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SOME EXAMPLES IN VECTOR INTEGRATION

JOSÉ RODRÍGUEZ

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Abstract

Some classical examples in vector integration due to Phillips, Hagler and Talagrand are revisited from the point of view of the Birkhoff and McShane integrals.

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1. Introduction and preliminaries

Nowadays vector integration is considered as a classical and fruitful branch of functional analysis. While the Bochner and Pettis integrals have been the preeminent notions in this context, recent studies make clear that others, such as the Birkhoff and McShane integrals, play a relevant role and can be an interesting alternative to the Pettis integral in several situations; see, for example, [1, 3–5, 8–10, 17–20, 22, 23].

In this note we revisit some classical examples which illuminated some aspects of the Pettis integral theory, but now discussing their relevance to the Birkhoff and McShane integrals. We pay attention to:

- Phillips' example of a Pettis integrable function which is not Birkhoff integrable [15, Example 10.2];
- Hagler's example of a scalarly measurable ℓ^∞-valued function which is not strongly measurable [6, p. 43];
- Talagrand's example of a bounded Pettis integrable function having no conditional expectation [24, Example 6-4-2].

Moreover, our discussion leads to several open questions which might stimulate further research on vector integration.

The books [6] and [24] are two standard references on this topic. Let us recall the definitions of the Birkhoff and McShane integrals. A function $f : \Omega \to X$, defined on a probability space (Ω, Σ, μ) and taking values in a Banach space X, is called *Birkhoff*

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integrable, with 'integral' $x \in X$, if for every $\varepsilon > 0$ there is a countable partition (A_m) of Ω in Σ such that, for any choice of points $t_m \in A_m$, the series $\sum_m \mu(A_m) f(t_m)$ converges unconditionally in X and $\|\sum_m \mu(A_m) f(t_m) - x\|_X \leq \varepsilon$.

A function $f : K \to X$, defined on a compact Radon probability space (K, Σ, μ) and taking values in a Banach space X, is called *McShane integrable*, with 'integral' $x \in X$, if for every $\varepsilon > 0$ there is a gauge δ on K (that is, a function that maps each $t \in K$ to some open set $\delta(t) \subset K$ containing t) such that the inequality

$$\left\| \sum_{i=1}^{p} \mu(E_i) f(t_i) - x \right\|_X \leq \varepsilon$$

holds for every finite partition $E_1, \ldots, E_p$ of K in Σ and every choice of points $t_i \in K$ with $E_i \subset \delta(t_i)$ for all $1 \leq i \leq p$; such a collection $\{(E_i, t_i)\}_{1 \leq i \leq p}$ is called a *McShane partition* of K *subordinate* to δ.

The relationship between these notions of integrability is

$$\text{Bochner} \Longrightarrow \text{Birkhoff} \Longrightarrow \text{McShane} \Longrightarrow \text{Pettis},$$

the corresponding 'integrals' coincide and none of the reverse arrows holds in general, see for example [8–10].

As usual, the topological dual and the closed unit ball of a Banach space X are denoted by X^* and B_X, respectively. The norm of X is denoted by $\| \cdot \|_X$.

2. Phillips' example

The first example of a Pettis integrable function which is not Birkhoff integrable was given by Phillips in [15, Example 10.2]. His function (see Example 2.1 below) is of the form $f : [0, 1] \to \ell^\infty([0, 1])$, where $[0, 1]$ is equipped with the Lebesgue measure λ. Such a function cannot be strongly measurable, since Pettis and Birkhoff integrability are always equivalent for strongly measurable functions, [14, Corollary 5.11].

Phillips' example was revisited by Riddle and Saab [16] who proved that f is *universally* Pettis integrable (that is, Pettis integrable with respect to every Radon probability on $[0, 1]$) while the family of compositions of f with elements of $B_{\ell^1([0,1])}$

$$\{\langle f, y \rangle \mid y \in B_{\ell^1([0,1])}\} \subset \mathbb{R}^{[0,1]}$$

fails the so-called *Bourgain property* with respect to λ. In fact, this failure is equivalent to the non-Birkhoff integrability of f with respect to λ, as follows from the recent results of Cascales and the present author [3].

