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On a class of difference operator and its applications to a family of analytic functions

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This paper mainly considers the problem of generalizing a certain class of analytic functions by means of a class of difference operators. We consider some relations between starlike or convex functions and functions belonging to such classes. Some other useful properties of these classes are also considered.

Keywords: q-calculus; p,q-difference operator; q-derivative; analytic functions; q-starlike functions

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1. Introduction and preliminaries

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{D}$. Let $f(z) \in \mathcal{H}$ and let $f(z)$ be univalent in $\mathbb{D}$. Then $f(z)$ maps $\mathbb{D}$ onto a convex domain if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{D}). \tag{1.1}$$

Such function f is said to be convex in $\mathbb{D}$ (or briefly convex). Let $\mathcal{K}$ denote the subclass of $\mathcal{H}$ consisting of functions satisfying (1.1) and normalized by $f(0) = 0$,

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$f'(0) = 1$. A function $f \in \mathcal{S}$ is said to be starlike of order α if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathbb{D}), \tag{1.2}$$

for some $0 \leqslant \alpha < 1$. The class of functions starlike of order α is denoted by $\mathcal{S}^*(\alpha)$.

Jackson in [13, 14] introduced and studied the q-difference operator, $0 < q < 1$, as

$$\mathrm{d}_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad z \neq 0 \text{ and } \quad \mathrm{d}_q f(0) = f'(0). \tag{1.3}$$

This operator is the q-analogue of the derivative, which is also called the q-derivative or the Jackson derivative. Obviously, $\mathrm{d}_q f(z) \to f'(z)$, when $q \to 1^-$. This q-derivative is useful in the theory of hypergeometric series and quantum physics. Jackson's derivative is a part of a field called q-calculus (quantum calculus), which has many applications in combinatorics, number theory, fluid mechanics, quantum mechanics and physics. The quantum calculus has many applications in the fields of special functions and many other areas (see [1–6]). Further there is possibility of extension of the q-calculus to post quantum calculus denoted by the p, q-calculus. In [15] Chakrabarti and Jagannathan introduced a consideration of the p, q-integer in order to generalize or unify several forms of q-oscillator algebras well known in the physics literature related to the representation theory of single-parameter quantum algebras (see also [3–5, 16]).

1.1. ξ, η-difference operator

Let us begin by defining a basic number $[n]_{\xi,\eta}$ called a ξ, η number by

$$[n]_{\xi,\eta} = \frac{\xi^n - \eta^n}{\xi - \eta}, \quad (\xi, \eta \in \mathbb{C},\ \xi \neq \eta,\ n \in \mathbb{N} \setminus \{1\}).$$

We consider the function

$$h_{\xi,\eta}(z) = \sum_{n=1}^{\infty} [n]_{\xi,\eta} z^n = z + \sum_{n=2}^{\infty} \frac{\xi^n - \eta^n}{\xi - \eta} z^n, \tag{1.4}$$

for $\xi, \eta \in \mathbb{D}$. It is easy to see that if $\xi, \eta \in \mathbb{D}$, then (1.4) converges for $|z| < 1$. The ξ, η-difference operator on $f \in \mathcal{A}$ is

$$\mathrm{d}_{\xi,\eta} f(z) = \frac{f(\xi z) - f(\eta z)}{\xi z - \eta z}, \quad z \neq 0 \qquad \text{and} \quad \mathrm{d}_{\xi,\eta} f(0) = f'(0). \tag{1.5}$$

It is easy to find that for $f \in \mathcal{A}$

$$\mathrm{d}_{\xi,\eta} f(z) = \frac{1}{z} \sum_{n=1}^{\infty} [n]_{\xi,\eta} a_n z^n = \frac{1}{z} \left\{ h_{\xi,\eta}(z) * f(z) \right\}, \quad z \in \mathbb{D}, \tag{1.6}$$

where $*$ denotes the Hadamard product of power series (1.4) and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.7}$$

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Lemma 1.1. [19] *If* $f \in \mathcal{S}^*(1/2)$ *and* $g \in \mathcal{S}^*(1/2)$ [*or if* $f \in \mathcal{K}$ *and* $g \in \mathcal{S}^*$], *then*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\mathrm{co}}\{F(\mathbb{D})\}, \quad z \in \mathbb{D}, \tag{1.8}$$

where $F \in \mathcal{H}$ *and* $\overline{\mathrm{co}}\{F(\mathbb{D})\}$ *denotes the closed convex hull of* $F(\mathbb{D})$.

