma = a\alpha\gamma = z = b\alpha\gamma = b\gamma$. Since c is arbitrary, γ is a constant map, which is a contradiction.

For the sufficiency, see Theorem 1.3(ii) and Corollary 2.4. □

Furthermore, by Theorems 1.3 and 3.1, we have the following corollaries.

Corollary 3.2. *The elements of $AM(X)$ that are left and right compatible with $\leqslant$ are the permutations of X.*

Corollary 3.3. *For $\gamma \in AM(X)$, γ is left (respectively, right) compatible on $(AM(X), \leqslant)$ if and only if γ is left (respectively, right) compatible on $(T(X), \leqslant)$.*

Theorem 3.4. *Let $\gamma \in AE(X)$. Then:*

(i) *γ is left compatible on $(AE(X), \leqslant)$ if and only if γ is onto and $x\gamma^{-1}$ is a finite set for all $x \in X$;*
(ii) *γ is right compatible on $(AE(X), \leqslant)$ if and only if γ is one-to-one.*

Proof. (i) Assume that γ is left compatible with $\leqslant$ on $AE(X)$. Then, according to the proof of Theorem 3.1(i), γ is onto. Let $x \in X$. Then choose $y \in X \setminus \{x\}$ and define an idempotent β in $AE(X)$ by

$$\beta = \begin{pmatrix} x & a \\ y & a \end{pmatrix}_{a \in X\setminus\{x\}}.$$

By Remark 1.1(ii), we have $\beta \leqslant 1_X$ on $AE(X)$. Since γ is left compatible, $\gamma\beta \leqslant \gamma$ on $AE(X)$. A surjection γ and Theorem 2.6 imply that $x\gamma^{-1} = (\operatorname{ran}\gamma \setminus \operatorname{ran}\gamma\beta)\gamma^{-1}$ is finite.

Conversely, suppose that γ is onto and $x\gamma^{-1}$ is finite for all $x \in X$. From Theorem 1.3(i), we have that γ is left compatible with $\leqslant$ on $T(X)$. Let $\alpha \leqslant \beta$ on $AE(X)$. Remark 1.1(iii), Theorem 1.3(i) and Theorem 2.6 imply that $\alpha \leqslant \beta$ and $\gamma\alpha \leqslant \gamma\beta$ on $T(X)$ and $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite, respectively. By assumption, $(\operatorname{ran}\gamma\beta \setminus \operatorname{ran}\gamma\alpha)(\gamma\beta)^{-1} = (\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)(\gamma\beta)^{-1} = (\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}\gamma^{-1}$ is finite. Then, by Corollary 2.7, $\gamma\alpha \leqslant \gamma\beta$ on $AE(X)$. Hence γ is left compatible with $\leqslant$ on $AE(X)$.

(ii) The necessity is obtained from the proof of Theorem 3.1(ii).

For the sufficiency, suppose that γ is one-to-one. Let $\alpha, \beta \in AE(X)$ be such that $\alpha \leqslant \beta$ on $AE(X)$. By Corollary 2.7, $\alpha \leqslant \beta$ on $T(X)$ and $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is finite. From Theorem 1.3(ii), it follows that $\alpha\gamma \leqslant \beta\gamma$ on $T(X)$. We claim that $\alpha\gamma \leqslant \beta\gamma$ on $AE(X)$. By Corollary 2.7, it suffices to show that $(\operatorname{ran}\beta\gamma \setminus \operatorname{ran}\alpha\gamma)(\beta\gamma)^{-1}$ is finite. Consider

$$\begin{aligned}(\operatorname{ran}\beta\gamma \setminus \operatorname{ran}\alpha\gamma)(\beta\gamma)^{-1} &= (\operatorname{ran}\beta\gamma \setminus \operatorname{ran}\alpha\gamma)\gamma^{-1}\beta^{-1} \\ &= (\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\gamma\gamma^{-1}\beta^{-1} \quad \text{(since } \gamma \text{ is one-to-one)} \\ &= (\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}.\end{aligned}$$

Since $(\operatorname{ran}\beta \setminus \operatorname{ran}\alpha)\beta^{-1}$ is a finite set, we have the claim. Therefore, γ is right compatible on $(AE(X), \leqslant)$. □

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Corollary 3.5. *For $\gamma \in AE(X)$, the following assertions hold:*

(i) *γ is left compatible on $(AE(X), \leqslant)$ if and only if γ is left compatible on $(T(X), \leqslant)$ and $x\gamma^{-1}$ is finite for all $x \in X$;*

(ii) *γ is right compatible on $(AE(X), \leqslant)$ if and only if γ is right compatible on $(T(X), \leqslant)$.*

Next, we will show that the left elements compatible with $\leqslant$ on $AE(X)$ and $T(X)$ are different. Choose A, B to be disjoint subsets of X such that $|A| = |B| = |X|$ and $A \cup B = X$. Let $a \in A$ and $b \in B$. Recall $\alpha, \beta \in AE(X)$ with $\alpha \leqslant \beta$ on $AE(X)$ from Example 2.9. That is,

$$\alpha = \begin{pmatrix} \{a, b\} & x \\ a & x \end{pmatrix}_{x \in X \setminus \{a,b\}}, \quad \beta = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \setminus \{a,b\}}.$$

