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CONTINUITY PROPERTIES OF VECTOR-VALUED CONVEX FUNCTIONS

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Abstract

We extend various characterizations of scalar-valued lower semicontinuity and determine their relationship to the continuity of vector-valued convex functions. Order completeness of the range space is not assumed.

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0. Introduction

The continuity properties of scalar-valued convex functions are well documented (for example [9], [10]). Motivated by much recent activity in abstract convex optimization (for example [3], [4], [5], [15]) and vector-valued subdifferential calculus (for example [12], [13], [14]), this note develops a corresponding theory for the vector-valued case.

We extend and compare various characterizations of scalar-valued lower semicontinuity. Vector-valued lower semicontinuity has been considered by Théra [12], though only with a subdifferential calculus in mind. This requires order completeness of the range space, a restriction not required in our present context. Borwein [2], [3] has also considered continuity properties of vector-valued convex functions, mainly via the medium of multifunctions. Our approach, however, is more in the spirit of classical convexity-continuity arguments. Indeed, the abstract setting helps to elucidate the essential features of the classical arguments. A crucial result in our development is that a vector-valued convex function on $\mathbf{R}^n$ is continuous throughout the relative interior of its effective domain (Corollary 3.2), a property well known in the scalar case.

The generalized definitions of semicontinuity are given in Section 2, and their relationship to the continuity of convex functions is discussed in Section 3. In Section 4 we introduce a topology of uniform convergence which is useful in many applications (for example, optimization subject to cone constraints), and for which the results of the preceding sections are considerably simplified.

1. Preliminaries

Throughout this paper X shall denote a (real Hausdorff) *topological vector space* (t.v.s.) and (Y, K) shall denote a *partially ordered topological vector space* (p.o.t.v.s.), where K is a closed convex cone in Y and the partial order, $\geqslant_K$, is given by $y_1 \geqslant_K y_2$ if and only if $y_1 - y_2 \in K$. We denote $\tilde{Y} := Y \cup \{\infty\}$, where ∞ is an abstract maximal element. Adjoining ∞ permits the consideration of functions defined on the whole of X rather than just on a subset of X. Accordingly, the (*essential*) *domain* of a function $f\colon X \to (\tilde{Y}, K)$ is

$$\operatorname{dom} f := \{x \in X : f(x) \in Y\}.$$

For a set $A \subset X$, co A, aff A, sp A, int A, and $\bar{A}$ shall denote the *convex hull*, *affine hull*, *linear hull*, *interior* and *closure* of A respectively. The *relative interior* of dom f, ri(dom f), is the interior of dom f relative to $\overline{\operatorname{aff}}(\operatorname{dom} f)$. The *intrinsic core* of dom f, icr(dom f), is defined by

$$\text{(1)} \qquad \operatorname{icr}(\operatorname{dom} f) := \{x \in X : (\forall z \in \overline{\operatorname{aff}}(\operatorname{dom} f))(\exists \lambda_0 > 0)(\forall |\lambda| \leqslant \lambda_0)\ x + \lambda z \in \operatorname{dom} f\}.$$

It should be noted that many references (for example [2], [10]) use instead the *core* of dom f, cor(dom f), which is defined as in (1) except that $\overline{\operatorname{aff}}(\operatorname{dom} f)$ is replaced by X. Usually, however, only $\overline{\operatorname{aff}}(\operatorname{dom} f)$ is of any interest, so that the latter definition is unduly restrictive. Moreover, if dom f is convex then ri(dom f) = icr(dom f) under any one of the following conditions: (a) ri(dom f) $\neq \varnothing$, (b) $X = \mathbf{R}^n$, (c) dom f is closed and $\overline{\operatorname{sp}}(\operatorname{dom} f - \operatorname{dom} f)$, with the topology inherited from X, is barreled (see [10, page 31]).