1.2. Other known difference operators

If ξ and η are real then $\mathrm{d}_{\xi,\eta}$ becomes a p,q-difference operator. Corcino [7] developed the theory of a p,q-extension of the binomial coefficients and also established some properties parallel to those of the ordinary and q-binomial coefficients, which comprised horizontal generating function, the triangular, vertical and the horizontal recurrence relations and the inverse and the orthogonality relations. Sadjang [20] investigated some properties of the p,q-derivatives and the p,q-integrations. Sadjang [20] also provided two suitable polynomial bases for the p,q-derivative and gave various properties of these bases.

The p,q-derivative operator

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad (p \neq q, x \neq 0) \tag{1.9}$$

was perhaps first used in [8]. The operator is also mentioned in the paper [9].

In [10] Hahn introduced the Hahn difference operator:

$$D_{q,\omega}f(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, \quad (0 < q < 1, \omega > 0) \tag{1.10}$$

and defined the q-analogues of the trigonometric functions. When $\omega = 0$, (1.10) becomes Jackson's derivative operator. On the other hand, when $q \to 1$, (1.10) becomes the familiar forward difference operator of difference calculus. The operator (1.10) is a useful tool in constructing families of orthogonal polynomials and investigating certain approximation problems, see, for instance the paper [4].

If E is a given interval and $\beta : E \to E$, $\beta \neq x$, is strictly monotonically increasing function, i.e., $x > t \Rightarrow \beta(x) > \beta(t)$, then β derivative of a function $f(x)$ is defined by

$$D_\beta f(x) = \frac{f(\beta(x)) - f(x)}{\beta(x) - x}. \tag{1.11}$$

If $\beta(x) = qx$, then (1.11) becomes the q-derivative operator. Further, for $\beta(x) = qx + \omega$, the operator (1.11) reduces to the Hahn difference operator (1.10). The operator (1.11) is found in the papers [11].

2. Main results

Theorem 2.1. *If* f *is in the class* $\mathcal{K}$ *of convex univalent functions, then for* ξ, η *such that* $\xi, \eta \in \mathbb{D}$, *the function given by*

$$z\mathrm{d}_{\xi,\eta}f(z) = z + \sum_{n=2}^{\infty}[n]_{\xi,\eta}a_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

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is in the class $\mathcal{S}^*(\alpha)$ of starlike univalent functions of order $\alpha = \frac{1-|\xi\eta|}{(1+|\xi|)(1+|\eta|)}$. The order α is the best possible.

Proof. It is easy to check that

$$h_{\xi,\eta}(z) = \sum_{n=1}^{\infty} [n]_{\xi,\eta} z^n = \frac{z}{(1-\xi z)(1-\eta z)}, \quad z \in \mathbb{D}. \tag{2.2}$$

Hence

$$\frac{zh'_{\xi,\eta}(z)}{h_{\xi,\eta}(z)} = 1 + \frac{\xi z}{1-\xi z} + \frac{\eta z}{1-\eta z}.$$

Because

$$\mathfrak{Re}\frac{\xi z}{1-\xi z} > \frac{-|\xi|}{1+|\xi|}, \quad \mathfrak{Re}\frac{\eta z}{1-\eta z} > \frac{-|\eta|}{1+|\eta|}, \quad z \in \mathbb{D},$$

we have

$$\begin{aligned}\mathfrak{Re}\frac{zh'_{\xi,\eta}(z)}{h_{\xi,\eta}(z)} &> 1 - \frac{|\xi|}{1+|\xi|} - \frac{|\eta|}{1+|\eta|} \\ &= \frac{1-|\xi||\eta|}{(1+|\xi|)(1+|\eta|)}, \quad z \in \mathbb{D}.\end{aligned}$$

Therefore, if $\xi, \eta \in \mathbb{D}$, then $h_{\xi,\eta}$ is starlike univalent in $\mathbb{D}$ of order $\frac{1-|\xi||\eta|}{(1+|\xi|)(1+|\eta|)}$. On the other hand

$$z\mathrm{d}_{\xi,\eta}f(z) = \left\{z + \sum_{n=2}^{\infty} a_n z^n\right\} * \sum_{n=1}^{\infty}[n]_{\xi,\eta} z^n = f(z) * \frac{z}{(1-\xi z)(1-\eta z)}. \tag{2.3}$$