Let ϕ be a bijection from $B \setminus \{b\}$ onto $X \setminus \{a, b\}$. Define a surjection

$$\gamma = \begin{pmatrix} A & b & x \\ a & b & x\phi \end{pmatrix}_{x \in B \setminus \{b\}}.$$

By Theorem 1.3(i), we have that γ is left compatible on $T(X)$. Consider

$$\gamma\alpha = \begin{pmatrix} A \cup \{b\} & x \\ a & x\phi \end{pmatrix}_{x \in B \setminus \{b\}} \quad \text{and} \quad \gamma\beta = \begin{pmatrix} A & b & x \\ b & a & x\phi \end{pmatrix}_{x \in B \setminus \{b\}}.$$

Then $(\operatorname{ran} \gamma\beta \setminus \operatorname{ran} \gamma\alpha)(\gamma\beta)^{-1} = A$ is infinite. Also, by Theorem 2.6, we have $\gamma\alpha \nleqslant \gamma\beta$ on $AE(X)$. Hence γ is not left compatible with $\leqslant$ on $AE(X)$.

Then the cardinality of X and the left compatibility are related as follows.

Corollary 3.6. *The following assertions are equivalent.*

(i) *X is a finite set.*

(ii) *For $\gamma \in AE(X)$, γ is left compatible on $(AE(X), \leqslant)$ if and only if γ is left compatible on $(T(X), \leqslant)$.*

4. Minimal and maximal elements

In [4], the authors characterised the minimal and maximal elements in $(T(X), \leqslant)$. As stated before, if X is finite, $T(X)$, $AM(X)$ and $AE(X)$ are the same semigroup. Then, throughout this section, we assume that X is an infinite set, and we will characterise the minimal and maximal elements in $(AM(X), \leqslant)$ and $(AE(X), \leqslant)$.

Theorem 4.1. *Let $\alpha \in AM(X)$. Then:*

(i) *$AM(X)$ has no minimal elements;*

(ii) *α is a maximal element in $(AM(X), \leqslant)$ if and only if α is one-to-one or onto.*

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Proof. (i) Since X is an infinite set, α is not a constant map. Then there exist distinct elements $a, b \in \operatorname{ran}\alpha$. Since $\alpha \in AM(X)$, $a\alpha^{-1} \cup b\alpha^{-1}$ is finite and $\beta \in AM(X)$, where

$$\beta = \begin{pmatrix} a\alpha^{-1} \cup b\alpha^{-1} & x \\ a & x\alpha \end{pmatrix}_{x \in X \setminus (a\alpha^{-1} \cup b\alpha^{-1})}.$$

We claim that $\beta \leqslant \alpha$ on $AM(X)$. Clearly, $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. Since $\alpha^{-1}|_{\operatorname{ran}\beta} \subseteq \beta^{-1}$, we have $\beta\alpha^{-1} \subseteq \beta\beta^{-1}$. Let $(x, y) \in \alpha\alpha^{-1}$. Then $(x, z) \in \alpha$ and $(z, y) \in \alpha^{-1}$ for some $z \in X$.

Case 1: $x \in a\alpha^{-1} \cup b\alpha^{-1}$. Then $z = x\alpha \in \{a, b\}$ and $(x, a) \in \beta$. Since $y \in z\alpha^{-1}$, we have $(a, y) \in \beta^{-1}$.

Case 2: $x \notin a\alpha^{-1} \cup b\alpha^{-1}$. Then $x\beta = x\alpha = z = y\alpha = y\beta$, so $(x, x\alpha) \in \beta$ and we have $(x\alpha, y) \in \beta^{-1}$.

In either case, $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Thus $\beta \leqslant \alpha$ on $AM(X)$ and $\beta \neq \alpha$.

(ii) Suppose α is neither one-to-one nor onto. Then there exists $d \in X \setminus \operatorname{ran}\alpha$ and there are distinct elements $a, b \in X$ such that $a\alpha = b\alpha$. Let $\beta \in AM(X)$ be such that

$$\beta = \begin{pmatrix} b & x \\ d & x\alpha \end{pmatrix}_{x \in X \setminus \{b\}}.$$

Clearly, $\alpha \neq \beta$. Theorem 1.2 shows that $\alpha \leqslant \beta$ on $T(X)$. By Corollary 2.4, we have $\alpha \leqslant \beta$ on $AM(X)$, which implies that α is not a maximal element in $(AM(X), \leqslant)$.

The converse follows immediately from Theorem 1.4(ii). □

The next corollary follows directly from Theorems 1.4 and 4.1.