A function $f\colon X \to (\tilde{Y}, K)$ is *convex* (often called *K-convex*) if

$$\text{(2)} \qquad f(\lambda x_1 + (1-\lambda)x_2) \leqslant_K \lambda f(x_1) + (1-\lambda) f(x_2)$$

whenever $x_1, x_2 \in \operatorname{dom} f$ and $0 < \lambda < 1$. Clearly (2) holds if $x_1 \notin \operatorname{dom} f$ or $x_2 \notin \operatorname{dom} f$. Also note that dom f is convex if f is convex.

A p.o.t.v.s. (Y, K) is *normal* (often said K is *normal* in Y) if there is a base $\mathcal{V}$ of neighbourhoods of 0 in Y such that

$$(\forall V \in \mathcal{V}) V = [V] := (V + K) \cap (V - K).$$

Most commonly occuring cones are normal, so that this is not a severe restriction. Note that we lose no generality in assuming that each $V \in \mathcal{V}$ is symmetric. If Y is a *locally convex space* (l.c.s.), we may also assume that each $V \in \mathcal{V}$ is closed (see [8, Chapter 2, 1.5]). There are many equivalences and consequences of normality for which the reader is referred to [6], [8], [11].

2. Semicontinuity and quasicontinuity

A function f: $X \to (\tilde{Y}, K)$ is *lower semicontinuous* (l.s.c.) at a, relative to a set $A \subset X$ containing a, if for each neighbourhood V of 0 in Y there is a neighbourhood N of 0 in X such that

$$f((a + N) \cap A) \subset f(a) + V + K; \tag{3}$$

if $A = X$, reference to A shall be omitted. We shall say that f is *lower quasicontinuous* (l.q.c.) if there is a base $\mathcal{V}$ of neighbourhoods of 0 in Y such that

$$L(y, V) := \{x : f(x) \in y + V - K\} \quad \text{is closed in } X, \tag{4}$$

for each $y \in Y$ and each $V \in \mathcal{V}$. Corresponding definitions of *upper semicontinuity* (u.s.c.) and *upper quasicontinuity* (u.q.c.) are obtained by replacing K by $-K$ in (3) and (4) respectively.

We shall also say that f satisfies property (E) if it has a closed *epigraph*, that is,

$$\text{epi } f := \{(x, y) : f(x) \leq_K y\} \quad \text{is closed in } X \times Y; \tag{E}$$

and that f satisfies property (L) if it has closed (sub-) *level sets*, that is,

$$L(y) := \{x : f(x) \leq_K Y\} \quad \text{is closed in } X, \tag{L}$$

for each $y \in Y$.

The definitions of l.s.c., (E) and (L) are immediate extensions of lower semicontinuity for scalar-valued functions. The motivation for l.q.c. is less apparent and will be made clearer in Section 3 (see Theorem 3.5). For the moment it will suffice to observe that l.s.c., l.q.c., (E) and (L) are all equivalent in the scalar case. More generally, we have the following implications.

THEOREM 2.1. (i) *l.q.c.* $\Rightarrow (E) \Rightarrow (L)$.

(ii) *If Y has a neighbourhood base $\mathcal{V}$ in which each neighbourhood possesses a majorant of itself, then* $(E) \Rightarrow$ *l.q.c.*

(iii) *If K has non-empty interior, then* $(L) \Rightarrow (E)$.

(iv) *If K is normal and has non-empty interior, then*

$$l.q.c. \Leftrightarrow (E) \Leftrightarrow (L).$$

(v) *l.s.c. (at each $x \in X$) $\Rightarrow (E)$.*

PROOF. (i) [l.q.c. $\Rightarrow (E)$] Let $\{(x_\alpha, y_\alpha)\} \subset \operatorname{epi} f$ and $(x_\alpha, y_\alpha) \to (x, y)$; here $\{(x_\alpha, y_\alpha)\}$ denotes a net, indexed by α. Fix $V \in \mathcal{V}$. There exists α_0 such that $y_\alpha \in y + V$, and hence

$$f(x_\alpha) \in y_\alpha - K \subset y + V - K,$$

for all $\alpha \geqslant \alpha_0$. Then $f(x) \in y + V - K$ by l.q.c. Since V is arbitrary and K is closed, it follows that $f(x) \in y - K$, so that $(x, y) \in \operatorname{epi} f$.