Hence $z\mathrm{d}_{\xi,\eta}f$ is a convolution of f with a starlike function of order $\alpha = \frac{1-|\xi||\eta|}{(1+|\xi|)(1+|\eta|)}$. Because of the famous result [**19**] that $\mathcal{K} * \mathcal{S}^*(\delta) = \mathcal{S}^*(\delta)$, $\delta \in [0,1)$, we finally obtain that the function in (2.1) is in the class $\mathcal{S}^*(\frac{1-|\xi||\eta|}{(1+|\xi|)(1+|\eta|)})$. The order α is the best possible because if we take convex function $f(z) = z/(1-z)$, then

$$z\mathrm{d}_{\xi,\eta}f(z) = h_{\xi,\eta}(z)$$

and the order of starlikeness of $z\mathrm{d}_{\xi,\eta}f(z)$ is equal to the order of starlikeness of $h_{\xi,\eta}(z)$ which is α as we just have proved. □

It is known that

$$\forall f \in \mathcal{S}^* \exists\ g \in \mathcal{K}: \quad f(z) = zg'(z).$$

A question worth considering here is the following:

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Is it true that

$$\forall f \in \mathcal{S}^* \ \exists\, g \in \mathcal{K}\ \exists \xi, \eta \in \mathbb{D}: \quad f(z) = z d_{\xi,\eta} g(z) \quad ?$$

In terms of the convolution, this problem becomes: Are there $\xi, \eta \in \mathbb{D}$ such that for given $f(z) = z + a_2 z^2 + \cdots \in \mathcal{S}^*$, the function

$$z + \sum_{n=2}^{\infty} \frac{a_n}{[n]_{\xi,\eta}} z^n \in \mathcal{K} \quad ? \tag{2.4}$$

The answer of the question (2.4) is 'no'. Namely, for the starlike function

$$\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D},$$

$a_n = n$ and the function in (2.4) becomes

$$z + \sum_{n=2}^{\infty} \frac{n}{[n]_{\xi,\eta}} z^n,$$

which is not in the class $\mathcal{K}$ because it has the coefficients $n/[n]_{\xi,\eta}$ and

$$\left| \frac{n}{[n]_{\xi,\eta}} \right| = \frac{n}{|\xi^{n-1} + \xi^{n-2}\eta + \cdots + \xi\eta^{n-2} + \eta^{n-1}|} > 1.$$

THEOREM 2.2. *If f is in the class $\mathcal{K}$ of convex univalent functions, then*

$$\mathfrak{Re}\left\{ \frac{1}{1-\eta} \frac{\mathrm{d}_{\xi,\eta} f(z)}{\mathrm{d}_{\xi} f(\eta z)} \right\} = \mathfrak{Re}\left\{ \frac{1}{1-\eta} \frac{\mathrm{d}_{\xi,\eta} f(z)}{\mathrm{d}_{\xi\eta,\eta} f(z)} \right\} > \mathfrak{Re}\left\{ \frac{1+\eta}{2(1-\eta)} \right\}, \quad z \in \mathbb{D}, \tag{2.5}$$

for all ξ, $\eta \in \mathbb{D}$.

Proof. It is known from [19], [p.10], that if $f \in \mathcal{K}$, then for all ζ, v and $w \in \mathbb{D}$, we have

$$\mathfrak{Re}\left\{ \frac{\zeta}{\zeta - v} \frac{v-w}{\zeta - w} \frac{f(\zeta) - f(w)}{f(v) - f(w)} - \frac{v}{\zeta - v} \right\} > \frac{1}{2}. \tag{2.6}$$

If we put $\zeta = \xi z$, $w = \eta z$ and $v = \xi\eta z$ in (2.6), then we obtain

$$\mathfrak{Re}\left\{ \frac{1}{1-\eta} \frac{f(\xi z) - f(\eta z)}{z(\xi - \eta)} \left[\frac{f(\xi\eta z) - f(\eta z)}{\eta z(\xi - 1)} \right]^{-1} \right\} - \mathfrak{Re}\left\{ \frac{\eta}{1-\eta} \right\} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Trivial calculations give

$$\mathfrak{Re}\left\{ \frac{1}{1-\eta} \frac{f(\xi z) - f(\eta z)}{z(\xi - \eta)} \left[\frac{f(\xi\eta z) - f(\eta z)}{\eta z(\xi - 1)} \right]^{-1} \right\} > \mathfrak{Re}\left\{ \frac{1+\eta}{2(1-\eta)} \right\}, \quad z \in \mathbb{D}.$$