Corollary 4.2. *For $\alpha \in AM(X)$, α is a minimal (respectively, maximal) element in $(AM(X), \leqslant)$ if and only if α is a minimal (respectively, maximal) element in $(T(X), \leqslant)$.*

Theorem 4.3. *Let $\alpha \in AE(X)$. Then:*

(i) *α is a minimal element in $(AE(X), \leqslant)$ if and only if $x\alpha^{-1}$ is an infinite set for all $x \in \operatorname{ran}\alpha$;*

(ii) *α is a maximal element in $(AE(X), \leqslant)$ if and only if α is one-to-one or onto.*

Proof. (i) Suppose there exists $b \in \operatorname{ran}\alpha$ such that $b\alpha^{-1}$ is finite. Since X is an infinite set and $X \setminus \operatorname{ran}\alpha$ is finite, we can choose $a \in \operatorname{ran}\alpha \setminus \{b\}$. Define

$$\beta = \begin{pmatrix} a\alpha^{-1} \cup b\alpha^{-1} & x \\ a & x\alpha \end{pmatrix}_{x \in X \setminus (a\alpha^{-1} \cup b\alpha^{-1})}.$$

Since $\alpha \in AE(X)$, $X \setminus \operatorname{ran}\beta = (X \setminus \operatorname{ran}\alpha) \cup \{b\}$ is finite, which implies that $\beta \in AE(X)$. By the proof of Theorem 4.1(i), $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, $\beta\alpha^{-1} \subseteq \beta\beta^{-1}$ and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. We know that $(\operatorname{ran}\alpha \setminus \operatorname{ran}\beta)\alpha^{-1} = b\alpha^{-1}$ is finite. Thus $\beta \leqslant \alpha$ on $AE(X)$ and $\beta \neq \alpha$.

Conversely, suppose $x\alpha^{-1}$ is infinite for all $x \in \operatorname{ran}\alpha$. Assume that there is $\beta \in AE(X)$ such that $\beta \leqslant \alpha$ on $AE(X)$. We claim that $\operatorname{ran}\beta = \operatorname{ran}\alpha$. By Theorem 2.6(i), $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. Suppose that there is $b \in \operatorname{ran}\alpha \setminus \operatorname{ran}\beta$. By Theorem 2.6(iv), $b\alpha^{-1} \subseteq (\operatorname{ran}\alpha \setminus \operatorname{ran}\beta)\alpha^{-1}$, which is a finite set. This is a contradiction. Then $\operatorname{ran}\beta = \operatorname{ran}\alpha$. Let $x \in X$. Since $\operatorname{ran}\alpha = \operatorname{ran}\beta$, there is $a \in X$ such that $a\beta = x\alpha$. By Theorem 2.6(ii),

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we have $(a, x) \in \beta\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $x\beta = a\beta = x\alpha$. Therefore, $\alpha = \beta$. Hence α is a minimal element with respect to $\leqslant$.

(ii) Suppose that α is neither one-to-one nor onto. By the proof of Theorem 4.1(ii), we have $\beta \in AE(X) \setminus \{\alpha\}$ such that $\alpha \leqslant \beta$ on $AM(X)$. It follows from Corollary 2.4 that $\alpha \leqslant \beta$ on $T(X)$. Then

$$\beta = \begin{pmatrix} b & x \\ d & x\alpha \end{pmatrix}_{x \in X \setminus \{b\}},$$

where a and b are distinct elements such that $a\alpha = b\alpha$ and $d \notin \operatorname{ran} \alpha$. Consider $(\operatorname{ran} \beta \setminus \operatorname{ran} \alpha)\beta^{-1} = d\beta^{-1} = \{b\}$. By Corollary 2.7, $\alpha \leqslant \beta$ on $AE(X)$. Hence α is not maximal in $(AE(X), \leqslant)$.

The converse follows directly from Theorem 1.4(ii). □

We will show that $(AE(X), \leqslant)$ certainly has a minimal element if X is infinite.

EXAMPLE 4.4. Since X is an infinite set, $|X| = |X \times X|$. Let ϕ be a bijection from X onto $X \times X$. Observe that:

(i) $\bigcup_{x \in X}(\{x\} \times X)\phi^{-1} = X$;
(ii) $(\{x\} \times X)\phi^{-1} \cap (\{y\} \times X)\phi^{-1} = \emptyset$ for distinct elements x and y in X, since ϕ is one-to-one.

Define $\alpha \in AE(X)$ by

$$\alpha = \begin{pmatrix} (\{x\} \times X)\phi^{-1} \\ x \end{pmatrix}_{x \in X}.$$

Since $\{x\} \times X$ is infinite and ϕ is a bijection, $x\alpha^{-1} = (\{x\} \times X)\phi^{-1}$ is an infinite set for all $x \in X$. Then α is a minimal element in $(AE(X), \leqslant)$.

The following two corollaries are obtained from Theorems 1.4 and 4.3.

COROLLARY 4.5. *The following assertions are equivalent.*

(i) *X is a finite set.*
(ii) *For $\alpha \in AE(X)$, α is a minimal element in $(AE(X), \leqslant)$ if and only if α is a minimal element in $(T(X), \leqslant)$.*

COROLLARY 4.6. *For $\alpha \in AE(X)$, α is a maximal element in $(AE(X), \leqslant)$ if and only if α is a maximal element in $(T(X), \leqslant)$.*

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