[$(E) \Rightarrow (L)$] Let $y \in Y$, $\{x_\alpha\} \subset L(y)$ and $x_\alpha \to x$. Then $(x_\alpha, y) \in \operatorname{epi} f$, so that $(x, y) \in \operatorname{epi} f$ by (E); thus $x \in L(y)$.

(ii) Fix $y \in Y$ and $V \in \mathcal{V}$. Let $\{x_\alpha\} \subset L(y, V)$ and $x_\alpha \to x$. Let $\bar{v} \in V$ be a majorant of V. Then

$$f(x_\alpha) \in y + V - K \subset y + \bar{v} - K,$$

so that $(x_\alpha, y + \bar{v}) \in \operatorname{epi} f$. Then $(x, y + \bar{v}) \in \operatorname{epi} f$ by (E), so that $x \in L(y, V)$.

(iii) Let $\{x_\alpha, y_\alpha\} \subset \operatorname{epi} f$ and $(x_\alpha, y_\alpha) \to (x, y)$. Fix $k_0 \in \operatorname{int} K$ and let V be a symmetric neighbourhood of 0 in Y such that $k_0 + V \subset K$. Then there exists α_0 such that $y_\alpha \in y + V$, and hence

$$\begin{aligned} f(x_\alpha) \in y_\alpha - K \subset y + V - K &= y + k_0 - (k_0 + V) - K \\ &\subset y + k_0 - K, \end{aligned}$$

for all $\alpha \geqslant \alpha_0$. Thus $f(x) \in y + k_0 - K$ by (L). Since k_0 is arbitrary and K is closed, it follows that $f(x) \in y - K$, so that $(x, y) \in \operatorname{epi} f$.

(iv) The proof depends on the fact that the hypothesis in (ii) is satisfied, and is best deferred until the end of Section 4.

(v) Let $\{(x_\alpha, y_\alpha)\} \subset \operatorname{epi} f$ and $(x_\alpha, y_\alpha) \to (x, y)$. Fix $V \in \mathcal{V}$ and choose a circled neighbourhood U of 0 in Y such that $U + U \subset V$. Using l.s.c. at x, there exists α_0 such that $f(x_\alpha) \in f(x) + U + K$ and $y_\alpha \in y + U$ for all $\alpha \geqslant \alpha_0$. Then, since $f(x_\alpha) \in y_\alpha - K$, it follows that

$$f(x) \in y_\alpha + U - K \subset y + U + U - K \subset y + V - K.$$

Then since V is arbitrary and K is closed, we have $f(x) \in y - K$, so that $(x, y) \in \operatorname{epi} f$.

The following examples show that the hypotheses made in Theorem 2.1 cannot in general be removed.

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EXAMPLE 2.2. Let $Y = l^1$ and $K = \{y \in l^1\colon (\forall i \in \mathbf{N}) y_i \geqslant 0\}$. For $n \in \mathbf{N}$, let $e_n = (y_i)$ where $y_i = \delta_{in}$, and define $f\colon \mathbf{R} \to \tilde{Y}$ by

$$f(x) = \begin{cases} e_n, & \text{if } |x| = 1/n, \\ \infty, & \text{otherwise.} \end{cases}$$

Then f satisfies (E) (note that $(0, y) \notin \overline{\text{epi}}\, f$ for any $y \in Y$) but f is not lower quasicontinuous. For example, if V is the unit ball in Y, then $1/n \in L(0, V)$ but $0 \notin L(0, V)$. This example shows that (ii) can fail without the 'majorant' hypothesis.