We next present Phillips' function and prove that, in fact, it satisfies a stronger integrability condition. Given two sets $A \subset B$, a Banach space X and $h : B \to X$, we write $h\chi_A$ to denote the X-valued function on B which agrees with h on A and vanishes on $B \setminus A$.

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EXAMPLE 2.1. The function $f : [0, 1] \to \ell^\infty([0, 1])$ defined by

$$f(t)(s) := \begin{cases} 1 & \text{if } t - s \text{ is a dyadic rational} \\ 0 & \text{otherwise} \end{cases}$$

is universally McShane integrable.

PROOF. Let μ be a Radon probability on $[0, 1]$. Then we can find a countable set (maybe empty) $C \subset [0, 1]$ such that $\mu(\{s\}) = 0$ for all $s \in D := [0, 1] \setminus C$. Since $f\chi_C$ is bounded and C is countable, $f\chi_C$ is Bochner integrable and so McShane integrable with respect to μ. Since $f = f\chi_C + f\chi_D$, it only remains to show that $f\chi_D$ is also McShane integrable with respect to μ.

Define an equivalence relation $\sim$ on $[0, 1]$ by saying that $t \sim s$ if and only if $t - s$ is a dyadic rational, and choose a set $G \subset [0, 1]$ such that for each $s \in [0, 1]$ there is a unique $t \in G$ with $t \sim s$. For each $t \in [0, 1]$, let A_t be the (countable) set made up of all $s \in [0, 1]$ for which $t \sim s$. Thus, $f(t) = \chi_{A_t}$ for every $t \in [0, 1]$ and $[0, 1]$ is the disjoint union of $\{A_t \mid t \in G\}$. Write $A_t = \{a_{t,1}, a_{t,2}, \ldots\}$ for all $t \in G$.

Fix $\varepsilon > 0$. Given $s \in D$, we have $s = a_{t,n}$ for some $t \in G$ and $n \in \mathbb{N}$, and we can choose an open set $\delta(s) \subset [0, 1]$ containing s such that $\mu(\delta(s)) \leq \varepsilon/2^n$. On the other hand, given $s \in C$, we define $\delta(s) := [0, 1]$. Then δ is a gauge on $[0, 1]$.

Let $\{(E_i, s_i)\}_{1 \leq i \leq p}$ be any McShane partition of $[0, 1]$ subordinate to δ. Assume without loss of generality that $s_i \neq s_j$ whenever $i \neq j$. For each $t \in G$, let I_t be the set (maybe empty) of all $i \in \{1, \ldots, p\}$ such that $s_i \in A_t \cap D$. Since $[0, 1]$ is the disjoint union of $\{A_t \mid t \in G\}$, we can write

$$\sum_{i=1}^{p} \mu(E_i) f\chi_D(s_i) = \sum_{t \in G} \left(\sum_{i \in I_t} \mu(E_i) \chi_{A_{s_i}} \right) = \sum_{t \in G} \left(\sum_{i \in I_t} \mu(E_i) \right) \chi_{A_t} \tag{2.1}$$

(bear in mind that $A_t = A_s$ whenever $t \sim s$). Fix $t \in G$. For each $i \in I_t$ there is $n_i \in \mathbb{N}$ such that $s_i = a_{t,n_i}$. Since $n_i \neq n_j$ whenever $i, j \in I_t$ are distinct, we have

$$\sum_{i \in I_t} \mu(E_i) \leq \sum_{i \in I_t} \mu(\delta(a_{t,n_i})) \leq \sum_{i \in I_t} \frac{\varepsilon}{2^{n_i}} \leq \varepsilon.$$

From the previous inequality and (2.1) it follows that

$$\left\| \sum_{i=1}^{p} \mu(E_i) f\chi_D(s_i) \right\|_{\ell^\infty([0,1])} \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $f\chi_D$ is McShane integrable (with integral $0 \in X$) with respect to μ. □

It should be mentioned that, if K is a compact Hausdorff topological space and X is a *separable* Banach space, then a bounded function from K to X^* is universally scalarly measurable if and only if it is universally Birkhoff integrable,

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see [18, Corollary 2]. This result relies on the work by Bourgain, Fremlin and Talagrand [2]. Without the separability assumption such equivalence fails in general (just bear in mind Phillips' example), but one might ask whether universal Pettis integrability is equivalent to universal McShane integrability.