This gives (2.5). □

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THEOREM 2.3. *If f is in the class $\mathcal{K}$ of convex univalent functions, then, we have*

$$\mathfrak{Re}\left\{\frac{\xi^2}{\eta-\xi^2}\frac{\mathrm{d}_{\xi^2,\xi}f(z)}{\mathrm{d}_{\xi,\eta}f(z)}\right\}=\mathfrak{Re}\left\{\frac{\xi^2}{\eta-\xi^2}\frac{\mathrm{d}_{\xi}f(\xi z)}{\mathrm{d}_{\xi,\eta}f(z)}\right\}<\mathfrak{Re}\left\{\frac{\eta+\xi^2}{2(\eta-\xi^2)}\right\},\quad z\in\mathbb{D}, \tag{2.7}$$

for all $\xi,\eta\in\mathbb{D}$.

Proof. From (2.6), we have that for $f\in\mathcal{K}$, for all t,v and $w\in\mathbb{D}$, we have

$$\mathfrak{Re}\left\{\frac{t}{t-v}\frac{v-w}{t-w}\frac{f(t)-f(w)}{f(v)-f(w)}-\frac{v}{t-v}\right\}>\frac{1}{2}. \tag{2.8}$$

If we put $v=\eta z$, $w=\xi z$ and $t=\xi^2 z$ in (2.8), then we obtain

$$\mathfrak{Re}\left\{\frac{\xi^2 z}{\xi^2 z-\eta z}\frac{\eta z-\xi z}{\xi^2 z-\xi z}\frac{f(\xi^2 z)-f(\xi z)}{f(\eta z)-f(\xi z)}-\frac{\eta z}{\xi^2 z-\eta z}\right\}>\frac{1}{2}.$$

After some calculations, we obtain

$$\mathfrak{Re}\left\{\frac{\xi^2}{\xi^2-\eta}\frac{f(\xi^2 z)-f(\xi z)}{\xi^2 z-\xi z}\frac{\eta z-\xi z}{f(\eta z)-f(\xi z)}-\frac{\eta}{\xi^2-\eta}\right\}>\frac{1}{2}$$

or

$$\mathfrak{Re}\left\{\frac{\xi^2}{\xi^2-\eta}\frac{\mathrm{d}_{\xi^2,\xi}f(z)}{\mathrm{d}_{\xi,\eta}f(z)}-\frac{\eta}{\xi^2-\eta}\right\}>\frac{1}{2}.$$

This proves (2.7). □

COROLLARY 2.4. *If f is in the class $\mathcal{K}$ of convex univalent functions, then we have*

$$\mathfrak{Re}\left\{\frac{\xi^2}{\eta-\xi^2}\frac{\frac{1}{1-\xi z}*\mathrm{d}_{\xi}f(z)}{\mathrm{d}_{\xi,\eta}f(z)}\right\}<\mathfrak{Re}\left\{\frac{\eta+\xi^2}{2(\eta-\xi^2)}\right\},\quad z\in\mathbb{D},$$

for all $\xi,\eta\in\mathbb{D}$, $\eta\neq\xi^2$.

COROLLARY 2.5. *If f is in the class $\mathcal{K}$ of convex univalent functions, then we have*

$$\mathfrak{Re}\left\{\frac{\eta z\mathrm{d}_{\xi,\eta}f(z)}{f(\eta z)}\right\}>\frac{1}{2},\quad z\in\mathbb{D} \tag{2.9}$$

and

$$\mathfrak{Re}\left\{\frac{\xi}{\xi-\eta}\left(\mathrm{d}_{\xi,\eta}f(z)-\frac{\eta}{\xi}\right)\right\}>\frac{1}{2},\quad z\in\mathbb{D}, \tag{2.10}$$

for all $\xi,\eta\in\mathbb{D}$.