EXAMPLE 2.3. Let $Y = \mathbf{R}^2$ and $K = \mathbf{R}_+ \times \{0\}$. Define $f\colon \mathbf{R} \to \tilde{Y}$ by

$$f(x) = \begin{cases} (0, x), & \text{if } x > 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then f is K-convex and satisfies (L) (if $y = (y_1, y_2)$, then $L(y) = \{y_2\}$ if $y_1 \geqslant 0$ and $y_2 > 0$, otherwise $L(y) = \varnothing$) but f does not satisfy (E). For example, $(1/n, (0, 1/n)) \in \text{epi}\, f$ but $(0, (0, 0)) \notin \text{epi}\, f$. Thus (iii) can fail without the 'non-empty interior' hypothesis.

EXAMPLE 2.4. Let $X = Y = \{(x_i)\colon x_i = 0$ for all but finitely many $i \in \mathbf{N}\}$ and let $K = \{x \in X\colon (\forall i \in \mathbf{N}) x_i \geqslant 0\}$. Norm X by $\|x\| = \sup_i |x_i|$. Let $C = \{f_n\colon n \in \mathbf{N}\}$ where $f_n\colon X \to X$ is defined by

$$f_n(x) = (x_1, 2x_2, \ldots, nx_n, 0, \ldots).$$

If $x \in X$ then $x = (x_1, \ldots, x_m, 0, \ldots)$ for some $m = m(x)$. Hence

$$f_n(x) \leqslant_K \|x\|(1, 2, \ldots, m, 0, \ldots)$$

so that $\{f_n(x)\colon n \in \mathbf{N}\}$ is majorized for each $x \in X$. Since (X, K) is order complete [8, Chapter 1, Example 1.6] we can define a function $f\colon X \to X$ by

$$f(x) = \sup_n f_n(x).$$

It is routine to check that f is K-convex and lower quasicontinuous, and hence that f satisfies (E) and (L). However f is not continuous, and hence not lower semicontinuous. For example, $n^{-1} e_n \to 0$ but $f(n^{-1} e_n) = e_n \not\to 0 = f(0)$. This example shows that the reverse implication in (v) need not be true.

If X is locally convex and f is K-convex, it is evident that (E) and (L) are preserved when passing to the weak topology on X (for closed convex sets in a locally convex space are weakly closed). The same is true for lower quasicontinuity if, in addition, Y is locally convex (for then each $L(y, V)$ is convex). These observations lead to the following variation of Theorem 2.1 (ii).

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THEOREM 2.5. *If Y is locally weak compact (for example, if Y is a reflexive Banach space) and f is K-convex, then $(E) \Rightarrow$ l.q.c.*

PROOF. Fix $y \in Y$ and $V \in \mathcal{V}$ (we may assume that V is weak compact). Let $\{x_\alpha\} \subset L(y, V)$ and $x_\alpha \to x$. Hence, for each α, there exists $v_\alpha \in V$ such that $(x_\alpha, y + v_\alpha) \in \operatorname{epi} f$. By passing to a subnet of $\{v_\alpha\}$, if necessary, we may assume that $v_\alpha \to v \in V$ (weakly). Then $(x, y + v) \in \operatorname{epi} f$ by (E) and the remarks preceding the theorem, so that $x \in L(y, V)$.

If (Y, K) is normal, then upper and lower semicontinuity jointly characterize continuity, though this is not necessarily the case for upper and lower quasicontinuity (whence our notation).

Note also that upper semicontinuity, expressed in another way, is just the condition that the *multifunction* $H_f\colon X \to 2^Y$, defined by $H_f(x) := f(x) + K$, is *lower semicontinuous* (see, for example, [1], [3]). This prompts the warning that function and multifunction semicontinuity are not equivalent notions. In the present context we shall only be concerned with functions so that no confusion should arise.

Finally, we recall that scalar-valued lower semicontinuity has also a "limit inferior" characterization. The vector-valued case, however, is much less tractable. A particular problem is that the infimum (or supremum) of a set need no longer be "topologically" close to the set. In the case where (Y, K) is order complete, and assuming a certain compactness property, Penot has managed to define a limit inferior which is the infimum of cluster points [7, Definition 24 and Proposition 29]. Using this it is possible to show that a function f satisfies property (L) if and only if

$$(\forall x \in X) f(x) \leqslant_K \liminf_{z \to x} f(z).$$

We mention this only in passing, however, for it will have no direct bearing on our present work.