In general, this question has a negative answer. Indeed, under the Continuum Hypothesis, the author has constructed in [20, Example 4.1] a bounded function $h : [0, 1] \to Y$ (where Y is a Banach space) which is not McShane integrable with respect to λ and such that, for each $y^* \in Y^*$, the composition $\langle h, y^* \rangle$ vanishes up to a countable set. It is easy to check that such an h is universally Pettis integrable. Clearly, when h is considered as a Y^{**}-valued function, h is universally Pettis integrable but not universally McShane integrable.

QUESTION 2.2. Is there a ZFC (Zermelo–Fraenkel with Choice) example of a universally Pettis integrable function which is not universally McShane integrable?

Any Pettis integrable function (defined on a compact Radon probability space) which is not McShane integrable would be a natural candidate to test the previous question. The example of Fremlin and Mendoza [10, 3C] does not give information about this matter, as their function takes values in $\ell^\infty \cong (\ell^1)^*$. Recently, Deville and the present author [4] have given another ZFC example of a Pettis integrable function which is not McShane integrable, now taking values in $\ell^1(\mathfrak{c}^+)$ (as usual $\mathfrak{c}^+$ denotes the smallest cardinal greater than the continuum).

3. Hagler's example

Hagler's example [6, p. 43] exhibits a 'nontrivial' scalarly measurable ℓ^∞-valued function which we denote by g (see Example 3.1 below). A suitable modification of the range space allowed Edgar to construct a scalarly bounded function which is not scalarly equivalent to a bounded function, see [24, Example 3-3-5]. Recently, the present author [21] benefited from Edgar's ideas to provide, for instance, a negative answer to [13, Problem 4] by showing that the real-valued function $\|g(\cdot)\|$ is not measurable for some equivalent norm $\|\cdot\|$ on ℓ^∞.

We next present Hagler's function and point out that it is not only Pettis integrable (as shown in [24, Example 4-2-4]) but also universally Birkhoff integrable. We first give an elementary proof of its integrability with respect to the usual product probability on $\{0, 1\}^{\mathbb{N}}$ (denoted by μ_c). We write $T := \bigcup_{n\in\mathbb{N}}\{0, 1\}^n$ and, given $u = (u_i)_{i\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$, we write $B_u := \{u|n : n \in \mathbb{N}\} \subset T$, where $u|n := (u_i)_{i=1}^n \in \{0, 1\}^n$. For each $m \in \mathbb{N}$ and each $\tau \in \{0, 1\}^m$, set

$$V_\tau := \{u \in \{0, 1\}^{\mathbb{N}} : u|m = \tau\}.$$

EXAMPLE 3.1. The function $g : \{0, 1\}^{\mathbb{N}} \to \ell^\infty(T)$ defined by

$$g(u) := \chi_{B_u}$$

is Birkhoff integrable with respect to μ_c.

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PROOF. Fix $\varepsilon > 0$. We can choose $n \in \mathbb{N}$ large enough such that $2^{-n} \leq \varepsilon$. Observe that $\{V_\sigma \mid \sigma \in \{0, 1\}^n\}$ is a partition of $\{0, 1\}^{\mathbb{N}}$ into finitely many clopen (so Borel) sets. Take arbitrary points $t_\sigma, t'_\sigma \in V_\sigma$ for every $\sigma \in \{0, 1\}^n$. We claim that

$$\Big\| \sum_{\sigma\in\{0,1\}^n} \mu_c(V_\sigma) g(t_\sigma) - \sum_{\sigma\in\{0,1\}^n} \mu_c(V_\sigma) g(t'_\sigma) \Big\|_{\ell^\infty(T)} \leq \varepsilon. \tag{3.1}$$