Proof. If we put $v=0$, $w=\eta z$ and $t=\xi z$ in (2.8), then we obtain

$$\mathfrak{Re}\left\{\frac{\eta z}{f(\eta z)}\frac{f(\xi z)-f(\eta z)}{\xi z-\eta z}\right\}>\frac{1}{2},\quad z\in\mathbb{D}.$$

After some calculations, we obtain (2.9). If we put $v\to w=\eta z$ and $t=\xi z$ in (2.8), then we obtain (2.10), in the same way. □

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The class of starlike functions of order α is defined by condition (1.2). We want to consider here this condition with the operator $\mathrm{d}_{\xi,\eta}$ instead of derivative f'. For the purpose of this paper, we represent by $\mathcal{S}^*_{\xi,\eta}(\alpha)$ a class which is defined by

Definition 2.6. *Let $f \in \mathcal{A}$. For given $\xi, \eta \in \mathbb{D}$, we say that f is in the class $\mathcal{S}^*_{\xi,\eta}(\alpha)$ of ξ, η-starlike functions of order α, $0 \leqslant \alpha < 1$, if*

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_{\xi,\eta}f(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}, \tag{2.11}$$

where the operator $\mathrm{d}_{\xi,\eta}$ is defined in (1.5).

Condition (2.11) may be written as

$$\mathfrak{Re}\left\{\frac{f(\xi z) - f(\eta z)}{\xi z - \eta z}\frac{z}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}. \tag{2.12}$$

Remark 2.7. For $\xi \to \eta$ condition (2.11) becomes

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}, \tag{2.13}$$

and the class $\mathcal{S}^*_{\xi,\eta}(\alpha)$ tends to the well-known class $\mathcal{S}^*(\alpha)$ of starlike functions of order α.

Remark 2.8. It is known that condition (2.13) implies the univalence of f, whenever $f \in \mathcal{A}$. Notice that the condition (2.11) does not imply that f is univalent in $\mathbb{D}$. For example, it is known that $f(z) = z + (3/4)z^2$ is not univalent in $\mathbb{D}$, while $f \in \mathcal{S}^*_{1/2,1/4}(0)$ because for this function f we have

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_\zeta f(z)}{f(z)}\right\} = \mathfrak{Re}\left\{\frac{1 + (3/4)^2 z}{1 + (3/4)z}\right\} > \frac{25}{28}, \quad z \in \mathbb{D}.$$

Theorem 2.9. *The function $g(z) = z + cz^2$ is in the class $\mathcal{S}^*_{\xi,\eta}(\alpha)$, if and only if,*

$$\mathfrak{Re}\left\{\frac{1 - |c|^2(\xi + \eta) - |c||\xi + \eta - 1|}{1 - |c|^2}\right\} > \alpha. \tag{2.14}$$

Proof. We have

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_{\xi,\eta}g(z)}{g(z)}\right\} = \mathfrak{Re}\left\{\frac{1 + c[2]_{\xi,\eta}z}{1 + cz}\right\} = \mathfrak{Re}\left\{\frac{1 + c(\xi + \eta)z}{1 + cz}\right\}.$$

The function

$$z \mapsto \frac{1 + c(\xi + \eta)z}{1 + cz}$$

maps $\mathbb{D}$ onto a disc centred at S with radius R, where

$$S = \frac{1 - |c|^2(\xi + \eta)}{1 - |c|^2}, \quad R = \frac{|c||\xi + \eta - 1|}{1 - |c|^2}.$$

Therefore, $g \in \mathcal{S}^*_{\xi,\eta}(\alpha)$, if and only if, $\mathfrak{Re}(S - R) > \alpha$, which gives (2.14). □

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COROLLARY 2.10. *For all $\xi, \eta \in \mathbb{D}$ in the classes $\mathcal{S}^*_{\xi,\eta}(0)$ are not univalent functions in $\mathbb{D}$.*

Proof. It suffices to consider here a function $h(z)$ defined by

$$h(z) = z + \frac{1}{\xi + \eta} z^2, \quad z \in \mathbb{D}.$$

In view of the definition 2.6, we have

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_{\xi,\eta}h(z)}{h(z)}\right\} = \mathfrak{Re}\left\{\frac{1+z}{1+\frac{1}{\xi+\eta}z}\right\} > 0, \quad z \in \mathbb{D},$$

which implies that $h \in \mathcal{S}^*_{\xi,\eta}(0)$. But we observe that for the function $h(z)$ we have $|1/(\xi+\eta)| > 1/2$, therefore the function $h(z)$ is not univalent which validates the assertion of the corollary 2.10. □