3. Continuity of convex functions

The classical condition (necessary and) sufficient for continuity of scalar-valued convex functions is that of being "bounded above on an open set" (see, for example, [10, Theorem 8]). In the vector-valued case, this suggests the existence of $y \in Y$ and an open set $N \subset X$ such that

$$f(N) \subset y - K \tag{5}$$

(see [2, Corollary 2.4(a)]). This assumption, however, is far too strong, especially in the case where int $K = \varnothing$. For example, if $X = Y = \mathbf{R}^2$ and $K = \mathbf{R}_+ \times \{0\}$, then $f(x_1, x_2) = (0, x_2)$ is a continuous linear function which does not satisfy (5). As another example, the identity map on l^p $(1 \leq p < \infty)$ does not satisfy (5). Before giving the appropriate modification of (5), we give a special case which motivates the transition.

THEOREM 3.1. *Let X be a t.v.s., let (Y, K) be a normal p.o.t.v.s., let f: $X \to (\tilde{Y}, K)$ be K-convex, let $a \in X$. If there is a (topologically) bounded set $C \subset Y$ and an open set $N \subset X$ containing a such that*

$$f(N) \subset C - K, \tag{6}$$

then f is continuous at a.

PROOF. Without loss of generality we may assume that $a = 0$, $f(0) = 0$ and that N and C are symmetric. Let $0 < \lambda < 1$. If $x \in \lambda N$ then, by convexity of f,

$$f(x) \in (1 - \lambda) f(0) + \lambda f(x/\lambda) - K \subset \lambda(C - K), \quad \text{and}$$
$$f(x) \in (1 + \lambda) f(0) - \lambda f(-x/\lambda) + K \subset \lambda(C + K),$$

so that

$$f(\lambda N) \subset \lambda((C + K) \cap (C - K)) = \lambda[C]. \tag{7}$$

Since K is normal and C is bounded, $[C]$ is bounded by [6, 3.2.6]. Hence, for each neighbourhood V of 0 in Y, there is $\lambda > 0$ such that $\lambda[C] \subset V$ which, together with (7), shows that f is continuous at 0.

COROLLARY 3.2. *Let (Y, K) be a normal p.o.t.v.s., let f: $\mathbf{R}^n \to (\tilde{Y}, K)$ be convex. Then f is continuous, relative to* aff(dom f), *throughout* ri(dom f) = icr(dom f).

PROOF. If ri(dom f) $= \varnothing$ there is nothing to prove. If ri(dom f) $\neq \varnothing$ then it contains $m + 1$ affinely independent points x_i, $i = 1, 2, \ldots, m + 1$, where m is the dimension of sp(dom f − dom f). Then (6) is satisfied by

$$C = \operatorname{co}\{f(x_i): i = 1, 2, \ldots, m + 1\} \quad \text{and} \quad N = \operatorname{ri}\operatorname{co}\{x_i: i = 1, 2, \ldots, m + 1\}.$$

Moreover, if $a \in$ ri(dom f), the points x_i can be chosen so that $a \in N$.

Corollary 3.2 generalizes a well-known scalar result (see, for example, [10, Corollary 8A]). One important consequence is that a convex function is always continuous along any "line" in the relative interior of its effective domain. An alternative proof of this, based on monotone convergence properties in normally ordered spaces, has been given by us in [15, Proposition 3.3].

We are now in a position to show that upper semicontinuity (that is, (3) with K

replaced by $-K$) is the appropriate modification of (5). Note that (6) is a strong way of ensuring upper semicontinuity.

THEOREM 3.3. *Let X be a t.v.s., let (Y, K) be a normal p.o.t.v.s., let f: $X \to (\tilde{Y}, K)$ be convex, let $a \in X$. Then f is continuous at a if (and only if) f is upper semicontinuous at a. In this case f is actually continuous throughout* int(dom f).