Indeed, fix $\tau \in T$ and let $\pi_\tau \in B_{\ell^\infty(T)^*}$ be the associated 'evaluation functional'. Observe that

$$\langle g, \pi_\tau\rangle(u) = \chi_{B_u}(\tau) = \chi_{V_\tau}(u) \quad \text{for every } u \in \{0, 1\}^{\mathbb{N}},$$

hence

$$\Big| \pi_\tau \Big(\sum_{\sigma\in\{0,1\}^n} \mu_c(V_\sigma)(g(t_\sigma) - g(t'_\sigma)) \Big) \Big| \leq \frac{1}{2^n} \sum_{\sigma\in\{0,1\}^n} |\chi_{V_\tau}(t_\sigma) - \chi_{V_\tau}(t'_\sigma)|.$$

Take any $\sigma \in \{0, 1\}^n$. Then $|\chi_{V_\tau}(t_\sigma) - \chi_{V_\tau}(t'_\sigma)|$ is either 0 or 1. In the second case we would have $t_\sigma \in V_\tau$ and $t'_\sigma \notin V_\tau$ or vice versa, so that $V_\sigma \cap V_\tau \neq \emptyset$ and $V_\sigma \setminus V_\tau \neq \emptyset$, hence $V_\tau \subset V_\sigma$. Since V_τ can be contained at most in one of the V_σ, it follows that

$$\Big| \pi_\tau \Big(\sum_{\sigma\in\{0,1\}^n} \mu_c(V_\sigma)(g(t_\sigma) - g(t'_\sigma)) \Big) \Big| \leq \frac{1}{2^n} \leq \varepsilon.$$

As $\tau \in T$ is arbitrary, (3.1) holds. Therefore, g is Birkhoff integrable with respect to μ_c (see [3, Proposition 2.6]). □

To prove that g is universally Birkhoff integrable (Proposition 3.3 below) we need the result isolated in Lemma 3.2, whose proof was kindly suggested by A. Avilés. Recall that a bounded sequence (x_n) in a Banach space X is called an ℓ^1-*sequence* if there is a constant $C > 0$ such that

$$\sum_{n=1}^N |a_n| \leq C \Big\| \sum_{n=1}^N a_n x_n \Big\|_X$$

for every sequence (a_n) in $\mathbb{R}$ and every $N \in \mathbb{N}$.

LEMMA 3.2. *The set $\{\chi_{V_\tau} \mid \tau \in T\} \subset C(\{0, 1\}^{\mathbb{N}})$ does not contain ℓ^1-sequences.*

PROOF. For $\tau = (u_i)_{i=1}^n$ and $\tau' = (v_i)_{i=1}^m$ in T, we write $\tau \prec \tau'$ if $n < m$ and $u_i = v_i$ for all $1 \leq i \leq n$. Given any set A, the symbol $[A]^2$ stands for the set of all subsets of A with cardinality two.

Fix an infinite set $S \subset T$. Define a function $F : [S]^2 \to \{0, 1\}$ as follows. Given $\{\tau, \tau'\} \in [S]^2$, set

$$F(\{\tau, \tau'\}) := \begin{cases} 0 & \text{if } \tau \prec \tau' \text{ or } \tau' \prec \tau, \\ 1 & \text{otherwise.} \end{cases}$$

Ramsey's theorem (see [11, Theorem 9.1]) ensures the existence of an infinite set $M \subset S$ such that F is constant on $[M]^2$. There are two cases to be considered.

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(a) *For any $\tau, \tau' \in M$ with $\tau \neq \tau'$, we have either $\tau \prec \tau'$ or $\tau' \prec \tau$.* This implies that M can be written as $M = \{\tau_n \mid n \in \mathbb{N}\}$ with $\tau_n \prec \tau_{n+1}$ for all $n \in \mathbb{N}$. Hence, $V_{\tau_{n+1}} \subset V_{\tau_n}$ for all $n \in \mathbb{N}$ and, therefore, we have

$$\left\| \sum_{k=1}^{n} (-1)^k \chi_{V_{\tau_k}} \right\|_\infty = 1 \quad \text{for all } n \in \mathbb{N}$$

(as usual, $\|\cdot\|_\infty$ denotes the supremum norm on $C(\{0, 1\}^{\mathbb{N}})$). It follows that $\{\chi_{V_\tau} \mid \tau \in S\}$ is not an ℓ^1-sequence.