THEOREM 2.11. *If f is in the class $\mathcal{S}^*(1/2)$ of starlike functions of order $1/2$, then*

$$\frac{z\mathrm{d}_{\xi,\eta}f(z)}{f(z)} \in \overline{\mathrm{co}}\{F(\mathbb{D})\},$$

for all $\xi, \eta \in \mathbb{D}$ where

$$F(z) = \frac{1-z}{(1-\xi z)(1-\eta z)}, \quad z \in \mathbb{D}.$$

Proof. Note that

$$g(z) = \frac{z}{1-z} \in \mathcal{S}^*(1/2), \quad z \in \mathbb{D}.$$

Therefore, lemma 1.1 gives

$$\frac{z\mathrm{d}_{\xi,\eta}f(z)}{f(z)} = \frac{f(z) * \frac{z}{1-z}\,\frac{\frac{z}{(1-\xi z)(1-\eta z)}}{\frac{z}{1-z}}}{f(z) * \frac{z}{1-z}} \in \overline{\mathrm{co}}\{F(\mathbb{D})\}, \tag{2.15}$$

where

$$F(z) = \frac{\frac{z}{(1-\xi z)(1-\eta z)}}{\frac{z}{1-z}}, \quad z \in \mathbb{D},$$

for all $\xi, \eta \in \mathbb{D}$. □

COROLLARY 2.12. *If f is in the class $\mathcal{S}^*(1/2)$ of starlike functions of order $1/2$, then $f \in \mathcal{S}^*_{\xi,\eta}(0)$ for all real $\xi, \eta \in (-1, 1)$, such that*

$$1 + \xi + \eta - 3\xi\eta \geqslant 0 \quad \textit{and} \quad \xi\eta \geqslant 0 \tag{2.16}$$

or

$$1 + \xi + \eta + \xi\eta \geqslant 0 \quad \textit{and} \quad \xi\eta < 0. \tag{2.17}$$

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Proof. From theorem 2.11, we have for $\xi \neq \eta$

$$F(z) = \frac{1-z}{(1-\xi z)(1-\eta z)} = \frac{\frac{\xi-1}{\xi-\eta}}{1-\xi z} + \frac{\frac{1-\eta}{\xi-\eta}}{1-\eta z}$$

$$\mathfrak{Re}\left\{\frac{z\mathrm{d}_{\xi,\eta}f(z)}{f(z)}\right\} > \min_{z\in\mathbb{D}} \mathfrak{Re}\left\{\frac{\frac{\xi-1}{\xi-\eta}}{1-\xi z} + \frac{\frac{1-\eta}{\xi-\eta}}{1-\eta z}\right\}.$$

Assume that $z = \cos\phi + i\sin\phi$, then after some calculations, we have

$$\mathfrak{Re}\left\{\frac{\frac{\xi-1}{\xi-\eta}}{1-\xi z}\right\} = \frac{\xi-1}{\xi-\eta}\frac{1-\xi\cos\phi}{(1-\xi\cos\phi)^2+(\xi\sin\phi)^2} = \frac{\xi-1}{\xi-\eta}\frac{1-\xi\cos\phi}{1+\xi^2-2\xi\cos\phi}$$

also

$$\mathfrak{Re}\left\{\frac{\frac{1-\eta}{\xi-\eta}}{1-\eta z}\right\} = \frac{1-\eta}{\xi-\eta}\frac{1-\eta\cos\phi}{(1-\eta\cos\phi)^2+(\eta\sin\phi)^2} = \frac{1-\eta}{\xi-\eta}\frac{1-\eta\cos\phi}{1+\eta^2-2\eta\cos\phi}.$$

Therefore, we have

$$\begin{aligned}\mathfrak{Re}&\left\{\frac{1-z}{(1-\xi z)(1-\eta z)}\right\}\\ &= \frac{1}{\xi-\eta}\ \frac{(\xi-1)(1-\xi\cos\phi)(1+\eta^2-2\eta\cos\phi) -(\eta-1)(1-\eta\cos\phi)(1+\xi^2-2\xi\cos\phi)}{(1+\xi^2-2\xi\cos\phi)(1+\eta^2-2\eta\cos\phi)}\\ &= \frac{(1-\cos\phi)\,(1+\xi+\eta-\xi\eta-2\xi\eta\cos\phi)}{(1+\xi^2-2\xi\cos\phi)(1+\eta^2-2\eta\cos\phi)}.\end{aligned}$$