PROOF. The necessity of upper semicontinuity is immediate. Conversely, suppose that f is upper semicontinuous at a. As before we may assume that $a = 0$, $f(0) = 0$, and that each V and N (in the definition of u.s.c.) are symmetric. If $x \in N$ then

$$f(-x) \in K - f(x) \subset K - (V - K) = V + K$$

by convexity and upper semicontinuity of f. Hence, since K is normal,

$$f(N) \subset (V + K) \cap (V - K) = V,$$

so that f is continuous at 0.

Now let $x \in \operatorname{int}(\operatorname{dom} f)$; then there is $\lambda_0 > 0$ such that $\lambda x \in \operatorname{dom} f$ for all $|\lambda - 1| \leq \lambda_0$. Fix $V \in \mathcal{V}$ and choose a circled neighbourhood U of 0 in Y such that $U + U \subset V$. By the continuity of f at 0, there is a neighbourhood M of 0 in X such that $f(M) \subset U$. Then, by the convexity of f,

$$f\left(x + \frac{\lambda - 1}{\lambda} M\right) - f(x) \subset \frac{\lambda - 1}{\lambda} f(M) + \left(\frac{1}{\lambda} f(\lambda x) - f(x)\right) - K$$
$$\subset U + U - K \subset V - K$$

for λ sufficiently close to 1, noting that $\lim_{\lambda \to 1} f(\lambda x) = f(x)$ by Corollary 3.2. Thus f is upper semicontinuous, and hence continuous, at x.

It may often occur that int(dom f) $= \varnothing$ while icr(dom f) $\neq \varnothing$. In this event it is useful to consider the continuity of f relative to $\overline{\text{aff}}$(dom f). Indeed the proof of Theorem 3.3 still applies and we obtain:

COROLLARY 3.4. *Under the assumptions of Theorem 3.3, f is continuous at a, relative to* $\overline{\text{aff}}$(dom f), *if (and only if) f is upper semicontinuous at a, relative to* $\overline{\text{aff}}$(dom f). *In this case f is actually continuous, relative to* $\overline{\text{aff}}$(dom f), *throughout* icr(dom f).

The assertions of Theorem 3.3 are essentially also proved by Borwein [2, Corollary 2.4] and [3, Lemma 4], though via a multifunction approach which necessitates a slightly more complicated form of (3). Our proofs highlight the significance of upper semicontinuity, and also the "line" continuity of convex functions remarked at the end of Corollary 3.2.

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It is well known in the scalar case that lower semicontinuous convex functions with barreled domain space are continuous (see, for example, [10, Corollary 8B]). More generally, lower quasicontinuity is the appropriate hypothesis, as is shown below.

THEOREM 3.5. *Let* X *be a l.c.s., let* (Y, K) *be a normal p.o.l.c.s., let* f: $X \to (\tilde{Y}, K)$ *be convex, let* $\overline{\mathrm{sp}}(\mathrm{dom}\, f - \mathrm{dom}\, f)$ *be barreled* (*with the topology inherited from* X). *If* f *is lower quasicontinuous then* f *is continuous, relative to* $\overline{\mathrm{aff}}(\mathrm{dom}\, f)$, *throughout* $\mathrm{icr}(\mathrm{dom}\, f)$.

PROOF. Let $a \in \mathrm{icr}(\mathrm{dom}\, f)$ and fix $V \in \mathcal{V}$. Then $L(f(a), V)$ is a closed convex subset of $\overline{\mathrm{aff}}(\mathrm{dom}\, f)$ containing a (closed since f is l.q.c., convex since f is convex and V is convex). Let $d \in \overline{\mathrm{aff}}(\mathrm{dom}\, f)$; then $\lim_{\lambda \to 0} f(a + \lambda d) = f(a)$ by Corollary 3.2. Hence there is $\lambda_d > 0$ such that $a + \lambda d \in L(f(a), V)$ for all $|\lambda| \leqslant \lambda_d$, so that $L(f(a), V)$ is radial at a. Then

$$N = (L(f(a), V) - a) \cap (a - L(f(a), V))$$

is a barrel, and hence a neighbourhood of 0, in $\overline{\mathrm{sp}}(\mathrm{dom}\, f - \mathrm{dom}\, f)$ satisfying

$$f(a + N) \subset f(a) + V - K.$$

Thus f is upper semicontinuous, and hence continuous (by Corollary 3.4), at a relative to $\overline{\mathrm{aff}}(\mathrm{dom}\, f)$.