(b) *The relationship $\tau \prec \tau'$ fails whenever $\tau, \tau' \in M$.* This means that $\{V_\tau \mid \tau \in M\}$ are pairwise disjoint and so

$$\left\| \sum_{\tau \in P} \chi_{V_\tau} \right\|_\infty = 1 \quad \text{for every finite set } P \subset M.$$

Hence, $\{\chi_{V_\tau} \mid \tau \in S\}$ is not an ℓ^1-sequence.

The proof of the lemma is complete. □

Following [16], we say that a family $\mathcal{H}$ of real-valued functions defined on a probability space (Ω, Σ, μ) has the *Bourgain property* if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \ldots, A_n \in \Sigma$, $A_i \subset A$ with $\mu(A_i) > 0$, such that: for each $h \in \mathcal{H}$ there is at least one A_i on which the oscillation of h is smaller than ε. Recently, this property has been used by Cascales and the present author [3, 19] to characterize the Birkhoff integrability of vector-valued functions.

PROPOSITION 3.3. *The function g (defined in Example 3.1) is universally Birkhoff integrable.*

PROOF. Since the family $\{\chi_{V_\tau} \mid \tau \in T\} \subset C(\{0, 1\}^{\mathbb{N}})$ is bounded and contains no ℓ^1-sequence (Lemma 3.2), it has the Bourgain property with respect to any Radon probability μ on $\{0, 1\}^{\mathbb{N}}$, see [13, Proposition 12.2]. Now, since $\langle g, \pi_\tau \rangle = \chi_{V_\tau}$ for all $\tau \in T$ (as we observed in the proof of Example 3.1) and $\{\pi_\tau \mid \tau \in T\} \subset B_{\ell^\infty(T)^*}$ is norming, an appeal to [3, Corollary 2.5] establishes that g is Birkhoff integrable with respect to μ. □

REMARK 3.4. According to the comments following Example 2.1, another approach to Proposition 3.3 is to check directly that g is universally scalarly measurable (and we note that this fact is proved implicitly in Hagler's arguments). However, this alternative approach is less elementary since it appeals to the deep work of [2].

Observe that g takes its values in the Johnson–Lindenstrauss space

$$JL_0 := \overline{\operatorname{span}}(c_0(T) \cup \{\chi_{B_u} \mid u \in \{0, 1\}^{\mathbb{N}}\}) \subset \ell^\infty(T),$$

see [12, Example 2] and [25] for its basic properties. Since JL_0 is a subspace of ℓ^∞, we know that every McShane integrable JL_0-valued function defined on a compact

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Radon probability space is Birkhoff integrable, see [9, Theorem 10]. A question arises as follows.

QUESTION 3.5. Let X be a subspace of ℓ^∞ satisfying Corson's property (C) (like JL_0). Are Pettis and Birkhoff integrability equivalent for X-valued functions?

Recall that a Banach space satisfies Corson's *property (C)* if every family of convex closed subsets with empty intersection contains a countable subfamily with empty intersection. Every weakly Lindelöf Banach space fulfills this property. By a well-known result of Edgar [7] (see [24, Theorem 3-4-5]), every scalarly measurable function taking values in a weakly Lindelöf Banach space is scalarly equivalent to a strongly measurable function. Therefore, the previous question has affirmative answer when X is a weakly Lindelöf subspace of ℓ^∞.

4. Talagrand's example

It is well known that *conditional expectations* always exist within the Bochner integral theory (see, for example, [6, Ch. 3]). However, in general this is not the case for the Pettis integral, as Talagrand made clear in [24, Section 6-4]. This pathology might appear even for bounded functions defined on 'reasonable' probability spaces, such as that in Example 4.1 below, which is taken from [24, Example 6-4-2].

It is natural to ask whether conditional expectations exist within the Birkhoff integral theory. This question was kindly brought to our attention by J. Diestel. We next provide a negative answer by showing that, in fact, Talagrand's function in Example 4.1 is Birkhoff integrable.