For all $\phi \in [0, 2\pi)$, we have

$$1-\cos\phi \geqslant 0, \quad 1+\xi^2-2\xi\cos\phi > 0, \quad 1+\eta^2-2\eta\cos\phi > 0.$$

Furthermore,

$$[1+\xi+\eta-3\xi\eta \geqslant 0 \quad \text{and} \quad \xi\eta \geqslant 0] \quad \Rightarrow \quad 1+\xi+\eta-\xi\eta-2\xi\eta\cos\phi \geqslant 0$$

for all $\phi \in [0, 2\pi)$. Next,

$$[1+\xi+\eta+\xi\eta \geqslant 0 \quad \text{and} \quad \xi\eta < 0] \quad \Rightarrow \quad 1+\xi+\eta-\xi\eta-2\xi\eta\cos\phi \geqslant 0$$

for all $\phi \in [0, 2\pi)$. Finally,

$$\mathfrak{Re}\left\{\frac{1-z}{(1-\xi z)(1-\eta z)}\right\} = \frac{(1-\cos\phi)\,(1+\xi+\eta-\xi\eta-2\xi\eta\cos\phi)}{(1+\xi^2-2\xi\cos\phi)(1+\eta^2-2\eta\cos\phi)} \geqslant 0$$

for all $\phi \in [0, 2\pi)$ and for all real $\xi, \eta \in (-1, 1)$, $\xi \neq \eta$, satisfying (2.16) or (2.17). This means that in this case $f \in \mathcal{S}^*_{\xi,\eta}(0)$. For $\xi = \eta$, in the same way as above, we

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can obtain for $z = \cos\phi + i\sin\phi$

$$\begin{aligned}\mathfrak{Re}\{F(z)\} &= \mathfrak{Re}\left\{\frac{1-z}{(1-\xi z)^2}\right\} \\ &= \frac{(1-\cos\phi)\left(1+2\xi-\xi^2-2\xi^2\cos\phi\right)}{(1+\xi^2-2\xi\cos\phi)^2} \\ &\geqslant 0\end{aligned}$$

for all $\phi \in [0, 2\pi)$ and for all real $\xi \in (-1, 1)$, satisfying (2.16) or (2.17) with $\xi = \eta$. This means that also in this case $f \in \mathcal{S}^*_{\xi,\eta}(0)$. □

Corollary 2.12 provides some examples of functions in the class $\mathcal{S}^*_{\xi,\eta}(0)$ for all real $\xi, \eta \in (-1, 1)$, satisfying (2.16) or (2.17). It is known that $\mathcal{K} \subset \mathcal{S}^*(1/2)$, therefore corollary 2.12 leads to the following result.

Corollary 2.13. *If f is in the class $\mathcal{K}$ of convex univalent functions, then $f \in \mathcal{S}^*_{\xi,\eta}(0)$ for all real $\xi, \eta \in (-1, 1)$, satisfying (2.16) or (2.17).*

Recall here another definition of q-starlike functions of order α. Namely, making use of q-derivative (1.3), Argawal and Sahoo in [2] introduced the class $\mathcal{S}^*_q(\alpha)$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_q(\alpha)$, $0 \leqslant \alpha < 1$, if

$$\left|\frac{z\mathrm{d}_q f(z)}{f(z)} - \frac{1-\alpha q}{1-q}\right| \leqslant \frac{1-\alpha}{1-q}, \quad z \in \mathbb{D}. \tag{2.18}$$

If $q \to 1^-$ the class $\mathcal{S}^*_q(\alpha)$ reduces to the class $\mathcal{S}^*(\alpha)$. If $\alpha = 0$ the class $\mathcal{S}^*_q(\alpha)$ coincides with the class $\mathcal{S}^*_q(0) = \mathcal{S}^*_q$, which was first introduced in [**12**] by Ismail *et al.* and was considered in [**1**, **3**, **5**, **17**, **18**].

3. Conclusion

We have considered a certain classes of analytic functions by means of a difference operator which is a q-analogue of the derivative, which is also called the q-derivative or the Jackson derivative. Jackson's derivative is a part of a field called q-calculus (quantum calculus), which has many applications. Some relations between starlike or convex functions and functions belonging to the classes defined above, which have been investigated, may provide opportunity for further work on the subject.

Compliance with ethical standards

The authors declare they have no financial interests. The authors have no conflicts of interest to declare that are relevant to the content of this article.

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