Theorem 3.5 should be contrasted with [3, Corollary 9] which shows that (E) characterizes continuity for convex functions when X is barreled and (Y, K) is a normal Fréchet space. Example 2.4 shows that Theorem 3.5 and [3, Corollary 9] need not be true if X is not barreled.

4. A topology of uniform convergence

Throughout this section (Y, K) is a p.o.l.c.s. (not necessarily normal) and K is a *pointed* cone (that is, $K \cap (-K) = \{0\}$). We denote the (topological) *dual space* of Y by Y', and $K^* = \{y' \in Y': (\forall y \in K) y'(y) \geqslant 0\}$ is the *dual cone* of K. We shall also assume that $\mathrm{int}_\tau K \neq \varnothing$, where $\mathrm{int}_\tau K$ denotes the interior of K when Y is given the Mackey topology, $\tau(Y, Y')$, that is, the topology of uniform convergence on weak* compact convex circled subsets of Y' (see [11]). We show that there is a topology on Y for which the results of the preceding sections are considerably simplified.

Fix $k_0 \in \mathrm{int}_\tau K$. Then $B = \{y' \in K^*: y'(k_0) = 1\}$ is a weak* compact convex *base* for K^*, that is, $0 \notin B$ and $K^* = \{\lambda b: b \in B, \lambda \geqslant 0\}$ (see [4, Lemma 2.3]).

Let $G = \overline{\text{ext}}\, B$, where ext B is the set of *extreme* points of B. As in [15], we define a norm on Y, called the *G-norm*, by

$$\|y\|_G := \sup_{y' \in G} |y'(y)|;$$

if K is not pointed then $\|\cdot\|_G$ is only a seminorm. The reason for using G rather than B is that $\|\cdot\|_G$ collapses in standard cases to familiar norms. For example, if $(Y, K) = (\mathbf{R}^n, \mathbf{R}^n_+)$ and $k_0 = (1, 1, \ldots, 1)$, then

$$\|y\|_G = \max_{1 \leqslant i \leqslant n} |y_i|$$

is the l^∞-norm. If $Y = C(T)$, the space of continuous real-valued functions on a compact set T, and $k_0(t) \equiv 1$, then

$$\|y\|_G = \sup_{t \in T} |y(t)|$$

is the usual supremum norm.

Let Y_G denote Y equipped the G-norm topology, that is, the topology of uniform convergence on G. It is clearly coarser than $\tau(Y, Y')$, but it is not necessarily a topology of the dual pair $\langle Y, Y' \rangle$ (see the remarks following Theorem 4.1). As G completely specifies K, in the sense that $k \in K$ (respectively $k \in \text{int}_\tau K$) if and only if $y'(k) \geqslant 0$ (respectively $y'(k) > 0$) for each $y' \in G$, the G-norm topology is often the most convenient topology for Y. This is especially so in problems of optimization where the feasible set is related to the cone K (see, for example, [15]).

The following result collects some fundamental properties of Y_G.

THEOREM 4.1 [15, Proposition 3.1].
(i) *K is normal in Y_G;*
(ii) $K^* - K^* = (Y_G)' \subset Y'$;
(iii) $\text{int}_G K = \text{int}_\tau K\ (\neq \varnothing)$, *where* $\text{int}_G K$ *denotes the G-norm interior of K;*
(iv) $(K_G)^* = K^*$, *where* $(K_G)^*$ *denotes the dual cone of K taken in* $(Y_G)'$.