We first need to introduce some terminology. Let (A_n) be the sequence of all clopen subsets of $\{0, 1\}^{\mathbb{N}}$ such that $\mu_c(A_n) = 1/2$. Let $\theta : \{0, 1\}^{\mathbb{N}} \to \ell^\infty$ be the w^*-continuous function given by the formula $\theta(u) := (\chi_{A_n}(u))_{n\in\mathbb{N}}$. Then $L := \theta(\{0, 1\}^{\mathbb{N}}) \subset B_{\ell^\infty}$ is w^*-compact. We consider the associated image probability $\nu := \mu_c\theta^{-1}$ on $\mathrm{Borel}(L)$. For each $n \in \mathbb{N}$, we denote by $\rho_n \in B_{(\ell^\infty)^*}$ the nth coordinate projection.

EXAMPLE 4.1. The function $\varphi : \{0, 1\}^{\mathbb{N}} \times L \to \ell^\infty(T)$ defined by

$$\varphi(u, v)(\tau) := \begin{cases} \rho_n(v) & \text{if } \tau = u|n \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } \tau \notin B_u \end{cases}$$

is Birkhoff integrable with respect to the product probability $\mu_c \otimes \nu$.

PROOF. It suffices to check that the family $\{\langle\varphi, \pi_\tau\rangle \mid \tau \in T\}$ has the Bourgain property with respect to $\mu_c \otimes \nu$, because φ is bounded and $\{\pi_\tau \mid \tau \in T\} \subset B_{\ell^\infty(T)^*}$ is norming, see [3, Corollary 2.5].

Observe first that for each $m \in \mathbb{N}$ and each $\tau \in \{0, 1\}^m$, we have

$$0 \le \langle\varphi, \pi_\tau\rangle(u, v) = \rho_m(v)\chi_{V_\tau}(u) \le \chi_{V_\tau}(u) \tag{4.1}$$

for all $(u, v) \in \{0, 1\}^{\mathbb{N}} \times L$. Note also that $\langle\varphi, \pi_\tau\rangle$ is measurable.

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Fix $\varepsilon > 0$ and $A \in \mathrm{Borel}(\{0, 1\}^{\mathbb{N}}) \otimes \mathrm{Borel}(L)$ with $(\mu_c \otimes \nu)(A) > 0$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \tau_2 \in \{0, 1\}^n$ with $\tau_1 \neq \tau_2$ such that

$$(\mu_c \otimes \nu)(A \cap (V_{\tau_i} \times L)) > 0 \quad \text{for } i = 1, 2.$$

Take $m \geq n$ and $\tau \in \{0, 1\}^m$. Then $V_\tau \cap V_{\tau_i} = \emptyset$ for some $i \in \{1, 2\}$, so that χ_{V_τ} vanishes on V_{τ_i}. From (4.1) it follows that $\langle \varphi, \pi_\tau \rangle$ vanishes on $A \cap (V_{\tau_i} \times L)$. On the other hand, since any finite family of measurable functions has the Bourgain property, we can find sets $A_1, \ldots, A_p \subset A$, $A_j \in \mathrm{Borel}(\{0, 1\}^{\mathbb{N}}) \otimes \mathrm{Borel}(L)$ with $(\mu_c \otimes \nu)(A_j) > 0$, such that: for each $m < n$ and each $\tau \in \{0, 1\}^m$, there is at least one A_j for which the oscillation of $\langle \varphi, \pi_\tau \rangle$ on A_j is smaller than ε.

This shows that $\{\langle \varphi, \pi_\tau \rangle \mid \tau \in T\}$ has the Bourgain property with respect to $\mu_c \otimes \nu$ and the proof is complete. □

Let $\mathcal{A}$ be the σ-algebra on $\{0, 1\}^{\mathbb{N}} \times L$ made up of all sets $U \times L$ with $U \in \mathrm{Borel}(\{0, 1\}^{\mathbb{N}})$. It was shown in [24, Example 6-4-2] that the range of φ is contained in a subspace E of $\ell^\infty(T)$ such that φ does not admit a Pettis integrable E-valued conditional expectation with respect to $\mathcal{A}$. That is, there is no Pettis integrable function ψ from $(\{0, 1\}^{\mathbb{N}} \times L, \mathcal{A}, \mu_c \otimes \nu)$ to E such that $\int_A \psi \, d(\mu_c \otimes \nu) = \int_A \varphi \, d(\mu_c \otimes \nu)$ for all $A \in \mathcal{A}$. Of course, the same can be said if 'Pettis' is replaced by 'Birkhoff'. This fact and Example 4.1 open a door for future research.