PROOF. Statement (i) follows directly from the definition of normality. Since the G-norm topology is coarser than $\tau(Y, Y')$, (ii) $(\subset)$ and $\text{int}_G K \subset \text{int}_\tau K$ are evident. Conversely, if $y \in \text{int}_\tau K$ then $\inf_{y' \in G} y'(y) > 0$ (since G is weak* compact and $0 \notin G$), whence $y \in \text{int}_G K$; this completes (iii). Now each positive linear form on Y is continuous for every topology on Y for which K has non-empty interior [11, V.5.5]; this combined with (iii) proves (iv). Finally, (ii) $(=)$ follows from (i), (iv) and [8, Chapter 2, 1.21].

It is natural to compare Y_G to Y in the case where Y is itself a Banach space. Since K is normal precisely when $Y' = K^* - K^*$ [11, V.3.3 and V.3.5], (ii) of

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Theorem 4.1 shows that Y_G is a topology of the dual pair $\langle Y, Y' \rangle$ precisely when K is normal, in which case Y_G must be equivalent to Y. A direct proof of this, with equivalence constants, is given below.

THEOREM 4.2. *If (Y, K) is a normal normed space, with norm $\|\cdot\|$, then $\|\cdot\|$ and $\|\cdot\|_G$ are equivalent norms.*

PROOF. Since $B = \overline{\text{co}}\, G$ is convex and weak* compact, B is bounded [11, IV.5.1]. Thus there is $\alpha > 0$ such that $\|y'\|^{-1} \geqslant \alpha$ for all $y' \in G$. Hence

$$\|y\| = \sup_{\|y'\| \leqslant 1} |y'(y)| \geqslant \sup_{y' \in G} \|y'\|^{-1} |y'(y)| \geqslant \alpha \|y\|_G.$$

Since K is normal, [11, V.3.4] and [8, Chapter 2, 1.12] show that there is $\beta > 0$ such that each $y' \in Y'$ with $\|y'\| \leqslant 1$ has a decomposition $y' = \lambda_1 y_1' - \lambda_2 y_2'$, where $y_i' \in B = \overline{\text{co}}\, G$ and $\|\lambda_i y_i'\| \leqslant \beta$ $(i = 1, 2)$. Since

$$\|y'\| = \sup_{\|y\| \leqslant 1} |y'(y)| \geqslant \|k_0\|^{-1} |y'(k_0)| = \|k_0\|^{-1}$$

for all $y' \in B$, it follows that $|\lambda_i| \leqslant \beta \|k_0\|$ $(i = 1, 2)$ and that

$$\|y\| = \sup_{\|y'\| \leqslant 1} |y'(y)| \leqslant 2\beta \|k_0\| \sup_{y' \in B} |y'(y)| = 2\beta \|k_0\| \|y\|_G.$$

Hence

$$\alpha \|\cdot\|_G \leqslant \|\cdot\| \leqslant 2\beta \|k_0\| \|\cdot\|_G.$$

Theorem 4.1 shows that the normality hypotheses in the results of Section 3 can be omitted if Y is given its G-norm topology. Moreover, since Y_G satisfies the hypothesis of Theorem 2.1 (ii), it follows that lower quasicontinuity, (E) and (L) are all equivalent. Immediate consequences are a proof for Theorem 2.1 (iv) and a simpler version of Theorem 3.5:

PROOF OF THEOREM 2.1 (iv). Now follows from the above remarks and Theorem 4.2, noting that Y must be normable [8, Chapter 2, Proposition 1.10] and that K must be pointed [11, V.3.1].

COROLLARY 4.3. *Let X be a l.c.s., let (Y_G, K) be as above, let $f\colon X \to (\tilde{Y}_G, K)$ be K-convex, let* $\overline{\text{sp}}(\text{dom}\, f - \text{dom}\, f)$ *be barreled. If* epi f *is closed then f is continuous, relative to* $\overline{\text{aff}}(\text{dom}\, f)$, *throughout* icr(dom f).

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