QUESTION 4.2. When do Birkhoff integrable functions admit Birkhoff integrable conditional expectations?

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References

[1] M. Balcerzak and M. Potyrała, 'Convergence theorems for the Birkhoff integral', *Czech. Math. J.* **58**(4) (2008), 1207–1219.

[2] J. Bourgain, D. H. Fremlin and M. Talagrand, 'Pointwise compact sets of Baire-measurable functions', *Amer. J. Math.* **100**(4) (1978), 845–886.

[3] B. Cascales and J. Rodríguez, 'The Birkhoff integral and the property of Bourgain', *Math. Ann.* **331**(2) (2005), 259–279.

[4] R. Deville and J. Rodríguez, Integration in Hilbert generated Banach spaces, *Israel J. Math.* to appear.

[5] L. Di Piazza and D. Preiss, 'When do McShane and Pettis integrals coincide?', *Illinois J. Math.* **47**(4) (2003), 1177–1187.

[6] J. Diestel and J. J. Uhl Jr., *Vector Measures*, Mathematical Surveys, 15 (American Mathematical Society, Providence, RI, 1977), with a foreword by B. J. Pettis.

[7] G. A. Edgar, 'Measurability in a Banach space', *Indiana Univ. Math. J.* **26**(4) (1977), 663–677.

[8] D. H. Fremlin, 'The generalized McShane integral', *Illinois J. Math.* **39**(1) (1995), 39–67.

[9] ———, The McShane and Birkhoff integrals of vector-valued functions, Research Report 92–10, Mathematics Department, University of Essex, 1992.

[10] D. H. Fremlin and J. Mendoza, 'On the integration of vector-valued functions', *Illinois J. Math.* **38**(1) (1994), 127–147.
[11] T. Jech, *Set Theory*, Springer Monographs in Mathematics (Springer, Berlin, 2003), the third millennium edition, revised and expanded.
[12] W. B. Johnson and J. Lindenstrauss, 'Some remarks on weakly compactly generated Banach spaces', *Israel J. Math.* **17** (1974), 219–230.
[13] K. Musiał, 'Topics in the theory of Pettis integration', *Rend. Istit. Mat. Univ. Trieste* **23**(1) (1991), 177–262. School on Measure Theory and Real Analysis (Grado, 1991) (1993).
[14] B. J. Pettis, 'On integration in vector spaces', *Trans. Amer. Math. Soc.* **44**(2) (1938), 277–304.
[15] R. S. Phillips, 'Integration in a convex linear topological space', *Trans. Amer. Math. Soc.* **47** (1940), 114–145.
[16] L. H. Riddle and E. Saab, 'On functions that are universally Pettis integrable', *Illinois J. Math.* **29**(3) (1985), 509–531.
[17] J. Rodríguez, 'On the existence of Pettis integrable functions which are not Birkhoff integrable', *Proc. Amer. Math. Soc.* **133**(4) (2005), 1157–1163.
[18] ———, 'Universal Birkhoff integrability in dual Banach spaces', *Quaest. Math.* **28**(4) (2005), 525–536.
[19] ———, 'The Bourgain property and convex hulls', *Math. Nachr.* **280**(11) (2007), 1302–1309.
[20] ———, 'On the equivalence of McShane and Pettis integrability in non-separable Banach spaces', *J. Math. Anal. Appl.* **341**(1) (2008), 80–90.
[21] ———, 'Weak Baire measurability of the balls in a Banach space', *Studia Math.* **185**(2) (2008), 169–176.
[22] ———, 'Pointwise limits of Birkhoff integrable functions', *Proc. Amer. Math. Soc.* **137**(1) (2009), 235–245.
[23] ———, Convergence theorems for the Birkhoff integral, *Houston J. Math*. to appear.
[24] M. Talagrand, 'Pettis integral and measure theory', *Mem. Amer. Math. Soc.* **51**(307) (1984), ix+224.
[25] V. Zizler, *Nonseparable Banach Spaces*, Handbook of the Geometry of Banach Spaces, 2 (North-Holland, Amsterdam, 2003), pp. 1743–1816.

JOSÉ RODRÍGUEZ, Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
e-mail: joserr@um.es

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